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# ON THE CONTINUITY OF RANDOM OPERATORS

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*Abstract.* In this work we study when an arbitrary linear random operator between Banach spaces must behave as a continuous operator on some measurable set with positive measure. We deal with the continuity in probability as well as the continuity in $r$–mean.

**0. Introduction.** Linear random operators arise extensively in the theory of random equations which, at the present time, is a very active area of mathematical research [2], [5]. On the other hand, (non continuous) linear random operators behaving as continuous operators on some measurable set with positive measure were found recently [6], [9] in a natural way. These papers were devoted to show the continuity properties of some algebraically well behaved linear random operators defined on Banach algebras. In [8] we proved that the biggest measure of a measurable set on which an arbitrary linear random operator $T$ must behave as an operator which is continuous in probability, is given by

$$\lim_{\varepsilon\to 0}\varliminf_{x\to 0}\mathbb{P}[\|T(x)\|\le\varepsilon].$$

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The first section of the present paper is a brief survey about that fact, which became a very useful tool for the treatment of some continuity problems [6 – 9]. The aim of this paper is to obtain a similar approach for the continuity in $r$–mean of linear random operators with moment of order $r$.

The second section is devoted to measure the continuity in $r$–mean of a linear random operator with moment of order $r$.

Finally, in the third section, we get a characterization of operators behaving as continuous operators on some measurable set with positive measure. Here the continuity is considered as continuity in probability as well as continuity in $r$–mean.

**1. Continuity in probability.** Let $X$, $Y$ be Banach spaces and $(\Omega, \Sigma, \mathbb{P})$ a probability space. We denote by $\mathcal{L}_0(\mathbb{P}, Y)$ the space of all $Y$–valued Bochner random variables on $\Omega$. A mapping $T : X \to \mathcal{L}_0(\mathbb{P}, Y)$ is said to be a *random operator* from $X$ to $Y$. *Linear random operators* are those $T$ such that

$$\mathbb{P}[T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)] = 1,$$

for all $x, y$ in $X$ and $\alpha, \beta$ constants.

The usual convergence in probability provides a complete semimetrizable linear topology on $\mathcal{L}_0(\mathbb{P}, Y)$ (see [3] Section II.2.2) which can be derived from the paranorm

$$\|\mathbf{y}\|_0 = \int_\Omega \frac{\|\mathbf{y}\|}{1 + \|\mathbf{y}\|} d\mathbb{P}, \ \forall \mathbf{y} \in \mathcal{L}_0(\mathbb{P}, Y).$$

We obtain the complete metrizable topological linear space $L_0(\mathbb{P}, Y)$ from the *almost surely* identification on $\mathcal{L}_0(\mathbb{P}, Y)$ (abbr. *a.s.*). Note that every linear random operator from $X$ to $Y$ can be regarded as a linear $L_0(\mathbb{P}, Y)$–valued operator on $X$. The continuity of $T$ as a $\mathcal{L}_0(\mathbb{P}, Y)$–valued operator, that is the *continuity in probability*, is equivalent to the continuity of $T$ as a $L_0(\mathbb{P}, Y)$–valued operator. The continuity in probability also will be referred as the $\| \cdot \|_0$*–continuity*.

We inquire on the continuity of a linear random operator $T$ by considering, as it is usual, its so-called *separating subspace*,

$$\mathcal{S}_0(T) = \{\mathbf{y} \in \mathcal{L}_0(\mathbb{P}, Y) : \ \exists x_n \to 0 \text{ with } \|T(x_n) - \mathbf{y}\|_0 \to 0\},$$

which is $\| \cdot \|_0$–closed in $\mathcal{L}_0(\mathbb{P}, Y)$. The closed graph theorem [10], Section 5.3, can be established as follows.

**Theorem 1.** *Let $T$ be a linear random operator from a Banach space $X$ to a Banach space $Y$. Then $T$ is $\| \cdot \|_0$–continuous if, and only if,* $\mathbb{P}[\mathbf{y} = 0] = 1, \forall \mathbf{y} \in \mathcal{S}_0(T)$.

