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OPTIMALITY CONDITIONS FOR CONTROL PROBLEMS GOVERNED BY ABSTRACT SEMILINEAR DIFFERENTIAL EQUATIONS IN COMPLEX BANACH SPACES

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Abstract. We consider the problem to minimize an integral functional defined on the space of absolutely continuous functions and measurable controls with values in an infinite–dimensional complex Banach space. The states are governed by abstract first order semilinear differential equations and are subject to periodic or anti–periodic type boundary conditions. We derive necessary conditions for optimality and introduce the notion of a dual field of extremals to obtain sufficient conditions for optimality. Such a dual field of extremals is constructed and a dual optimal synthesis is proposed. The paper is an extension of an earlier paper written for real Banach spaces. This extension covers optimal control problems which are governed by equations like the Schrödinger equation and other equations arising in Quantum mechanics.

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1. Introduction. Let X and U be complex separable reflexive Banach spaces. Let X^* be the dual to X and let $\langle x^*, x\rangle$ denote the duality pairing between X^* and X . Let $A(t) : X \to X, t \in [0,T]$, be a family of densely defined linear operators with domains $D(A(t))$ and let $\mathbf{U}(t), t \in [0,T]$, be a family of subsets of U. We consider the problem to minimize

$$J(x,u) = \int_0^T L(t, x(t), u(t))dt + l(x(0), x(T)) \tag{1}$$

over all absolutely continuous functions $x : [0,T] \to X$ and Lebesgue measurable functions $u : [0,T] \to U$, the controls, subject to

$$\dot{x}(t) + A(t)x(t) = f(t, x(t), u(t)) \qquad \text{a.e. in } [0,T], \tag{2}$$

and

$$u(t) \in \mathbf{U}(t) \qquad \text{a.e. in } [0,T]. \tag{3}$$

The maps are defined between the following spaces $L : [0,T] \times X \times U \to \mathbb{R}$, $f : [0,T] \times X \times U \to X$, and

$$l(a,b) = \begin{cases} +\infty & \text{if } a \neq b, \\ 0 & \text{if } a = b, \end{cases} \quad \text{or} \quad \begin{cases} +\infty & \text{if } a \neq -b, \\ 0 & \text{if } a = -b, \end{cases} \tag{4}$$

depending on whether periodic or anti–periodic boundary conditions are imposed.

Necessary conditions for optimal control problems in abstract spaces have been considered in several papers starting with references [4, 7]. In [7] Lions considered a general distributed parameter control problem in a Banach space without a priori given initial conditions, so–called systems with insufficient data. Papageorgiou extended this result in [12] to the case of a general convex integral functional by using the Dubovitskii–Milyutin method (see [3]). His results found continuations in the analysis of problems with terminal data [8] as well as for abnormal problems and problems with nonoperator type equality constraints [9] using some extensions of the Dubovitskii-Milyutin method. In [10] following the ideas in references [12, 8, 9], the Dubovitskii–Milyutin method as well as its generalization by Walczak [13] are applied to the above abstract optimal control problem and the local Maximum Principle is formulated for the cases without and with nonoperator equality constraints. However, the case considered there concerned problems in real Banach spaces which limited its applications. In the paper the case of complex Banach spaces is considered and using a complex formulation of the Dubovitskii–Milyutin theory the results of [10] are generalized to this setting. Necessary conditions for optimality in the form of the local and global versions of the Maximum Principle for this problem are formulated here.

Classical sufficiency theorems which apply field theories (Weierstrass type sufficiency theorems) [1, 10, 15] or dynamic programming as well as (optimal) feedback control require assumptions imposed not only on the data of the problem, but also on objects which are constructed during the study of these problems. Quite often it is rather difficult to verify these assumptions. This is so because constraints appear and this only allows at best for piecewise regularity of the constructed objects (fields of extremals, feedback control, or value function). A dual approach to these objects is an attempt to overcome these difficulties (compare [11]). Now the whole construction is done in the dual space (in the space of multipliers). Some regularity is required too, however, only for objects which are dual to the classical objects and so these need not be regular. In this paper an extension of of the dual field theory and the dual feedback control [10] to more general fields of extremal — chains of flights — is described.

The main contribution of the paper lies in its setting of the problem in a complex Banach spaces. In this setting we can use our model for optimal control problems of systems governed by many nonlinear equations of quantum mechanics, e.g. the Schrödinger equation.

2. Necessary conditions for optimality. Let $X \subset H$ be a complex separable reflexive Banach space embedded continuously and densely in a Hilbert space H. Let U be a complex separable reflexive Banach space modelling the control space. Furthermore, let $||\cdot||$, $|\cdot|$, and $||\cdot||_*$ denote the norms in the spaces X, H, and X^* respectively where X^* denotes the dual space to X. We denote the inner product in H by $(\cdot,\cdot)$ and the duality pairings from X^* on X and from U^* on U by $\langle\cdot,\cdot\rangle$. Then we have for every $x \in X \subset H$ and $h \in H \subset X^*$ that $(x,h) = \langle h,x\rangle$. Let $A(t) : X \to X^*$ and $f : [0,T] \times X \times U \to X$ be operators, $L : [0,T] \times X \times U \to \mathbb{R}$ a functional, $\mathbf{U} : [\mathbf{0},\mathbf{T}] \to \mathcal{P}(\mathbf{U})$ a multifunction with nonempty, closed, convex subsets as images.

We make the following basic assumptions:

(H0) For each $(x,u) \in X \times U$, the functions $t \to L(t,x,u)$, $t \to f(t,x,u)$ are $\mathbb{L}$–measurable. For each $t \in [0,T]$, the functions $(x,u) \to L(t,x,u)$, $(x,u) \to f(t,x,u)$ are continuously differentiable. The set $\{(t,u) \in [0,T] \times U : u(t) \in \mathbf{U}(t)\}$ is $\mathbb{L} \times \mathbb{B}$–measurable, i.e. belongs to the σ–algebra of subsets of $[0,T] \times U$ generated by products of Lebesgue measurable subsets of $[0,T]$ and Borel measurable subsets of U.

(H1) For each $x \in D(A(t))$ the function $t \to A(t)x$ is X–measurable in $[0,T]$. For $t \in [0,T]$, the linear operators $A(t) : X \to X^*$ are continuous.

In order to derive the necessary optimality conditions below, we also need some technical assumptions on the operators A and f and the function L which we list separately.

(A1) There exist functions $a^i(\cdot) \in L^2_+, i = 1, 2, 3$ and numbers $p \geq 2$, $q \geq 2$ such that $|L(t, x, u)| \leq a_1(t) + a_2(t)|x|^p + a_3(t)||u||^q$ a.e. in $[0, T]$.

(A2) $||f(t, x, u)||_* \leq b_1(t) + b_2(t)|x| + b_3(t)||u||$ with some functions $b_1, b_2 \in L^2_+$ and $b_3 \in L^\infty_+$.

Let us call the problem (1)–(4) under assumptions (H0)–(H1) and (A1)–(A2) Problem I. If $u(t)$ is a control function subject to (3) and $x(t)$ is an absolutely continuous function corresponding to $u(t)$ (by (2)) and $L(t, x(t), u(t))$ is integrable, then the pair $(x(t), u(t))$ will be called admissible and $x(t)$ is an admissible trajectory. Along an admissible pair $(x(t), u(t))$ we need the following assumptions:

(B1) For $t \in [0, T]$, the linear operators $B(t) = A(t) - f_x(t, x(t), u(t)) : X \to X^*$ are coercive uniformly in $[0, T]$ i.e. there are $\lambda > 0, \alpha > 0$ such that

$$\langle B(t)h, h\rangle + \lambda |h|^2 \geq \alpha ||h||^2.$$

(B2) There exists a constant $c > 0$ such that for $t \in [0, T]$ we have

$$||B(t)||_{L(X,X^*)} \leq c.$$

(B3) There exists a constant $\eta > 0$ such that for $t \in [0, T]$ we have

$$||f_u(t, x(t), u(t))||_{L(X,X^*)} \leq \eta.$$

2.1. **The Local Extremum Principle.** Denote by $W([0, T])$ the Banach space

$$W([0, T]) = \{x \in L^2(X) : \dot{x} \in L^2(X^*)\}$$

with norm

$$||x||^2_{W([0,T])} = \int_0^T ||x(t)||^2 dt + \int_0^T ||\dot{x}(t)||^2_* dt. \tag{5}$$

Since the space of absolutely continuous functions $x : [0, T] \to X$ is dense in $W([0, T])$ and $J(x, u)$ is continuous in $W([0, T])$, Problem I considered in both spaces has the same value. However, to derive the necessary conditions it is more convenient to work in the space $W([0, T])$. Thus we seek a solution of the above problem in the subspaces of periodic or antiperiodic solutions,

$$WP([0, T]) = \{x \in W([0, T]) : x(0) = x(T)\},$$

respectively

$$WA([0, T]) = \{x \in W([0, T]) : x(0) = -x(T)\}.$$

We simply use the notation $\tilde{W}([0, T])$ to denote either of them depending on the circumstances. We also denote the space of square-integrable Lebesgue measurable functions mapping $[0, T]$ into U by $L^2(U)$. Using the

