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NOTE ON DECREASING REARRANGEMENT OF FOURIER SERIES

S. KOSTYUKOVSKY AND A. OLEVSKII

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Abstract. In this note we construct a complete orthonormal system (ONS) of uniformly bounded functions, such that for any function $f \in L^2[0,1]$ its Fourier series with respect to the system, taken in decreasing order of magnitude of the coefficients, converges almost everywhere.

1. The famous Kolmogorov "rearrangement theorem" (1927) shows that there exists a function $f \in L^2[0,1]$ such that the Fourier series

$$f \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi int} \tag{1}$$

after an appropriate rearrangement of its terms diverges almost everywhere.

In 1961, Olevskiĭ and Uljanov proved that such a divergence theorem holds for any orthonormal basis in L^2. Namely:

Theorem A. *Let* $\{\psi_n\}$ *be an orthonormal basis in* $L^2[0,1]$. *Then there exists a series*

$$\sum a_n\psi_n(x), \qquad \sum |a_n|^2 < \infty \tag{2}$$

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which diverges a.e. after some rearrangement.

See [4], Ch. III for the proof and historical details.

Recently, T. Körner [1] gave an interesting improvement of Kolmogorov's original theorem. Namely, he constructed a series

$$\sum a_n e^{2\pi int} \quad , \qquad \sum |a_n|^2 < \infty$$

such that its decreasing coefficient rearrangement diverges:

$$\limsup_{\eta \to 0+} \Big| \sum_{|a_n| \geq \eta} a_n e^{2\pi int} \Big| = \infty \quad \text{a.e .}$$

The proof of Körner's result combines the general approach developed in [4] with a new delicate construction in which specific properties of character's system are used in an essential way.

Therefore, the natural question arises: whether such an improvement can be done in a general setting?

We prove here that this is not the case.

Theorem 1. *There exists an orthonormal basis* $\{\psi_n\}$ *in* $L^2[0,1]$*, consisting of uniformly bounded functions, such that every series* (2)*, ordered by decreasing magnitude of the coefficients, converges almost everywhere.*

By a rearrangement of (2) in decreasing order of magnitude of the coefficients we mean a rearrangement that puts the term whose coefficient has maximum modulus over all coefficients first, followed by the term whose coefficient has maximum modulus over all remaining coefficients, and so on. If there are terms with coefficients of the same modulus those terms may be placed in any order relative to each other.

2. Our construction of $\{\psi_n\}$ is a slight modification of one used in [3].

We use the following results:

Lemma 1 (*See* [4]*, Ch. IV*)**.** *There exist square matrices* $A_k = \|a_{ij}^{(k)}\|$ $(1 \leq i, j \leq 2^k)$ $k = 1, 2, \ldots$ *such that:*

1) *Each matrix is orthogonal.*
2) $a_{i1}^{(k)} = \frac{1}{\sqrt{2^k}}$ *for* $i = 1, 2, ..., 2^k$.
3) *The inequality*

$$\sum_j |a_{ij}^{(k)}| < C \qquad \forall i, k$$

holds (where C *is an absolute constant).*

Lemma 2 *(See [3], §1). Let $\{p_k(x)\}$ $(1 \le k \le n)$ be an orthonormal system of polynomials with respect to the Rademacher system. Then for any numbers $\{d_k\}$ the following inequality holds:*

$$\int_0^1 \max_{1\le m\le n} \left| \sum_{k=1}^{m} d_k p_k(x) \right|^2 dx \le A \cdot \sum_{k=1}^{n} |d_k|^2 \ .$$

(A is an absolute constant).

Consider the Walsh system $\{w_n\}$ on $[0,1]$. (For definitions of Walsh and Rademacher systems see, for example, [4]).

It is well known that $\{w_n\}$ is a basis in $L^2[0,1]$, consisting of uniformly bounded functions, and that Fourier series with respect to this system converges a.e. for any function $f \in L^2$. The Rademacher system is a subsystem of $\{w_n\}$.

Our system $\{\psi_n\}$ will consist of polynomials with respect to the Walsh system.