Let $\Omega_0$ be a measurable set in the probability space $(\Omega, \Sigma, \mathbb{P})$ such that $\mathbb{P}[\Omega_0] > 0$. In order to study the behaviour of a random operator $T$ on $\Omega_0$, we consider $\Omega_0$ as a new probability space with the inherited structure from $\Omega$ (the induced probability on $\Omega_0$ is the conditional probability $\mathbb{P}_0$). Therefore, we define the random operator

$$T_{\Omega_0} : X \to \mathcal{L}_0(\mathbb{P}_0, Y)$$

as the restriction

$$T_{\Omega_0}(x) = T(x)/\Omega_0, \ \forall x \in X.$$

Such an operator $T_{\Omega_0}$ is said to be a *conditional operator* of $T$.

In order to know "how continuous" a random operator $T$ is, we calculate the biggest measure of a measurable set on which $T$ must behave as a $\|\cdot\|_0$–continuous operator. That is the meaning of the number

$$\alpha_0(T) = \sup\{\mathbb{P}[\Omega_0] : \ T_{\Omega_0} \ \text{ is } \ \|\cdot\|_0 \text{ –continuous}\},$$

which is considered as *the probability with which $T$ is $\|\cdot\|_0$–continuous.* On the other hand, from the last theorem, whenever $T$ is linear can be also expect to determine "how continuous" $T$ is through its separating subspace. The "stochastic size" of the separating subspace, given by

$$\sup\{\delta : \mathbb{P}[\mathbf{y} = 0] \geq \delta, \ \forall \mathbf{y} \in \mathcal{S}_0(T)\},$$

is considered as *the probability with which the graph of $T$ is closed.*

The next result establishes that the probability with which a linear random operator is continuous coincides with the probability with which its graph is closed.

**Theorem 2.** [8] *For a linear random operator $T$ from $X$ to $Y$ the following assertions are equivalent:*

1. *$T$ has a $\|\cdot\|_0$–continuous conditional operator $T_{\Omega_0}$ with $\mathbb{P}[\Omega_0] \geq \delta$.*
2. *For every $0 < \delta' < \delta$ there exists a constant $M_{\delta'}$ such that*
$$\mathbb{P}[\|T(x)\| \leq M_{\delta'}\|x\|] \geq \delta', \forall x \in X.$$
3. $\underline{\lim}_{x \to 0} \mathbb{P}[\|T(x)\| \leq \varepsilon] \geq \delta, \ \forall \varepsilon > 0.$
4. $\mathbb{P}[\mathbf{y} = 0] \geq \delta, \ \forall \mathbf{y} \in \mathcal{S}_0(T).$

*In fact, the sets*

$$\{\delta \geq 0 : \exists M > 0 \text{ with } \mathbb{P}[\|T(x)\| \leq M\|x\|] \geq \delta \ \forall x \in X\}$$

*and*

$$\{\delta \geq 0 : \mathbb{P}[\mathbf{y} = 0] \geq \delta, \ \forall \mathbf{y} \in \mathcal{S}_0(T)\}$$

*attain a maximum respectively and*

$$\begin{aligned}\alpha_0(T) &= \max\{\delta \geq 0 : \exists M > 0 \textit{ with } \mathbb{P}[\|T(x)\| \leq M\|x\|] \geq \delta\ \forall x \in X\} \\ &= \{\delta \geq 0 : \mathbb{P}[\mathbf{y} = 0] \geq \delta,\ \forall \mathbf{y} \in \mathcal{S}_0(T)\} \\ &= \lim_{\varepsilon \to 0} \underline{\lim}_{x \to 0} \mathbb{P}[\|T(x)\| \leq \varepsilon].\end{aligned}$$

Classically, linear random operators are classified into continuous and non continuous operators. However we just got some additional information about non continuous linear random operators. Indeed some of them seem to be "more non continuous" than others, in a sense determined by mean a number. Now, random operators can be arranged according with the probability with which they are continuous, in a very reasonable way.

The above result was obtained from the following principle which will be applied again in the next section.

**Theorem 3.** [8] *Let $\mathcal{M}$ be an additive subgroup of random variables.*

1. *If there exists $\delta \in ]0,1[$ such that $\mathbb{P}[\mathbf{y} = 0] \geq \delta$ for all $\mathbf{y}$ in $\mathcal{M}$, then there exists a measurable subset $\Omega_0 \subset \Omega$ with $\mathbb{P}[\Omega_0] \geq \delta$ such that $\mathbf{y} = 0$ a. s. on $\Omega_0$, for all $\mathbf{y}$ in $\mathcal{M}$.*
2. *The sets $\{\mathbb{P}[\Omega_0] : \Omega_0$ is measurable and $\mathbf{y} = 0$ a. s. on $\Omega_0,\ \forall\ \mathbf{y} \in \mathcal{M}\}$ and $\{\delta \geq 0 : \mathbb{P}[\mathbf{y} = 0] \geq \delta,\ \forall \mathbf{y} \in \mathcal{M}\}$ attain a maximum respectively, and these values coincide.*