Dubovitskii–Milyutin method necessary conditions for optimality for Problem I can be derived in the form of the extremum principle given below. In its proof we need the following auxiliary result (see [6, Ch. 3, Vol. 1]):

Lemma 1. *Suppose* (H0) *and* (H1), *and let* $(x(t), u(t))$ *be an admissible pair such that assumptions* (B1), *and* (B2) *hold. Furthermore suppose the mapping* $t \to g(t)$ *belongs to* $L^2(X^*)$. *Then there exist a unique solution* $h \in \tilde{W}([0,T])$ *to the variational equation*

$$\dot{h}(t) + A(t)h(t) - f_x(t, x(t), u(t))h(t) = g(t) \tag{6}$$

and a unique solution $y \in \tilde{W}([0,T])$ *to the covariational or adjoint equation*

$$-\dot{y}(t) + A^*(t)y(t) - f_x^*(t, x(t), u(t))y(t) = g(t) \quad a.e. \tag{7}$$

For the Dubovitskii–Milyutin formalism we introduce the operator $F : \tilde{W}([0,T]) \times L^2(U) \to L^2(X^*)$ given by

$$F(x, u) = \dot{x}(t) + A(t)x(t) - f(t, x(t), u(t)). \tag{8}$$

In terms from optimization theory, our optimal control problem can be formulated as minimizing a functional $I(x, u)$ under the constraints $(x, u) \in Z_1 \cap Z_2$ where

$$Z_1 = \{(x, u) : F(x, u) = 0\}, \tag{9}$$

$$Z_2 = \{(x, u) : u \in V\} \tag{10}$$

and

$$V = \{u \in L^2(U) : u(t) \in \mathbf{U}(t) \text{ a.e.}\}.$$

We say the pair $(x, u) \in \tilde{W}([0,T]) \times L^2(U)$ is admissible if $(x, u) \in Z_1 \cap Z_2$. Note that the set Z_1 is an equality constraint, i.e. $intZ_1 = \emptyset$, and Z_2 can be either an inequality constraint (i.e. $intZ_2 \neq \emptyset$) or equality constraint (if $intZ_2 = \emptyset$).

Theorem 2 (Local Extremum Principle). *Assume* (H0), (H1), *and* (A1) *to* (A2) *hold and let the admissible process* (x, u) *be optimal for Problem I. Furthermore, suppose that conditions* (B1) *to* (B3) *are satisfied along* (x, u). *Then there exist* $y^0 \leq 0$ *and* $y \in \tilde{W}([0,T])$ *with the property that*

$$\|y(t)\|_{X^*} + |y^0| \tag{11}$$

does not vanish and such that they satisfy the adjoint equation

$$\dot{y}(t) = -f_x^*(t, x(t), u(t))y(t) + A^*(t)y(t) - y^0 L_x(t, x(t), u(t)) \quad a.e. \tag{12}$$

and the following maximum condition

$$\int_0^T \mathrm{Re}\langle y^0 L_u(t,x(t),u(t)) + f_u^*(t,x(t),u(t))y(t), u-u(t)\rangle dt \le 0 \tag{13}$$

for all $u \in V$.

Proof. If Z_2 is an inequality constraint, i.e. when $intZ_2 \neq \emptyset$, then we can derive the extremum principle for this problem using the Dubovitskii–Milyutin framework. However, since the classical Dubovitskii–Milyutin formalism is formulated in real Banach spaces, some modifications become necessary to adjust to the complex setting. More generally consider the problem to

> minimize $I : X \to \mathbb{R}$ in a complex Banach space X subject to $x \in Z = \bigcap_{i=1}^{n+1} Z_i$ where $Z_i \subset X$, $i = 1, \ldots, n$ represent inequality constraints (i.e. int $Z_i \neq \emptyset$) and $Z_{n+1} = \{x \in X : F(x) = 0\}$ represents an equality constraint.

Recall that a subset $C \subset X$ is a cone if $\lambda C \subset C$ for all $\lambda > 0$.

Definition 1. If C is a cone in a complex Banach space, then the **dual or polar cone** C^* consists of all continuous linear functionals for which their real part is nonnegative on C,

$$C^* = \{f \in X^* : \mathrm{Re} f(x) \ge 0 \quad \forall\, x \in C\}.$$

Then the following extremum principle is valid:

Theorem 3. *Suppose* I *attains a local minimum on* $Z = \bigcap_{i=1}^{n+1} Z_i$ *at* x_0 *and the cones* $DC(I, x_0)$, $FC(Z_i, x_0)$, $i = 1, \ldots, n$, *and* $TC(Z_{n+1}, x_0)$ *are nonempty and convex. Then there exist linear functionals* $f_0 \in DC(I; x_0)^*$, $f_i \in FC(Z_i; x_0)^*$, *for* $i = 1, 2, \ldots, n$, *and* $f_{n+1} \in TC(Z_{n+1}; x_0)^*$, *which do not all vanish identically, such that*

$$f_0 + f_1 + \ldots + f_n + f_{n+1} \equiv 0. \tag{14}$$

Thus we need to determine the cone of decrease for the functional I at (x,u), $DC(I,(x,u))$, the tangent cone to the set Z_1 at (x,u), $TC(Z_1,(x,u))$ and the feasible cone to the set Z_2 at (x,u), $FC(Z_2,(x,u))$ as well as the corresponding dual cones.

Repeating arguments from [12] and [8] and applying [3, Theorem 7.4], it follows that

$$DC(I,(x,u)) = \{(h,v) \in \tilde{W}([0,T]) \times L^2(U) : \\ \int_0^T \mathrm{Re}\left(\langle L_x(t,x(t),u(t)), h(t)\rangle + \langle L_u(t,x(t),u(t)), v(t)\rangle\right) dt < 0\} \tag{15}$$

and

$$DC(I,(x,u))^* = \{f_0 : \text{ there exists a } y^0 \leq 0 \text{ such that}$$
$$f_0(h,v) = y^0 \int_0^T \langle L_x(t,x(t),u(t)), h(t)\rangle + \langle L_u(t,x(t),u(t)), v(t)\rangle\, dt\}. \quad (16)$$

In view of the assumptions of the theorem the operator F defined in (8) is continuously Fréchet differentiable at (x,u) with derivative given by

$$F'(x,u)(h,v)(t) = \dot{h}(t) + A(t)h(t) - f_x(t,x(t),u(t))h(t) - f_u(t,x(t),u(t))v(t). \quad (17)$$

It follows from Lemma 1 that in fact for arbitrary $v \in L^2(U)$ and $z \in L^2(X^*)$, the equation

$$\dot{h}(t) + A(t)h(t) - f_x(t,x(t),u(t))h(t) - f_u(t,x(t),u(t))v(t) = z(t) \quad (18)$$

has a unique solution $h \in \tilde{W}([0,T])$. Hence $F'(x,u)$ maps $\tilde{W}([0,T]) \times L^2(U)$ onto $L^2(X^*)$ and the classical Lusternik Theorem applies. We obtain that

$$TC(Z_1,(x,u)) = \{(h,v) \in \tilde{W}([0,T]) \times L^2(U) :$$
$$\dot{h}(t) + A(t)h(t) - f_x(t,x(t),u(t))h(t) - f_u(t,x(t),u(t))v(t) = 0\}. \quad (19)$$

The structure of the corresponding dual cone will not be required in the remainder of the proof.