Divide the system $\{w_n\}$ into two systems $\{u_n\}$ and $\{v_n\}$, where $\{v_n\}$ is the Rademacher system with the natural ordering. Define

$$N_k = 2^{10^k} \qquad k = 1, 2, \dots \ .$$

Divide the system $\{v_n\}$ into blocks

$$B''_k = \{v_1^{(k)}, v_2^{(k)}, \dots, v_{N_k}^{(k)}\} \qquad k = 1, 2, \dots \ .$$

From each block B''_k take the function $v_1^{(k)}$ and combine them to form the system $\{y_n\}$, with the same order. In each block B''_k replace $v_1^{(k)}$ with u_k. So we get a new system, consisting of blocks

$$B'_k = \{u_k, v_2^{(k)}, \dots, v_{N_k}^{(k)}\}.$$

Using the matrices A_k from Lemma 1 we put:

$$g_i^{(k)} = a_{i1}^{(\nu_k)} \cdot u_k + \sum_{j=2}^{N_k} a_{ij}^{(\nu_k)} \cdot v_j^{(k)} = \frac{u_k}{\sqrt{N_k}} + \sum_{j=2}^{N_k} a_{ij}^{(\nu_k)} v_j^{(k)}$$

$$1 \le i \le N_k \ , \quad \nu_k = 10^k \ , \quad k = 1, 2, \dots \ .$$

In this way, we get a system $\{g_n\}$, consisting of blocks

$$B_k = \{g_1^{(k)}, g_2^{(k)}, \dots, g_{N_k}^{(k)}\}.$$

Now combine $\{g_n\}$ and $\{y_n\}$ in one system $\{\psi_n\}$ in the following way: Put

$$\psi_{2n} = g_n \ , \quad \psi_{2n-1} = y_n \quad n = 1, 2, \dots \ .$$

Thus we construct the system $\{\psi_n\}$. The following properties of $\{\psi_n\}$ are clear:

1) $\{\psi_n\}$ is an orthonormal basis in $L^2[0,1]$.
2) $\{\psi_n\}$ consists of uniformly bounded functions. (This follows from the uniform boundedness of Walsh system and property 3 of matrices A_k).

3. Now we will prove that a series (2) with respect to the system $\{\psi_n\}$, taken in decreasing order of magnitude of the coefficients, will converge a.e.

As a series from L^2 with respect to the Rademacher system converges a.e. after any rearrangement, we need only to prove that the series

$$\sum_n b_n g_n(x) \quad , \qquad \sum_n |b_n|^2 \le 1 \tag{3}$$

converges a.e. when taken in decreasing order of magnitude of the coefficients.

First of all, we prove that $\{g_n\}$ (and so $\{\psi_n\}$) is a system of convergence. (Remember that an ONS $\{g_n\}$ is called a system of convergence if each series (3) converges a.e.).

We have:

$$g_i^{(k)} = \frac{u_k}{\sqrt{N_k}} + \sum_{j=2}^{N_k} a_{ij}^{(\nu_k)} v_j^{(k)} = \frac{u_k - y_k}{\sqrt{N_k}} + \sum_{j=1}^{N_k} a_{ij}^{(\nu_k)} v_j^{(k)} =$$
$$= \frac{u_k - y_k}{\sqrt{N_k}} + p_i^{(k)} \qquad 1 \le i \le N_k \ ; \quad k = 1, 2, \dots$$

where $p_i^{(k)} = \sum_{j=1}^{N_k} a_{ij}^{(\nu_k)} v_j^{(k)}$, $1 \le i \le N_k; k = 1, 2, \dots$, forming orthonormal polynomials with respect to Rademacher system. So:

$$\sum_{n=1}^{\infty} b_n g_n = \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} b_i^{(k)} g_i^{(k)} =$$
$$\sum_k \sum_{i=1}^{N_k} b_i^{(k)} p_i^{(k)}(x) + \sum_k \sum_{i=1}^{N_k} \frac{b_i^{(k)}}{\sqrt{N_k}} u_k(x) - \sum_k \sum_{i=i}^{N_k} \frac{b_i^{(k)}}{\sqrt{N_k}} y_k(x).$$

We prove that all three series (denoted by S^1, S^2, S^3) converge a.e.

S^1 is a series from L^2 with respect to $\{p_i^{(k)}\}$ and its convergence after any rearrangement follows from Lemma 2 as shown in [3], §1.