**2. Continuity in $r$–mean.** As it is well known, essential subspaces of $\mathcal{L}_0(\mathbb{P}, Y)$ are the spaces $\mathcal{L}_r(\mathbb{P}, Y)$ of all $Y$–valued Bochner random variables having finite moment of order $r$, *i.e.*

$$\mathcal{L}_r(\mathbb{P}, Y) := \{\mathbf{x} \in \mathcal{L}_0(\mathbb{P}, Y) : \int_\Omega \|\mathbf{x}\|^r d\mathbb{P} < \infty\}.$$

In addition to the inherited topology from $\mathcal{L}_0(\mathbb{P}, Y)$, every space $\mathcal{L}_r(\mathbb{P}, Y)$ also has its own topology which is given by the paranorm

$$\|\mathbf{x}\|_r = \int_\Omega \|\mathbf{x}\|^r d\mathbb{P},\ \mathbf{x} \in \mathcal{L}_r(\mathbb{P}, Y),$$

when $0 < r < 1$, while for $r \geq 1$ this topology is given by the seminorm

$$\|\mathbf{x}\|_r = \left(\int_\Omega \|\mathbf{x}\|^r d\mathbb{P}\right)^{\frac{1}{r}},\ \mathbf{x} \in \mathcal{L}_r(\mathbb{P}, Y).$$

The usual almost surely identification in $\mathcal{L}_r(\mathbb{P}, Y)$ leads to the complete metrizable topological linear space $L_r(\mathbb{P}, Y)$.

We say that a random operator $T : X \to \mathcal{L}_0(\mathbb{P}, Y)$ has *moment of order $r$* whenever $T$ is a $\mathcal{L}_r(\mathbb{P}, Y)$–valued mapping. Therefore (apart from the $\|\cdot\|_0$–continuity) random operators with $r$–moment have their specific notion of continuity, namely the $\|\cdot\|_r$–*continuity* also called *continuity in* $r$–*mean*. We note that a random operator $T$ with $r$–moment is continuous in $r$–mean precisely when $T$ is $\|\cdot\|_r$–continuous considered as a $L_r(\mathbb{P},Y)$–valued mapping.

To estimate the $r$–mean continuity of a linear random operator $T$ with $r$–moment we have the *separating subspace*

$$\mathcal{S}_r(T)=\{\mathbf{y}\in\mathcal{L}_r(\mathbb{P},Y):\ \exists x_n\to 0 \text{ with } \|T(x_n)-\mathbf{y}\|_r\to 0\},$$

which is $\|\cdot\|_r$–closed in $\mathcal{L}_r(\mathbb{P},Y)$. In this frame, the classical closed graph theorem [10], Section 5.3, leads to the following result.

**Theorem 4.** *Let $T$ be a linear random operator having moment of order $r$. Then $T$ is continuous in $r$–mean if, and only if,* $\mathbb{P}[\mathbf{y}=0]=1$, $\forall\mathbf{y}\in\mathcal{S}_r(T)$.

Let $T$ be a random operator with moment of order $r$. In order to know "how continuous in $r$–mean" $T$ is, we calculate the biggest measure of a measurable set on which $T$ must behave as a $\|\cdot\|_r$–continuous operator. That is the meaning of the number

$$\alpha_r(T)=\sup\{\mathbb{P}[\Omega_0]:\ T_{\Omega_0}\ \text{ is }\ \|\cdot\|_r\text{–continuous}\}.$$

Finally we observe that every linear random operator $T$ with moment of order $r$, also has moment of order $q$, for every $0\le q<r$. We recall that the $\|\cdot\|_r$–convergence is stronger than the $\|\cdot\|_q$–convergence. However, in spite of that, we proves that *$T$ is $\|\cdot\|_r$–continuous if, and only if, $T$ is $\|\cdot\|_q$–continuous for some $0\le q\le r$.* This will be a straightforward consequence of the next result.

**Proposition 5.** *Let $T$ be a linear random operator with moment of order $r$. Then the following conditions are equivalent:*

1. *$T$ is $\|\cdot\|_0$–continuous.*
2. *$T$ is $\|\cdot\|_r$–continuous.*

*Proof.* $(ii)\Rightarrow(i)$ is obvious because the $\|\cdot\|_r$–convergence is stronger than the $\|\cdot\|_0$–convergence [1], Theorem 7.1.5.