We now analyze the inequality constraint Z_2 given by (10). Theorem 10.5 in [3] implies that

$$FC(Z_2,(x,u)) = \{(h,v) \in \tilde{W}([0,T]) \times L^2(U) : v = \lambda(w-u), w \in V\} \quad (20)$$

$$FC(Z_2,(x,u))^* = \{f_2(h,v) = (0, f_2'(v)) \text{ where } \mathrm{Re} f_2' \text{ is a functional supporting the set } V \text{ at } u\}. \quad (21)$$

Hence by Theorem 3 there exist linear functionals $f_0 \in DC(I,(x,u))^*$, $f_1 \in TC(Z_1,(x,u))^*$ and $f_2 \in FC(Z_2,(x,u))^*$, not all zero, such that

$$f_0(h,v) + f_1(h,v) + f_2(h,v) \equiv 0 \qquad \forall\, (h,v) \in \tilde{W}([0,T]) \times L^2(U).$$

Let $(h,v) \in TC(Z_1,(x,u))$. Since $TC(Z_1,(x,u))$ is a subspace, we have $f_1(h,v) = 0$. Hence, for $(h,v) \in TC(Z_1,(x,u))$ and using (15) and (21), the Euler–Lagrange equation implies

$$y^0 \int_0^T \mathrm{Re}\left(\langle L_x(t,x(t),u(t)), h(t)\rangle + \langle L_u(t,x(t),u(t)), v(t)\rangle\right) dt + \mathrm{Re} f_2'(v) = 0. \quad (22)$$

By Lemma 1 the adjoint equation (12) has a solution $y \in \tilde{W}([0,T])$. Using the adjoint equation in (22) we obtain

$$- y^0 \int_0^T \mathrm{Re}\,\langle L_x(t,x(t),u(t)), h(t)\rangle\, dt = \int_0^T \mathrm{Re}\langle \dot{y}(t), h(t)\rangle dt$$
$$- \int_0^T \mathrm{Re}\langle A^*(t)y(t), h(t)\rangle dt + \int_0^T \mathrm{Re}\langle f_x^*(t,x(t),u(t))y(t), h(t)\rangle dt.$$

Using Lemma 5.1 from [14] about integration by parts and the fact that $h \in \tilde{W}([0,T])$ we obtain

$$- y^0 \int_0^T \mathrm{Re}\,\langle L_x(t,x(t),u(t)), h(t)\rangle\, dt$$
$$= \int_0^T \mathrm{Re}\langle y(t), -\dot{h}(t) - A(t)h(t) + f_x(t,x(t),u(t))h(t)\rangle dt. \quad (23)$$

Since $(h,v) \in TC(Z_1,(x,u))$, using (19) in (23) we have that

$$y^0 \int_0^T \mathrm{Re}\,\langle L_x(t,x(t),u(t)), h(t)\rangle\, dt$$
$$= \int_0^T \mathrm{Re}\langle y(t), f_u(t,x(t),u(t))v(t)\rangle dt. \quad (24)$$

Combining equations (22) and (24) it follows therefore that

$$\mathrm{Re} f_2'(v) = -\int_0^T \mathrm{Re}\langle y^0 L_u(t,x(t),u(t)) + f_u^*(t,x(t),u(t))y(t), v(t)\rangle dt.$$

The maximum condition (13) of the theorem follows now directly from the definition of a supporting functional to V at (x,u).

This proves the theorem for the case when the cone of decrease for I at (x,u), $DC(I,(x,u))$, given by (15) is nonempty. If it is empty, then actually

$$\int_0^T \mathrm{Re}\,(\langle L_x(t,x(t),u(t)), h(t)\rangle + \langle L_u(t,x(t),u(t)), v(t)\rangle)\, dt = 0$$

for all $(h,v) \in \tilde{W}([0,T]) \times L^2(U)$. This equation takes the form of the Euler–Lagrange equation with multipliers equal to $y^0 = 0$ and $f_2' \equiv 0$. Now proceeding as above, the theorem follows.

If the set V has empty interior in $L^2(U)$, then the geometric model for the Dubovitskii–Milyutin method changes since we now have two equality constraints and one of them is in the nonoperator form. Such a case is not included in the classical Dubovitskii–Milyutin formalism. The difficulty stems from the fact that in the derivation of the Euler–Lagrange equation from the Hahn–Banach theorem we need the property that

$$\left(\bigcap_{i=1}^n C_i\right)^* = \sum_{i=1}^n C_i^*$$

which in general only holds for open cones, i.e. for inequality constraints. For closed convex cones we have

$$\left(\bigcap_{i=1}^{n} C_i\right)^* = \overline{\sum_{i=1}^{n} C_i^*}$$

where the bar denotes the closure in the weak-* topology. In [13] the concept of *cones of the same sense* has been introduced and it has been shown that the algebraic sum of closed cones of the same sense is weak-* closed. This allowed to formulate an extension of the Dubovitskii–Milyutin formalism to problems with many equality constraints [13]. In [10] this extension has been applied to our problem formulated in a real Banach space and it has been shown that the cones dual to the tangent cones to the constraints Z_1 and Z_2 in our problem are of the same sense. This remains true for complex Banach spaces and thus the identical argument can be made as in [10]. We omit this argument here. □

Remark 2.1. If we assume in addition that

(A6) the mapping $t \to |\mathbf{U}(t)| = sup\{||u|| : u \in \mathbf{U}(t)\}$ belongs to L^2_+,

then, proceeding like in Theorem 3.1 in [12], we can express the maximum condition in the following simpler form:

$$\mathrm{Re}\langle y^0 L_u(t, x(t), u(t)) + f_u^*(t, x(t), u(t))y(t), v - u(t)\rangle \leq 0 \qquad (25)$$

for all $v \in \mathbf{U}(t)$ and almost everywhere in $[0, T]$.

Remark 2.2. If $\mathbf{U}(t) = L^2(U)$ for all $t \in [0, T]$, i.e. there are no constraints on the control functions, then the maximum condition (25) becomes

$$\mathrm{Re}\left(y^0 L_u(t, x(t), u(t)) + f_u^*(t, x(t), u(t))y(t)\right) = 0 \qquad \text{a.e.}$$

2.2. **The Global Extremum Principle.** The Maximum Principle formulated in Theorem 2 gives the local form of the maximum condition. The stronger global versions can be derived using Milyutin's method of variable time–transformations as it is outlined in [3] by applying the local version (Theorem 2) to an auxiliary problem whose controls are time–transformations along trajectories. The assumptions are the same except that differentiability is no longer required in the control variable u and that we choose a constant control set $\mathbf{U}(t) = U_0$. But now U_0 can be an arbitrary control set. We denote the correspondingly modified assumption (H0) by (H0'). Then we have:

Theorem 4 (Global Extremum Principle)**.** *Assume* (H0'), (H1)*, and* (A1) *to* (A2) *hold and let the admissible process* (x, u) *be optimal for Problem I. Furthermore, suppose that conditions* (B1) *to* (B3) *are satisfied along* (x, u)*. Then there exist* $y^0 \leq 0$ *and* $y \in \tilde{W}([0, T])$ *with the property that*

$$\|y(t)\|_{X^*} + |y^0| \tag{26}$$

does not vanish and such that they satisfy the adjoint equation

$$\dot{y}(t) = -f_x^*(t, x(t), u(t))y(t) + A^*(t)y(t) - y^0 L_x(t, x(t), u(t)) \quad a.e. \tag{27}$$

and the following maximum condition

$$\operatorname{Re}\left(y^0 L(t, x(t), u(t)) + f^*(t, x(t), u(t))y(t)\right) = \max_{v \in U_0} \operatorname{Re}\left(y^0 L(t, x(t), v) + f^*(t, x(t), v)y(t)\right). \tag{28}$$

We briefly outline the main idea of the proof, but need to refer the reader to Girsanov's monograph [3] for the details of the method of variable time–transformations. A non–negative function $\tau : [0, 1] \to [0, T], s \mapsto t = \tau(s)$ of the form

$$\tau(s) = \int_0^s w(r)dr, \quad \tau(1) = T \tag{29}$$

with $w \in L^1_\infty(0, 1)$ defines a time transformation from $[0, 1]$ onto $[0, T]$. The function τ is one–to–one in intervals where $w(s) > 0$. But if $w(s) = 0$ for all s in some interval Δ then $\tau(s) = \text{const}$ on Δ and all of Δ is mapped into one point. Choosing the left endpoint in such a case defines a unique inverse

$$\sigma(t) = \inf\{s \geq 0 : \tau(s) = t\}.$$

Let us denote by $\mathcal{B}_T$ the set

$$\mathcal{B}_T = \{w \in L^1_\infty(0, 1) : \int_0^1 w(r)dr = T\}.$$

Let (x, u) be an admissible process for Problem I defined over $[0, T]$ and let $w \in L^1_\infty(0, 1)$ be a non-negative function in $\mathcal{B}_T$. Set $R_1 = \{s \in [0, 1] : w(s) > 0\}$ and $R_2 = \{s \in [0, 1] : w(s) = 0\}$. The reparametrised process $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is defined as $\tilde{x} : [0, 1] \to \mathbb{R}^n, s \mapsto \tilde{x}(s) = x(\tau(s))$ while $\tilde{u} : [0, 1] \to U$ is chosen of the form

$$\tilde{u}(s) = \begin{cases} u(\tau(s)) & \text{for } s \in R_1 \\ \text{arbitrary} & \text{for } s \in R_2 \end{cases} \tag{30}$$

leaving, for the moment, the values on R_2 unspecified. The triple $(\tilde{x}(\cdot), \tilde{u}(\cdot), w(\cdot))$ is then an admissible process for the reparametrised problem and the values of the corresponding objectives are equal. Conversely, if the process $(\tilde{x}, \tilde{u}, w)$ is admissible for the reparametrised problem, then (x, u) defined by $x(t) = \tilde{x}(\sigma(t))$ and $u(t) = \tilde{u}(\sigma(t))$ is admissible for the original problem and the values of the corresponding objectives agree. Therefore, if (x_0, u_0) defined over $[0, T]$ is optimal for Problem I, then the process $(\tilde{x}_0, \tilde{u}_0, w)$ is optimal for any nonnegative $w \in \mathcal{B}_T$ and vice versa. Furthermore, if we freeze the control u_0 in the reparametrised problem, then the pair $(\tilde{x}_0, w)$ with $w \in \mathcal{B}_T$ is still optimal for the corresponding reparametrised problem where we only minimize over w.