Now for

$$c_k = \sum_{i=1}^{N_k} \frac{b_i^{(k)}}{\sqrt{N_k}} \qquad k = 1, 2, \dots$$

we have:

$$\sum_{k=1}^{\infty} |c_k|^2 = \sum_k \Big| \sum_{i=1}^{N_k} \frac{b_i^{(k)}}{\sqrt{N_k}} \Big|^2 \le \sum_k \sum_{i=1}^{N_k} |b_i^{(k)}|^2 = \sum_{n=1}^{\infty} |b_n|^2 < \infty \ .$$

Thus the series $\sum_{k=1}^{\infty} c_k u_k(x)$ belongs to $L^2[0,1]$ and, therefore, converges a.e. Each partial sum $S_n^{(2)}$ of S^2 consists of some number m of entire blocks and, possibly, a part of the block with number $m+1$, so it has the form:

$$S_n^{(2)} = \sum_{k=1}^{m} c_k u_k(x) + \sum_{i=1}^{M} \frac{b_i^{(m+1)}}{\sqrt{N_{m+1}}} u_{m+1}(x)$$

where m and M depend on n. However,

$$\Big|\sum_{i=1}^{M} \frac{b_i^{(m+1)}}{\sqrt{N_{m+1}}} u_{m+1}(x)\Big| \leq \sum_i |b_i^{(m+1)}|^2 = o(1)$$

and we get that S^2 converges a.e. The same arguments tell us that S^3 also converges a.e. So we get that a series (3) converges a.e.

We prove now that the series

$$S^2 = \sum_k \sum_{i=1}^{N_k} \frac{b_i^{(k)}}{\sqrt{N_k}} u_k(x)$$

taken in decreasing order of magnitude of the numbers $b_i^{(k)}$, converges a.e. For $k = 1, 2, \ldots$ put

$$\begin{aligned} \Lambda_k &= \Big\{ i : \frac{1}{N_k} < |b_i^{(k)}| < N_k^{-1/10} \Big\} \\ \Lambda'_k &= \Big\{ i : |b_i^{(k)}| \leq \frac{1}{N_k} \Big\} \\ \Lambda''_k &= \Big\{ i : |b_i^{k)}| \geq N_k^{-1/10} \Big\} . \end{aligned} \tag{4}$$

As $\sum_{i \in \Lambda'_k} |b_i^{(k)}|/\sqrt{N_k} \leq 1/\sqrt{N_k}$, we get that the series

$$\sum_k \sum_{i \in \Lambda'_k} \frac{b_i^{(k)}}{\sqrt{N_k}} u_k(x)$$

converges absolutely.

From the definition of the set Λ''_k and the inequality in (3) we get that the cardinality of Λ''_k is at most $N_k^{1/5}$, so $\sum_{i \in \Lambda''_k} |b_i^{(k)}|/\sqrt{N_k} \leq N_k^{1/5}/\sqrt{N_k} = N_k^{-3/10}$ and the series $\sum_k \sum_{i \in \Lambda''_k} (b_i^{(k)}/\sqrt{N_k}) u_k(x)$ converges absolutely.

As is clear from (4)

$$|b_i^{(k)}| > \frac{1}{N_k} \geq \frac{1}{N_{k+1}^{1/10}} \geq |b_j^{(k+1)}| \qquad i \in \Lambda_k \ , \quad j \in \Lambda_{k+1} \ ; \quad k = 1, 2, \ldots$$

so when we arrange the series $\sum_k \sum_{i \in \Lambda_k} (b_i^{(k)}/\sqrt{N_k}) u_k(x)$ by decreasing order, the rearrangement can only take place "inside" the blocks. However such a series converges a.e. — the proof is the same as for convergence a.e. of S^2, taken in the original order.

From here the convergence a.e. of S^2 follows after the decreasing order rearrangement.

The same statement (with the same argument) is true for S^3.

Combining this with the result on S^1 we finish the proof.

4. Remarks.

1. A system $\{\psi_n\}$ in Theorem 1 can be constructed as trigonometrical polynomials as well. For this we take the trigonometric ONB as $\{w_n\}$ and subtract a lacunary subsystem $u_k (= w_{2^k})$. Then we proceed with the same construction as before. The only changes needed in the proof would be to use the original Carleson theorem instead of its Walsh version (due to Billard) and to prove the corresponding variant of Lemma 2.
2. Actually one can get Theorem 1 without using deep convergence results. It is enough to start the construction with an ONB $\{w_n\}$ which is a system of convergence, consisting of bounded functions and containing a uniformly bounded infinite subsystem. A simple example of such a $\{w_n\}$ can be found in [3], Section 1.2.
3. We mention an interesting open question posed by T. Körner in [2], p. 18: if one takes the Haar system as $\{\psi_n\}$, does every series (2) in decreasing coefficients rearrangement converge a.e.?

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SCHOOL OF MATHEMATICAL SCIENCES
SACKLER FACULTY OF EXACT SCIENCES
TEL AVIV UNIVERSITY
RAMAT AVIV 69978
ISRAEL