$(i)\Rightarrow(ii)$. Let $x_n$ be a sequence in $X$ converging to zero and such that $T(x_n)$ is $\|\cdot\|_r$–convergent to a random variable $\mathbf{y}$. Then $T(x_n)$ is also $\|\cdot\|_0$–convergent to $\mathbf{y}$. Since $T$ is $\|\cdot\|_0$–continuous, we have that $\mathbb{P}[\mathbf{y}=0]=1$, from Theorem 1. Now we apply Theorem 4 to deduce the $\|\cdot\|_r$–continuity of $T$. □

Let $T$ be a linear random operator with moment of order $r$. In the next result we obtain a precise estimation of the probability with which $T$ is continuous in $r$–mean. We get it through the separating subspace $\mathcal{S}_0(T)$ as well as through the most natural subspace $\mathcal{S}_r(T)$, respectively.

**Theorem 6.** *Let $T$ be a linear random operator from $X$ to $Y$ having moment of order $r$. Then the following assertions are equivalent:*

1. *$T$ has a $\|\cdot\|_r$–continuous conditional operator $T_{\Omega_0}$ with $\mathbb{P}[\Omega_0] \geq \delta$.*
2. *$T$ has a $\|\cdot\|_0$–continuous conditional operator $T_{\Omega_0}$ with $\mathbb{P}[\Omega_0] \geq \delta$.*
3. *For every $0 < \delta' < \delta$ there exists $M_{\delta'} > 0$ such that*
$$\mathbb{P}[\|T(x)\| \leq M_{\delta'}\|x\|] \geq \delta', \ \forall x \in X.$$
4. $\underline{\lim}_{x\to 0} \mathbb{P}[\|T(x)\| \leq \varepsilon] \geq \delta, \ \forall \varepsilon > 0.$
5. $\mathbb{P}[\mathbf{y} = 0] \geq \delta, \ \forall \mathbf{y} \in \mathcal{S}_0(T).$
6. $\mathbb{P}[\mathbf{y} = 0] \geq \delta, \ \forall \mathbf{y} \in \mathcal{S}_r(T).$

*In fact*
$$\begin{aligned}\alpha_r(T) &= \alpha_0(T)\\ &= \max\{\delta \geq 0 : \exists M > 0 \textit{ with } \mathbb{P}[\|T(x)\| \leq M\|x\|] \geq \delta \ \forall x \in X\}\\ &= \max\{\delta \geq 0 : \mathbb{P}[\mathbf{y} = 0] \geq \delta, \ \forall \mathbf{y} \in \mathcal{S}_0(T)\}\\ &= \max\{\delta \geq 0 : \mathbb{P}[\mathbf{y} = 0] \geq \delta, \ \forall \mathbf{y} \in \mathcal{S}_r(T)\}\\ &= \lim_{\varepsilon\to 0} \underline{\lim}_{x\to 0} \mathbb{P}[\|T(x)\| \leq \varepsilon].\end{aligned}$$

*Proof.* The equivalence $1 \Leftrightarrow 2$ and the equality $\alpha_r(T) = \alpha_0(T)$ follow from Proposition 5. The assertions $2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$ and the equalities
$$\begin{aligned}\alpha_0(T) &= \max\{\delta > 0 : \exists M > 0 \text{ with } \mathbb{P}[\|T(x)\| \leq M\|x\|] \geq \delta \ \forall x \in X\}\\ &= \max\{\delta \geq 0 : \mathbb{P}[\mathbf{y} = 0] \geq \delta, \ \forall \mathbf{y} \in \mathcal{S}_0(T)\}\\ &= \lim_{\varepsilon\to 0} \underline{\lim}_{x\to 0} \mathbb{P}[\|T(x)\| \leq \varepsilon]\end{aligned}$$
were obtained in Theorem 2.