Thus the Local Extremum Principle Theorem 2 can be applied to give necessary conditions for optimality. A judicious choice of the transformation w_0 allows to transform the conditions back to the original time-scale to obtain Theorem 4. Choose R_1 as a Cantor set (i.e. a perfect, nowhere dense subset of $[0, 1]$ which is obtained by deleting a countable number of disjoint intervals from $[0, 1]$) of positive measure $\nu(R_1)$ and let R_2 be its complement. Thus R_2 is a countable union of pairwise disjoint open intervals. In addition choose R_1 so that it has the additional property that, whenever $I \subset [0, 1]$ is any interval of positive measure so that $I \cap R_2$ is nonempty, then $I \cap R_2$ contains an interval. The construction of sets of this kind is a standard exercise in Real Variables. Then define w_0 as

$$w_0(s) = \begin{cases} T/\nu(R_1) & \text{for } s \in R_1 \\ 0 & \text{for } s \in R_2. \end{cases} \tag{31}$$

Finally, specifying $\widetilde{u}$ on R_2 so that it takes values in a countable dense subset of the control set U_0, the result follows. For more details of the argument we refer the reader to [3].

3. Canonical spray of flights. We now formulate sufficient conditions for optimality. We say that an admissible pair $(x(t), u(t))$, $t \in [0, T]$ is a **line of flight** (briefly l.f.) if it satisfies the following conditions (the maximum principle): along $x(t)$ there exist a conjugate vector function $y(t)$, absolutely continuous in $[0, T]$ with values in X^*, and a number $y^0 \leq 0$ such that $\|y(t)\|_{X^*} + |y^0|$ is nonvanishing and

$$\dot{y}(t) = -f_x^*(t, x(t), u(t))y(t) - y^0 L_x(t, x(t), u(t)) + A^*(t)y(t), \quad \text{a.e.} \tag{32}$$

$$\begin{aligned} &\mathrm{Re}\langle y(t), f(t, x(t), u(t))\rangle + y^0 L(t, x(t), u(t)) \\ &\qquad = \sup\{\mathrm{Re}\langle y(t), f(t, x(t), u)\rangle + y^0 L(t, x(t), u) : u \in \mathbf{U}(t)\}, \end{aligned} \tag{33}$$

$$l(x(0), x(T)) = l(y(0), y(T)) = 0 \tag{34}$$

or

$$\text{condition (34) is not satisfied.} \tag{35}$$

Since for the construction of a field of extremals the class of extremals which satisfy (34) would generate a too narrow field in our construction we will allow some extremals which do not satisfy this boundary condition.

For a given line of flight $(x(t), u(t))$ we write $z(t) = (-x^0(t), x(t))$ where

$$x^0(t) = \int_t^T L(\tau, x(\tau), u(\tau))d\tau, t \in [0, T]$$

and we write $p(t) = (y^0, y(t))$ where $(y(t), y^0)$ are the corresponding conjugate function satisfying conditions (32)–(34) or (35). We call a triple $(z(t), p(t), u(t))$ of functions such that $(x(t),\ u(t))$ define a line of flight and $(y(t), y^0)$ are the corresponding conjugate function satisfying conditions (32)–(34) or (35) a **canonical line of flight** (briefly c.l.f). In the usual way we define an open arc line of flight or canonical line of flight. Further, denote by $P \subset \mathbb{R}^2 \times X^*$ a set covered by graphs of $p\,(t)$ such that $z(t), p(t), u(t)$ is a c.l.f., which in the sequel may be reduced to a smaller one; let $Q \subset \mathbb{R} \times X$ denote a set covered by graphs of $x(t)$ such that $(x(t), u(t))$ is a line of flight. If $(t_0, p_0) \in P$, then we write $V(t_0, p_0)$ for the value of

$$y_0^0 \int_{t_0}^T L(t, x_0(t), u_0(t))dt - \mathrm{Re}\langle x_0(t_0), y_0(t_0)\rangle = -\mathrm{Re}\langle z_0(t_0), p_0(t_0)\rangle_z \tag{36}$$

where the $\mathrm{Re}\langle\cdot,\cdot\rangle_z$ in (30) is defined by the left hand side, and $z_0(t) = (-x_0^0(t), x_0(t)), p_0(t) = (y_0^0, y_0(t)), u_0(t)$ is a c.l.f. such that $p_0(t_0) = p_0$. Of course, the map $(t, p) \to V(t, p)$ in P might be a multifunction. Here we assume that

(HS) the set P is such that the map $(t, p) \to V(t, p)$ is single valued in P.

Further we shall consider only those c.l.f. which are subject to (HS). A rectifiable curve C lying in P is called bounded if $V(t, p)$ is bounded along it.

In order to construct a spray of flights from canonical lines of flights defined above, we must choose a certain family of arcs of canonical lines of flight which satisfy extra regularity hypotheses. In this section we describe such a family of arcs of canonical lines of flight Let W be a separable reflexive Banach space and on an open set $G \subset W$ define a pair of differentiable continuous functions

$$t^-(\sigma), t^+(\sigma), \qquad t^-(\sigma) < t^+(\sigma), \qquad \sigma \in G,$$

with values in [0,T]. Let

$$\begin{aligned} S^- &= \{(t, \sigma) : t = t^-(\sigma) \geq 0,\ \sigma \in G\}, \\ S &= \{(t, \sigma) : t^-(\sigma) < t^+(\sigma),\ \sigma \in G\}, \\ S^+ &= \{(t, \sigma) : t = t^+(\sigma) \leq T,\ \sigma \in G\}, \end{aligned}$$

and set $[S] = S^- \cup S \cup S^+$.

The family of canonical lines of flight is described by the functions

$$z(t,\sigma), p(t,\sigma), u(t,\sigma), \qquad (t,\sigma) \in [S] \tag{37}$$

and will be denoted by Σ. The sets of pairs (t,x) where $x = x(t,\sigma)$ with (t,σ) belonging to $S^-, S, S^+, [S]$ will be denoted by $E^-, E, E^+, [E]$, respectively, and the set of pairs $(t, z(t,\sigma))$ with (t,σ) in $S^-, S, S^+, [S]$ by D^-, D, D^+,$[D]$; E^{*-}, E^*, E^{*+}, $[E^*]$ will denote the sets of values of $(t, p(t,\sigma))$ with (t,σ) in S^-, S, $S^+, [S]$. We shall write (when $\sigma \in G$) $\widetilde{V}^+(\sigma)$ for the expression $V(t^+(\sigma), p(t^+(\sigma),\sigma))$.

We shall make the following regularity hypotheses on Σ:

(H2) The functions $z(t,\sigma)$ and $p(t,\sigma)$ are C^1 in $[S]$ and $u(t,\sigma)$ is Borel measurable in $[S]$.

(H3) The functions $\tilde{L}(t,\sigma) = L(t, x(t,\sigma), u(t,\sigma))$, $\tilde{f}(t,\sigma) = f(t, x(t,\sigma), u(t,\sigma))$ are continuous in $[S]$; they have continuous derivatives $\tilde{L}_\sigma(t,\sigma)$, $\tilde{f}_\sigma(t,\sigma)$ in $[S]$ and for each fixed (t,x) in $[E]$, the partial derivatives $\frac{\partial}{\partial\sigma} L(t,x,u(t,\sigma))$, $\frac{\partial}{\partial\sigma} f(t,x,u(t,\sigma))$ at $x = x(t,\sigma)$ satisfy the following relations:

$$\frac{\partial \tilde{L}}{\partial\sigma}(t,\sigma) = \frac{\partial L(t,x,u(t,\sigma))}{\partial\sigma} + L_x(t,x,u(t,\sigma)) x_\sigma(t,\sigma),$$
$$\frac{\partial \tilde{f}}{\partial\sigma}(t,\sigma) = \frac{\partial f(t,x,u(t,\sigma))}{\partial\sigma} + f_x(t,x,u(t,\sigma)) x_\sigma(t,\sigma).$$

(H4) The maps $S^- \to D^-$, $S \to D$ defined by $(t,\sigma) \to (t, z(t,\sigma))$ have the following property: given any arc $C_z \subset D^-$ (or $C_z \subset D$) with the description $t_1 \le \tau \le t_2$, $(-x^0(\tau), x(\tau))$ where $x(t)$ is a trajectory of an admissible pair $(x(t), u(t))$, $t \in [0,T]$, $x^0(t) = \int_t^T L(\tau, x(\tau), u(\tau)) d\tau$, issuing from $(t_1, z(t_1,\sigma_1))$, there exists a rectifiable curve $\Gamma \subset S^-$ (or $\Gamma \subset S$) issuing from (t_1,σ_1) such that every small arc of C_z issuing from $(t_1, z(t_1,\sigma_1))$ is the image under the map $(t,\sigma) \to (t, z(t,\sigma))$ of a small arc of Γ issuing from (t_1,σ_1).