The inclusion $\mathcal{S}_r(T) \subset \mathcal{S}_0(T)$ shows $5 \Rightarrow 6$ and also that
$$\max\{\delta \geq 0 : \mathbb{P}[\mathbf{y} = 0] \geq \delta, \ \forall \mathbf{y} \in \mathcal{S}_0(T)\} \geq \sup\{\delta \geq 0 : \mathbb{P}[\mathbf{y} = 0] \geq \delta, \ \forall \mathbf{y} \in \mathcal{S}_r(T)\}.$$

To prove $6 \Rightarrow 1$ we apply Theorem 3, by considering $\mathcal{M} = \mathcal{S}_r(T)$, to get a measurable set $\Omega_0$ with $\mathbb{P}[\Omega_0] \geq \delta$ such that
$$\mathbf{y} = 0 \ \textit{a.s.} \text{ on } \Omega_0, \ \forall \mathbf{y} \in \mathcal{S}_r(T).$$

We observe that $T_{\Omega_0} = RT$ where $T$ is considered as a $L_r(\mathbb{P}, Y)$–valued operator and $R : L_0(\mathbb{P}, Y) \to L_0(\mathbb{P}_{\Omega_0}, Y)$ given by

$$R(\mathbf{y}) = \mathbf{y}/\Omega_0.$$

From the closed graph theorem (see [4], Lemma 1.3, which also holds for operators defined between metrizable complete linear spaces) it follows that $RT$ is continuous if and only if $\mathcal{S}_r(T) \subset \ker R$. Hence $T_{\Omega_0}$ is $\|\cdot\|_r$–continuous. On the other hand, from Theorem 3, the set

$$\{\delta \geq 0 : \mathbb{P}[\mathbf{y} = 0] \geq \delta, \ \forall \mathbf{y} \in \mathcal{S}_r(T)\}$$

attains a maximum which coincides with $\alpha_r(T)$. That concludes the proof. □

Consequently, in order to prove the existence of a measurable set $\Omega_0$ on which $T$ behaves as a continuous operator, we do not need to check the behaviour of $T$ on each measurable set $\Omega_0$. It is suffices to compute

$$\lim_{\varepsilon \to 0} \lim_{x \to 0} \mathbb{P}[\|T(x)\| \leq \varepsilon]$$

to know if such a measurable set $\Omega_0$ exists having positive measure, and also to determine the "biggest measure" that can be expected for it.

**3. Probably continuous linear random operators.** The aim of this section is to study when $\alpha_r(T) > 0$. Linear random operators with this property will be called *probably continuous operators.*

Previously we establish the following technical result which will be essential to our purpose.

**Lemma 7.** *Given $r \geq 0$, let $\mathcal{M}$ be a $\|\cdot\|_r$–closed linear subspace of random variables with moment of order $r$. Then, for every $0 < \delta < 1$, the set*

$$C_\delta := \{\mathbf{y} \in \mathcal{M} : \ \mathbb{P}[\mathbf{y} = 0] \geq \delta\}$$

*is $\|\cdot\|_r$–closed.*

*Proof.* Let $\mathbf{y}_n$ be a sequence in $C_\delta$ such that $\|\mathbf{y}_n - \mathbf{y}\|_r \to 0$, for a random variable $\mathbf{y}$. Then $\mathbf{y}_n$ converges to $\mathbf{y}$ in probability, that is

$$\int_\Omega \frac{\|\mathbf{y}_n\|}{1 + \|\mathbf{y}_n\|} d\mathbb{P} \to \int_\Omega \frac{\|\mathbf{y}\|}{1 + \|\mathbf{y}\|} d\mathbb{P},$$

from [1], Theorem 7.1.5. By defining $\Omega_n := \{\omega \in \Omega : \ \mathbf{y}_n(\omega) \neq 0\}$, we observe that

$$\int_\Omega \frac{\|\mathbf{y}_n\|}{1 + \|\mathbf{y}_n\|} d\mathbb{P} = \int_{\Omega_n} \frac{\|\mathbf{y}_n\|}{1 + \|\mathbf{y}_n\|} d\mathbb{P} \leq \mathbb{P}[\Omega_n].$$

Since $\mathbf{y}_n \in C_\delta$, for every $n \in \mathbb{N}$, it follows that $\mathbb{P}[\Omega_n] \leq 1-\delta$, so that

$$\int_\Omega \frac{\|\mathbf{y}\|}{1+\|\mathbf{y}\|} d\mathbb{P} \leq 1-\delta.$$

But, given $k \in \mathbb{N}$, it is obvious that $k\mathbf{y}_n \in C_\delta$, for every $n \in \mathbb{N}$. Moreover $k\mathbf{y}_n$ converges in probability to $k\mathbf{y}$. Therefore, from above,

$$\int_\Omega \frac{\|k\mathbf{y}\|}{1+\|k\mathbf{y}\|} d\mathbb{P} \leq 1-\delta.$$

Letting $k \to \infty$, we deduce that $\mathbb{P}[\Omega_\mathbf{y}] \leq 1-\delta$, where $\Omega_\mathbf{y} := \{\omega \in \Omega : \mathbf{y}(\omega) \neq 0\}$. Hence $\mathbf{y} \in C_\delta$, so that $C_\delta$ is closed. □

**Theorem 8.** *For $r \geq 0$, let $\mathcal{M}$ be a $\|\cdot\|_r$–closed linear subspace of $Y$–valued random variables with moment of order $r$. If $\mathbb{P}[\mathbf{y}=0] > 0$, $\forall \mathbf{y} \in \mathcal{M}$, then*

$$\inf\{\mathbb{P}[\mathbf{y}=0] : \ \mathbf{y} \in \mathcal{M}\} > 0.$$

*Proof.* Because $\mathcal{M} = \bigcup_{n=1}^{\infty} \{\mathbf{y} \in \mathcal{M} : \ \mathbb{P}[\mathbf{y}=0] \geq \frac{1}{n}\}$, we have that $\mathcal{M}$ is a numerable union of linear subspaces, which are $\|\cdot\|_r$–closed from the last lemma. We apply Baire's theorem to the canonical projection of $\mathcal{M}$ on $L_0(\mathbb{P}, X)$ in order to obtain $n_0 \in \mathbb{N}$ such that the set $\{\mathbf{y} \in \mathcal{M} : \mathbb{P}[\mathbf{y}=0] \geq \frac{1}{n_0}\}$ has an interior point, $\mathbf{y}_0$. Since for every $\mathbf{y} \in \mathcal{M}$ and every scalar $\lambda$, with $\mid \lambda \mid$ enough small,

$$\mathbb{P}[\mathbf{y}_0 + \lambda \mathbf{y} = 0] \geq \frac{1}{n_0},$$

it follows that

$$\begin{aligned} \frac{1}{n_0} &\leq \mathbb{P}[\mathbf{y}_0 = -\lambda \mathbf{y}] \\ &= \mathbb{P}[\mathbf{y}_0 = -\lambda\mathbf{y}, \mathbf{y}_0 = 0] + \mathbb{P}[\mathbf{y}_0 = -\lambda\mathbf{y}, \mathbf{y}_0 \neq 0] \\ &\leq \mathbb{P}[\mathbf{y}_0 = \mathbf{y} = 0] + \mathbb{P}[\|\mathbf{y}_0\| = \mid \lambda \mid \|\mathbf{y}\|, \|\mathbf{y}_0\| \neq 0] \\ &= \mathbb{P}[\mathbf{y}_0 = \mathbf{y} = 0] + \mathbb{P}\left[\frac{1}{\mid \lambda \mid} = \frac{\|\mathbf{y}\|}{\|\mathbf{y}_0\|}, \|\mathbf{y}_0\| \neq 0\right]. \end{aligned}$$

Let $\lambda \to 0$ to obtain

$$\frac{1}{n_0} \leq \mathbb{P}[\mathbf{y}_0 = \mathbf{y} = 0].$$

Thus,

$$\mathbb{P}[\mathbf{y}=0] \geq \frac{1}{n_0}, \ \forall \mathbf{y} \in \mathcal{M}$$

which proves the result. □

From the above theorem, probably $\|\cdot\|_r$–continuous operators (*i.e.* those whose probability of being continuous in $r$–mean is not zero) are characterized as follows.

**Theorem 9.** *Let $r \geq 0$ and let $T$ be a linear random operator $T$ from $X$ to $Y$ having moment of order $r$ (if it is $r > 0$). Then the next assertions are equivalent:*

1. *$T$ is probably continuous in $r$–mean.*
2. *There exists a constant $M$ such that*
$$\mathbb{P}[\|T(x)\| \leq M\|x\|] > 0, \ \forall x \in X.$$
3. $\underline{\lim}_{x \to 0} \mathbb{P}[\|T(x)\| \leq \varepsilon] > 0, \ \forall \varepsilon > 0.$
4. $\mathbb{P}[\mathbf{y} = 0] > 0, \ \forall \mathbf{y} \in \mathcal{S}_r(T).$
5. $\mathbb{P}[\mathbf{y} = 0] > 0, \ \forall \mathbf{y} \in \mathcal{S}_0(T).$

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