Definition 2. If hypotheses (H2)–(H4) are satisfied then Σ will be called a canonical spray of flights.

For $(t,p) \in [E^*]$ let $Z(t,p), U(t,p)$ stand for the sets of values of $z(t,\sigma)$ and $u(t,\sigma)$ at those $(t,\sigma) \in [S]$ for which $p(t,\sigma) = p$. For $(t,x) \in [E]$, $P(t,x), U(t,x)$ denote the sets of values of $p(t,\sigma)$ and $u(t,\sigma)$ at those $(t,\sigma) \in [S]$ for which $x(t,\sigma) = x$. By an admissible pair of functions

$$z(t,p) \in Z(t,p), \qquad u(t,p) \in U(t,p), \qquad (t,p) \in [E^*], \tag{38}$$

we shall mean single valued functions $(z(t,p), u(t,p))$ in $[E^*]$ such that for each $(t_0,p_0) \in [E^*]$ there exists a canonical line of flight $(z(t), p(t), u(t))$ for which $p(t_0) = p_0$, $z(t_0, p_0) = z(t_0)$, $u(t_0, p_0) = u(t_0)$. By an admissible pair of functions

$$p(t,x) \in P(t,x), \qquad u(t,x) \in U(t,x), \qquad (t,x) \in [E],$$

we mean single valued functions $(z(t,x), u(t,x))$ in $[E]$ such that for each $(t_0, x_0) \in [E]$ there is a canonical line of flight $z(t) = (x^0(t), x(t))$, $p(t) = (y^0, y(t))$, $u(t)$ for which $x(t_0) = x_0$, $p(t_0, x(t_0)) = p(t_0)$, $u(t_0, x(t_0)) = u(t_0)$.

On any bounded rectifiable curve C in $[E^*]$ with the arc length description $t(s), p(s), 0 \leq s \leq s_C$, we define the curvilinear integral

$$\begin{aligned}
&\int_C \{y^0 L(t, x(t,p), u(t,p)) + \mathrm{Re}\langle y, f(t, x(t,p), u(t,p)) - A(t)x(t,p)\rangle\}dt \\
&+\mathrm{Re}\,\langle z(t,p), dp\rangle_z = \\
&\int_0^{s_C} (\{y^0(s) L(t(s), x(t(s), p(s)), u(t(s), p(s))) \\
&+\mathrm{Re}\langle y(s), f(t(s), x(t(s), p(s)), u(t(s), p(s))) - A(t(s))x(t(s), p(s))\rangle\}\frac{dt}{ds} \\
&+\mathrm{Re}\langle z(t(s), p(s)), \frac{dp}{ds}\rangle_z)ds
\end{aligned}$$

for any admissible pair of functions (32) such that $(\{y^0 L + \mathrm{re}\langle y, f - Ax\rangle\} + \mathrm{re}\langle z, dp/ds\rangle_z$ is a measurable function of the arc length s along C.

We call a subset R of $[E^*]$ **an exact set**, if for each bounded rectifiable curve $C \subset R$ with ends $(t_1, p_1), (t_2, p_2)$, having the property that the expression $(\{y^0 L + \mathrm{re}\langle y, f - Ax\rangle\} + \mathrm{Re}\langle z, dp/ds\rangle_z$ at almost every point of C takes the same value for all admissible pair (38), we have

$$\int_C \left\{y^0 L(t, x(t,p), u(t,p)) + \mathrm{Re}\langle y, f(t, x(t,p), u(t,p)) - A(t)x(t,p)\rangle\right\} dt \\ + \mathrm{Re}\langle z(t,p), dp\rangle_z = V(t_1, p_1) - V(t_2, p_2) \quad (39)$$

for each admissble pair $z(t,p) \in Z(t,p), u(t,p) \in U(t,p), (t,p) \in [E^*]$.

In this section we assume that we are given a spray of flights for which *the set* E^{*+} *is exact.*

Remark 3.1. Let C_z denote any small arc contained in D^- or D, with the description $t_1 \leq \tau \leq t_2$, $(-x^0(\tau), x(\tau))$ where $x(t)$ is a trajectory of an admissible pair $(x(t), u(t))$, $t \in [0,T]$, with $l(x(0), x(T)) = 0$, $x^0(t) = \int_t^T L(\tau, x(\tau), u(\tau))d\tau$, issuing from $(t_1, z(t_1, \sigma_1))$. We also represent C_z in terms of its arc length s as $t = t(s)$ $z = z(s) = (x^0(s), x(s))$, $s \in [0, s_{C_z}]$. Furthermore, let Γ denote a rectifiable curve in S^- or S such that small arcs of C_z issuing from $(t_1, z(t_1, \sigma_1))$ are, in accordance with (H4), the images under the map $(t, \sigma) \to (t, z(t, \sigma))$ of small arcs of Γ issuing from (t_1, σ_1). We represent Γ in terms of its arc length λ by functions $\bar{t}(\lambda), \bar{\sigma}(\lambda)$, so that the point (t_1, σ_1) corresponds to $\lambda = 0$. We can then define a continuous increasing function $s(\lambda)$ having inverse $\lambda(s)$, which satisfies the relation

$$t(s(\lambda)) = \bar{t}(\lambda), \qquad z(s(\lambda)) = z(\bar{t}(\lambda), \bar{\sigma}(\lambda)).$$

In turn, let C_z be the image under the map $(t, \sigma) \to (t, p(t, \sigma))$ of Γ issuing from

$$(t_1, p(t_1, \sigma_1)) = (t_1, p_1) = (t_1, y_1^0, y_1).$$

We easily see that to small arcs of Γ issuing from from (t_1, σ_1) there correspond small arcs of C_p issuing from (t_1, p_1). Thus we can express the final points of the small arcs of C_p as a function of s, $(t(s), p(s))$. Denote by (t_2, p_2) the terminal point of C_p which corresponds to that of $C_z(t_2, -x^0(t_2), x(t_2))$.

Lemma 5. *Let C_p be one of the arcs described in Remark 3.1. Then along C_p, the function $V(t,p)$ is bounded and there exists a Borel measurable admissible pair of functions $(z(t,p),\ u(t,p))$ along C_p. Moreover, the functions $z(t,p)$, $L(t, x(t,p), u(t,p))$, $f(t, x(t,p), u(t,p))$ are bounded along it.*

Proof. The proof is identical to that in [10, Lemma 3]. □

Lemma 6. *Let any $\sigma_0 \in G$ be given. Then for each $w \in W$ the function*

$$t \to \mathrm{Re}\langle p(t, \sigma_0), z_\sigma(t, \sigma_0) w\rangle_z = \\ - y^0(\sigma_0)\mathrm{Re}\langle x_\sigma^0(t, \sigma_0), w\rangle + \mathrm{Re}\langle y(t, \sigma_0), x_\sigma(t, \sigma_0) w\rangle \quad (40)$$

is constant in $(t^-(\sigma_0), t^+(\sigma_0))$.

Proof. Proceeding analogously as in [10, Lemma 3] and taking partial derivative with respect to time of $\mathrm{re}\langle p_0(t), z_0(t, \sigma_0) w\rangle$. □

Lemma 7. *Let Γ denote any small rectifiable curve in $[S]$ with (t_0, σ_0) as the initial point and (t_1, σ_1) as the terminal one. Then*

$$\int_\Gamma \frac{d}{dt}\mathrm{re}\langle z(t,\sigma), p(t,\sigma)\rangle_z dt + \frac{d}{d\sigma}\mathrm{re}\langle z(t,\sigma), p(t,\sigma)\rangle_z d\sigma = \\ V(t_0, p(t_0, \sigma_0)) - V(t_1, p(t_1, \sigma_1)).$$

The proof follows directly from the definition of the function $V(t,p)$.

Lemma 8. *In the set S^+ the quantity*

$$\mathrm{re}\langle p(t,\sigma), z_\sigma(t,\sigma) w\rangle_z \quad (41)$$

is identically zero, $w \in W$.

Proof. Let (t_0, σ_0) be any point in S^+. Let Γ be any sufficiently small rectifiable curve in S^+ which starts from (t_0, σ_0) and has the description $t = t^+(\sigma^w)$ where σ^w varies from σ_0 to σ_1 along asegment with direction w. Now let C be the image of Γ in E^{*+} under the map $(t, \sigma) \to (t, p(t, \sigma))$ with ends $(t_0, p_0), (t_1, p_1)$. From the exactness of E^{*+} we get

$$\begin{aligned} V(t_0, p_0) - V(t_1, p_1) &= \int_C \Big\{ y^0 L(t, x(t,p), u(t,p)) + \mathrm{Re}\langle y, f(t, x(t,p), u(t,p)) \\ &\quad - A(t)x(t,p)\rangle \Big\}\, dt + \mathrm{Re}\langle z(t,p), dp\rangle_z \\ &= \int_\Gamma (y^0(\sigma)\tilde{L}(t,\sigma) + \mathrm{Re}\langle y(t,\sigma), \tilde{f}(t,\sigma) - A(t)x(t,\sigma)\rangle \\ &\quad + \mathrm{Re}\langle y_t(t,\sigma), x(t,\sigma)\rangle)dt + \mathrm{Re}\langle z(t,\sigma), p_\sigma(t,\sigma)d\sigma\rangle_z \\ &= \int_\Gamma \frac{d}{dt}\mathrm{Re}\langle z(t,\sigma), p(t,\sigma)\rangle_z dt + \frac{d}{d\sigma}\mathrm{Re}\langle z(t,\sigma), p(t,\sigma)\rangle_z d\sigma \\ &\quad - \int_\Gamma \mathrm{Re}\langle p(t,\sigma), z_\sigma(t,\sigma)d\sigma\rangle. \end{aligned}$$

From the above in view of Lemma 7 we obtain

$$\int_\Gamma \mathrm{Re}\langle p(t,\sigma), z_\sigma(t,\sigma)d\sigma\rangle = 0.$$

Because Γ is arbitrarily small and σ^w varies from σ_0 to σ_1 along a segment with direction w, we get the assertion of the lemma. □

As a direct consequence of Lemmas 6 and 8 we have the following corollary.

Corollary 9. *The quantity* (41) *is identically zero in* $[S]$.

Similarly as Lemma 5 we obtain the following lemma

Lemma 10. *Along an arc $t_1 \le \tau \le t_2$, $x(\tau)$ lying in E^- (or E), there exist Borel measurable functions $p(t,x), u(t,x)$, $(t,x) \in [E]$ with the property that the functions $p(t,x)$, $L(t,x,u(t,x))$, $f(t,x,u(t,x))$ are bounded along it.*

Theorem 11. *Let C_z and C_p be as described in Remark 3.1. Then the following relation holds for some admissible pair $p(t,x), u(t,x)$, $(t,x) \in [E]$:*

$$\begin{aligned} &V(t_1,p_1) - V(t_2,p_2) - \mathrm{Re}\langle x(t_2), y_2\rangle + \mathrm{Re}\langle x(t_1), y_1\rangle + x^0(t_2)y_2^0 - x^0(t_1)y_1^0 \\ &= \int_{C_z} (y^0(t,x)L(t,x,u(t,x)) + \mathrm{Re}\langle y(t,x), f(t,x,u(t,x)) - A(t)x\rangle)dt \\ &\quad - \mathrm{Re}\langle p(t,x), dz\rangle_z. \end{aligned}$$

Proof. Let $e(s) = (\frac{dt}{ds}, \frac{dz}{ds})$ stand for the direction of the tangent to C_z defined for a.e. s in $[0, s_{C_z}]$. Let s_0 be any point in $[0, s_{C_z}]$ such that $e(s)$ and $p(s)$ are approximately continuous at it. We set $t_0 = t(s_0)$, $x_0 = x(s_0)$, $e_0 = e(s_0)$, $\dot{t}_0 = dt(s_0)/ds$, $\dot{z}_0 = dz(s_0)/ds$, $A_0 = A(t_0)x_0$. Let $p_0 = (y_0^0, y_0)$, u_0 be any admissible vectors from the sets $P(t_0, x_0), U(t_0, x_0)$ such that $p(t_0, \sigma_0) = p_0$, $u(t_0, \sigma_0) = u_0$ for any (t_0, σ_0) belonging to the graph of Γ. We also put $f_0 = f(t_0, x_0, u_0)$, $L_0 = L(t_0, x_0, u_0)$ and let λ_0 be such that $\sigma_0 = \bar{\sigma}(\lambda_0)$.

Denote by γ a sufficiently small arc Γ starting from (t_0, σ_0) defined over an interval $I = [\lambda_0, \lambda_2]$ of values of λ, i.e. the functions $\bar{t}(\lambda)$, $\bar{\sigma}(\lambda)$ are restricted now to the interval I. Denote by ΔV the difference in $V(t, p)$ at the ends of the small arc C_p starting from (t_0, p_0) which is the image of γ, and denote by Δs the corresponding difference in s. By Lemma 7, Corollary 9, and taking into account Remark 3.1, we obtain

$$\begin{aligned}
\Delta R &= -\Delta V - \mathrm{Re}\langle x(\bar{t}(\lambda_2)), y(\bar{t}(\lambda_2)), \bar{\sigma}(\lambda_2))\rangle + \mathrm{Re}\langle x(t_0), y_0\rangle \\
&\qquad + x^0(\bar{t}(\lambda_2)) y^0(\bar{\sigma}(\lambda_2)) - x^0(t_0) y_0^0 \\
&= \int_\gamma (y^0(\sigma)\tilde{L}(t,\sigma) + \mathrm{Re}\langle y(t,\sigma), \tilde{f}(t,\sigma) - A(t)x(t,\sigma)\rangle \\
&\qquad + \mathrm{Re}\langle y_t(t,\sigma), x(t,\sigma)\rangle)dt + \mathrm{Re}\langle z(t,\sigma), p_\sigma(t,\sigma)d\sigma\rangle_z \\
&\qquad - \int_\gamma (\mathrm{Re}\langle z_t(t,\sigma), p(t,\sigma)\rangle_z + \mathrm{Re}\langle z(t,\sigma), p_t(t,\sigma)\rangle_z dt \\
&\qquad + \mathrm{Re}\langle p(t,\sigma), z_\sigma(t,\sigma)d\sigma\rangle_z + \mathrm{Re}\langle z(t,\sigma), p_\sigma(t,\sigma)d\sigma\rangle_z \\
&= \int_I [(y^0(\bar{\sigma}(\lambda))\tilde{L}(\bar{t}(\lambda), \bar{\sigma}(\lambda)) + \mathrm{Re}\langle y(\bar{t}(\lambda), \bar{\sigma}(\lambda)), \tilde{f}(\bar{t}(\lambda), \bar{\sigma}(\lambda)) \\
&\qquad - A(\bar{t}(\lambda))x(\bar{t}(\lambda), \bar{\sigma}(\lambda)))\rangle)\frac{dt}{ds} - \mathrm{Re}\langle p(\bar{t}(\lambda), \bar{\sigma}(\lambda)), \frac{dz}{ds}\rangle_z] ds(\lambda).
\end{aligned} \tag{42}$$

Since $p(t,\sigma)$, $\tilde{L}(t,\sigma)$, $\tilde{f}(t,\sigma)$, $z(t,\sigma)$, $A(t)x(t,\sigma)$ are continuous on γ, we deduce that they are bounded on I. This along with (40) implies the uniform boundedness of the ratio $\Delta R/\Delta s$ for all sufficiently small Δs. Thus the function $\bar{V}(s) = V(t(s), p(s))$ is locally Lipschitz (as $z(s)$ and $p(s)$ are locally Lipschitz). To prove the assertion of the theorem it is enough to show that

$$\lim \frac{\Delta R}{\Delta s} = \{y^0 L_0 + \mathrm{Re}\langle y_0, f_0 - A_0\rangle\}\dot{t}_0 - \mathrm{Re}\langle p_0, \dot{z}_0\rangle_z \qquad \text{as} \quad \Delta s \to 0.$$

But this is quite analogous to the corresponding part of the proof of Lemma 25.3 in [15] if we take there

$$\begin{aligned}
\varphi = \varphi(\lambda) &= \left(\left\{y^0\tilde{L} + \mathrm{Re}\left\langle y, \tilde{f} - Ax\right\rangle\right\}\frac{dt}{ds} - \mathrm{Re}\left\langle p, \frac{dz}{ds}\right\rangle_z\right) \\
&\qquad - \left(\left\{y_0^0 L_0 + \mathrm{Re}\left\langle y_0, f_0 - A_0\right\rangle\right\}\dot{t}_0 - \mathrm{Re}\langle p_0, \dot{z}_0\rangle_z\right).
\end{aligned}$$

□

4. Chain of flights. Now we are going to extend our argument from one spray of flights to a sequence of sprays forming a chain of flights.

Definition 3. A finite sequence of canonical sprays of flights

$$\Sigma_1, \Sigma_2, ..., \Sigma_n \tag{43}$$

will be **called a chain of flights** if for $i = 1, 2, ..., N$, they fit together in inverse order so that the set E_i^{*-}corresponding to $\sum_i^*$ contains E_{i+1}^{*+} corresponding to $\sum_{i+1}^*$.

In order to derive some more information on our sets E_i^{*-}, E_{i+1}^{*+} we need one more hypothesis for each $\sum_i$. For convenience we omit the indices i.

(H5) The maps $S^- \to E^{*-}$ defined by $(t,\sigma) \to (t, p(t,\sigma))$ are **descriptive**: a map $S^- \to E^{*-}$ is descriptive if for each $(t,\sigma) \in S^-$ and any given rectifiable curve $C \subset E^{*-}$ starting from $(t, p(t,\sigma))$ there exists a rectifiable curve $\Gamma \subset S^-$ starting from (t,σ) such that every small arc of C starting from $(t, p(t,\sigma))$ is the image under the above map of a small arc of Γ starting from (t,σ) (see also [15]).

We put $\bar{V}(s) = V(t(s), p(s))$ along any rectifiable curve C in E^{*-} with the arc length description $t = t(s)$, $p = p(s)$, $0 \le s \le s_C$.

Theorem 12. *The function $\bar{V}(s)$ is absolutely continuous along C and, for almost all s in $[0, s_C]$, and each admissible pair $(z(t,p), u(t,p))$, $(t,p) \in [E^{*-}]$ we have:*

$$\begin{aligned} \frac{d}{ds}\bar{V}(s) &= -(\{y^0(s)L(t(s), x(t(s), p(s)), u(t(s), p(s))) \\ &\quad +\mathrm{Re}\langle y(s), f(t(s), x(t(s), p(s)), u(t(s), p(s))) \\ &\quad -A(t(s))x(t(s), p(s))\rangle\}\frac{dt}{ds} + \mathrm{Re}\langle z(t(s), p(s)), \frac{dp}{ds}\rangle_z \end{aligned} \tag{44}$$

Proof. The proof proceeds analogously as in Theorem 4.1 of [10]. ☐

Integrating (44) along C, we obtain the following corollary:

Corollary 13. *For each admissible pair $z(t,p), u(t,p)$, $(t,p) \in [E^{*-}]$,*

$$\begin{aligned} V(t_1, p_1) - V(t_2, p_2) = \int_C &\{y^0 L(t, x(t,p), u(t,p)) \\ &+ \mathrm{Re}\langle y, f(t, x(t,p), u(t,p)) - A(t)x(t,p)\rangle\}dt + \mathrm{Re} z(t,p), dp\ \rangle_z, \end{aligned} \tag{45}$$

where $(t_1, p_1), (t_2, p_2)$ are the initial and final points of C.

A direct consequence of Corollary 13 we get the following corollary.

Corollary 14. *The set* E^{*+}_{i+1} *of* $\sum_{i+1}$ *is exact for* $\sum_i$ *as well for* $\sum_{i+1}$, $i = 1, \dots, N$.

If E^{*+}_1 of $\sum_1$ happens to be an exact set, then all sets E^{*+}_i, $i = 1, \dots, N$, are also exact sets and in each $[S_i]$ Corollary 9 holds.

Let G_N, G_1 be open sets of parameters σ^N, σ^1, respectively, associated with the spray $\sum_N$, and $\sum_1$. Let $(x^*(t), u^*(t)), t \in [0, T]$ be a l.f. satisfying (34). Suppose

(H6) There exists at least one $\sigma^N_0 \in G_N$, and $\sigma^1_0 \in G_1$such that $x^*(t) = x(t, \sigma^N_0), t \in [t^-(\sigma^N_0), t^+\sigma^N_0]$, $x^*(t) = x(t, \sigma^1_0)$, $t \in [t^-(\sigma^1_0), t^+(\ \sigma^1_0)]$. There exits a countinuously differentiable function $h : G_1 \to G_N$, such that $x(t, \sigma^N_0) = x(t, h(\sigma^1_0))$, and for each σ^1 the trajectory $x(t, \sigma^1)$, $t \in [t^-(\sigma^1), t^+(\ \sigma^1)]$ define the same l.f. as $x(t, h(\sigma^1)), t \in [t^-(h(\sigma^1)), t^+(h(\sigma^1))]$ and $h(G_1) = G_N$. Moreover, if condition (34) is not satisfied for a c.l.f. $z(t, \sigma^1), p(t, \sigma^1), u(t, \sigma^1)$ then

$$\mathrm{Re}\langle y(t^+(\sigma^1), \sigma^1), x_{\sigma^1}(t^+(\sigma^1), \sigma^1)\rangle = 0.$$

A chain of flights which satisfies hypothesis (H6) will be called a **distinguished chain of flights**.

Lemma 15. *For any distinguished chain of flights, the quantity* (41) *is identically zero in* S^+_1.

Proof. First notice that the Fenchel conjugate of the function l defined in (4) is the same function (for $g : X \to \bar{R}$, $g^*(x^*) = \sup\{\langle x^*, x\rangle - g(x),\ x \in X\}$, $x^* \in X$), thus condition (34) may equivalently be written as

$$(y(0), -y(T)) \in \partial l(x(0), x(T)) \tag{46}$$

where ∂l denotes the subdifferential of the convex function l. In view of (H6) we can confine ourselves to the case when our canonical lines of flight satisfy condition (34). Since the function $\sigma^1 \to \langle p(t^+(\sigma^1), \sigma^1), z_{\sigma^1}(t^+(\sigma^1), \sigma^1)w\rangle_z$ is continuous, the set G_0 of σ^1 defining those canonical lines of flight for which condition (34) is satisfied is open. Hence and by (34) $l_{\sigma^1}(x(0, h(\sigma^1)), x(T, \sigma^1))$ $= 0$. From this we infer the assertion of the lemma. □

Suppose we are given a distinguished chain of flights with the trajectory x^* described in (H6). Denote again by Q the set covered by graphs of trajectories of that chain.

Remark 4.1. Now let C_z be any trajectory with the description $0 \le t \le T$, $(-x^0(t), x(t))$ where $x(t)$ is a trajectory of an admissible pair $(x(t), u(t))$, $t \in [0,T]$, $l(x(0), x(T)) = 0$,whose graph contained in Q, $x^0(t) = \int_t^T L(\tau, x(\tau), u(\tau))d\tau$. Then we can divide C_z into a finite number of small arcs (see (H4)), which are described in Remark 3.1 and in the way presented there we are able to obtain an arc C_p with ends $(0, p(0)), (T, p(T))$, corresponding to C_z. Of course, for each such small arcs Theorem 4 holds. This implies the corollary below.

Corollary 16. *Let C_z and C_p be as in Remark 4.1. Then the following relation holds for some admissible pair $(p(t,x), u(t,x))$, $(t,x) \in Q$:*

$$\begin{aligned} &V(0,p(0)) + \mathrm{Re}\langle y(0), x(0)\rangle - x^0(0)y^0(0) \\ = &\int_{C_z} (y^0(t,x)L(t,x,u(t,x)) + \mathrm{Re}\langle y(t,x), f(t,x,u(t,x)) \\ &\quad -A(t)x\rangle)dt - \mathrm{Re}\langle p(t,x), dz\rangle_z. \end{aligned} \tag{47}$$

The last corollary allows us to derive sufficient conditions for a relative minimum of J. Denote by $G_p \subset G^1$ the set of those σ^1 for which $x(t,\sigma^1)$ satisfy (34) and let $E_p^- = \{(0,x) : x = x(0, h(\sigma^1),\ \sigma^1 \in G_p\}$, and denote by $x(t, k(\sigma^1))$, $t \in [t^-(h(\sigma^1), t^+(\sigma^1)]$, $\sigma^1 \in G_p$ the trajectories which correspond to $x(t,\sigma^1)$, $t \in [t^-(\sigma^1), t^+(\sigma^1)]$.

Theorem 17. *Assume that we are given a distinguished chain of flights. Let a triple of functions $z^*(t) = (-x^{0*}(t), x^*(t))$, $p^*(t) = (y^{0*}, y^*(t))$, $u^*(t)$, $t \in [0,T]$, satisfying* (34) *be a member of our chain and suppose that*

$$J(x^*, u^*) = \min_{\sigma^1 \in G_p} \int_0^T L(t, x(t, k(\sigma^1)), u(t, k(\sigma^1)))dt. \tag{48}$$

Then

$$-\bar{y}^0 J(x^*, u^*) \le -\bar{y}^0 J(x,u) \tag{49}$$

relative to all admissible pairs $(x(t), u(t))$, $t \in [0,T]$, with $x(t)$ satisfying (34)*, for which the graphs of $x(t)$ are contained in Q, $x(0) \in E_p^-$ and $\bar{y}^0 = y^0(\bar{\sigma}^N)$ where $\bar{\sigma}^N$ is such that $x(0, \bar{\sigma}^N) = x(0)$.*

Proof. Let $(x(t), u(t))$, $t \in [0,T]$, be any admissible pair with $x(t)$ satisfying (34) and its graph contained in Q, $x(0) \in E_p^-$. Put $l(x(0), x(T)) = 0$, $x^0(t) = \int_t^T L(\tau, x(\tau), u(\tau))d\tau$ and let C_z denote the trajectory $z(t) = (-x^0(t), x(t))$ and C_p the corresponding (see Remark 4.1) trajectory in the (t,p)-space with the initial point $(\bar{y}^0, \bar{y}(0)) = p(0)$. Let $\bar{z}(t), \bar{p}(t) = (\bar{y}^0, \bar{y}(t))$, $\bar{u}(t)$,

$t \in [0,T]$, $\bar{x}(0) = x(0)$, with $\bar{x}(t), \bar{y}(t)$ satisfying (34) be a member of our spray. Then by Corollary 16 and equation (33) we have

$$\begin{aligned} &\bar{y}^0 \int_0^T L(t,\bar{x}(t),\bar{u}(t))dt - \bar{y}^0 \int_0^T L(t,x(t),u(t))dt \\ = &\int_0^T [y^0(t,x(t))(L(t,x(t),u(t,x(t))) - L(t,x((t),u(t)))) \qquad (50) \\ &+\mathrm{Re}\langle y(t,x(t)), f(t,x,(t),u(t,x(t))) - f(t,x(t),u(t))\rangle]dt \geq 0. \end{aligned}$$

Hence $-\bar{y}^0 J(\bar{x},\bar{u}) \leq -y^0 J(x,u)$ and by (48) we get (46). □

Remark 4.2. In the assertion of Theorem 17, the multiplier $\bar{y}^0$ depends on $x(t)$, i.e. $\bar{y}^0$ is determined by $x(0)$ (see Remarks 3.1 and 4.1). If $y^0(\sigma^1) \neq 0$ for all $\sigma^1 \in G_p$, then (x^*,u^*) is a strong relative minimum for J.

5. Dual feedback control. In this section we assume all notations and assumptions of the previous sections. In Theorems 12 and 17 we used a Borel measurable selection $u(t,x)$ of the multifunction $U(t,x)$, $(t,x) \in Q$. In applications this function is considered a feedback control or synthesis. In practice it often plays a more useful role than minimizers of functionals and it is very important to have an algorithm to determine feedback functions. In fact, in Section 4 we gave a method to calculate the multifunction $U(t,x)$ whose existence is ensured by the existence of a chain of flights. However, from the theorems in Section 4 we cannot infer that there exists a selection of $U(t,x)$ which would be an optimal feedback control. That is so because relations (50) and (44) are satisfied only for some admissible pair $(p(t,x),u(t,x))$, $(t,x) \in [E]$, and hence we need the aditional requirement (48) in the sufficiency result Theorem 17.

The aim of this section is to study properties of selections $u(t,p)$ of the multifunction $U(t,p)$ introduced at the begining of Section 3 which we will call dual feedback controls. We give also sufficient conditions under which they become optimal feedback controls. To this effect we need one more hypothesis in each spray of flights of a distinguished chain of flights.

(H7) The map $S \to E^*$ defined by $(t,\sigma) \to (t,p(t,\sigma))$ is descriptive.

We put $\bar{V}(s) = V(t(s),p(s))$ along any rectifiable curve C in E^{*-} or E^* with the arc length description $t = t(s)$, $p = p(s)$, $0 \leq s \leq s_C$.

Theorem 18. *The function $\bar{V}(s)$ is absolutely continuous along C and, for almost all s in $[0,s_C]$, and each admissible pair $(z(t,p),u(t,p))$, $(t,p) \in [E^*]$*

we have:

$$\frac{d}{ds}\bar{V}(s) = -(\{y^0(s)L(t(s),x(t(s),p(s)),u(t(s),p(s))) + \mathrm{Re}\langle y(s), f(t(s),x(t(s),p(s)),u(t(s),p(s))) - A(t(s))x(t(s),p(s))\rangle\}\frac{dt}{ds} + \mathrm{Re}\langle z(t(s),p(s)), \frac{dp}{ds}\rangle_z. \tag{51}$$

Proof. The proof is exactly the same as that of Theorem 12. □

Integrating (51) along C, we obtain the following corollary:

Corollary 19. *For each admissible pair* $z(t,p), u(t,p)$, $(t,p) \in [E^*]$,

$$V(t_1,p_1) - V(t_2,p_2) = \int_C \{y^0 L(t,x(t,p),u(t,p)) + \mathrm{Re}\langle y, f(t,x(t,p),u(t,p)) - A(t)x(t,p)\rangle\}dt + \mathrm{Re}\langle z(t,p), dp\ \rangle_z, \tag{52}$$

where $(t_1,p_1), (t_2,p_2)$ *are the initial and final points of* C.

Now we give a precise definition of a dual feedback control and we show that the selections $u(t,p)$ of $U(t,p)$ are really dual feedback controls. Let a Borel measurable function $u = u(t,p)$ from a set $P \subset \mathbb{R}^2 \times X^*$ of the points $(t,p) = (t,y^0,y)$, $t \in [0,T]$, $y^0 \leq 0$, into $\mathbf{U}(t)$ be given. Then the differential equation

$$\dot{x} + A(t)x = f(t,x,u(t,p)), \tag{53}$$

has in general many solutions $x(t,p)$ in P. We say that $u = u(t,p)$ is a *dual feedback control* if we can choose any solution $x(t,p)$ of (53) such that for each admissible trajectory $x(t)$ lying in $Q = \{(t,x) : x = x(t,p),\ (t,p) \in P\}$, there exists a function $p(t) = (y^0, y(t))$ of bounded variation lying in P which satisfies $x(t) = x(t,p(t))$.

Proposition 20. *If* $z(t,p), u(t,p)$, $(t,p) \in P$, *is an admissible pair of functions, then* $u(t,p)$ *is a dual feedback control.*

Proof. By the definition of an admissible pair of functions we easily see that $x(t,p)$ from $z(t,p) = (-x^0(t,p), x(t,p))$ is a solution to (53). In Remarks 3.1 and 4.1 for each admissible trajectory $z(t) = (-x^0(t), x(t))$ with $x(t)$ lying in Q, the construction of a function $p(t)$ is described. From that construction we see that $p(t)$ lies in P and is of bounded variation and that $x(t,p(t)) = x(t)$. □

For a given dual feedback $u(t,p)$ with corresponding trajectory $x(t,p)$, $(t,p) \in P$, let us define the dual value function $S_D(t,p)$ in the set P as

$$S_D(t,p) = \inf\{-y^0 \int_t^T L(\tau, x(\tau), u(\tau))d\tau\}$$

where the infimum is taken over all admissible pairs $(x(\tau), u(\tau))$ restricted to $[t,T]$ with $x(\tau)$ satisfying (34) and having graph in Q.

A dual feedback $u(t,p)$ will be called optimal if for each $(t,p) \in P$ there exists an absolutely continuous function $\bar{p}(\tau) = (\bar{y}^0, \bar{y}(\tau))$, $\tau \in [0,T]$, with graph in P such that

$$S_D(t,p) = -y^0 \int_t^T L(\tau, x(\tau, \bar{p}(\tau)), u(\tau, p(\tau)))d\tau \tag{54}$$

($x(t,p)$ is a function corresponding to $u(t,p)$) and a function $V(t,p)$ in P such that the triple $V(t,p)$, $z(t,p) = (-x^0(t,p), x(t,p))$, $u(t,p)$ satisfies (48) for all rectifiable curves C lying in P.

The next theorem gives sufficient conditions for the existence of an optimal dual feedback control.

Theorem 21. *Assume we are given a canonical spray of flights satisfying* (H5) *and that there exists an admissible pair of functions $z(t,p) = (-x^0(t,p), x(t,p))$, $u(t,p)$, $(t,p) \in P$ for it such that*

$$\begin{aligned} &y^0 L(t, x(t,p), u(t,p)) + \mathrm{Re}\langle y, f(t, x(t,p), u(t,p)) - A(t)x(t,p)\rangle \\ &= \sup_{\tilde{u}(t,p)\in U(t,p)} \{y^0 L(t, x(t,p), \tilde{u}(t,p)) \\ &\qquad + \mathrm{Re}\langle y, f(t, x(t,p), \tilde{u}(t,p)) - A(t)x(t,p)\rangle\} \end{aligned} \tag{55}$$

where $U(t,p)$ is a multifunction corresponding to the chain of flights described in Section 4. Moreover, suppose that for each $(t,p) \in P$ there exists a $\bar{p}(\tau)$, $\tau \in [t,T]$, such that $\bar{x}(\tau) = x(\tau, \bar{p}(\tau))$, $\bar{p}(\tau), \bar{u}(\tau) = u(\tau, \bar{p}(\tau))$ is a member of our spray. Then $u(t,p)$ is an optimal dual feedback control.

Proof. It is quite anologous to the corresponding Theorem 4.2 in [10]. □

Remark 5.1. The existence of an optimal dual fedback control gives us more information about the problem under consideration then the sufficiency Theorem 17. However, to obtain a sufficiency theorem on the existence of an optimal feedback (Theorem 21) we need much stronger assumptions.

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