Journal of Algebraic Combinatorics **10** (1999), 29–45

On a New High Dimensional Weisfeiler-Lehman Algorithm

SERGEI EVDOKIMOV* evdokim@iias.spb.ru
St. Petersburg Institute for Informatics and Automation, Academy of Sciences, Russia

MAREK KARPINSKI† marek@cs.uni-bonn.de
Department of Computer Science, University of Bonn, Germany

ILIA PONOMARENKO* inp@pdmi.ras.ru
St. Petersburg Department of Mathematical Institute, Academy of Sciences, Russia

Received June 7, 1995; Revised January 6, 1998

Abstract. We investigate the following problem: how different can a cellular algebra be from its Schurian closure, i.e., the centralizer algebra of its automorphism group? For this purpose we introduce the notion of a Schurian polynomial approximation scheme measuring this difference. Some natural examples of such schemes arise from high dimensional generalizations of the Weisfeiler-Lehman algorithm which constructs the cellular closure of a set of matrices. We prove that all of these schemes are dominated by a new Schurian polynomial approximation scheme defined by the m-closure operators. A sufficient condition for the m-closure of a cellular algebra to coincide with its Schurian closure is given.

Keywords: graph isomorphism problem, cellular algebra, permutation group

1. Introduction

The starting point of the paper is the Graph Isomorphism Problem (ISO), a famous unsolved problem in computational complexity theory (see [8]). The problem is to test whether two finite graphs are isomorphic by means of an efficient algorithm. Despite many efforts, at present the best isomorphism test for graphs with n vertices makes at least $\exp(O(\sqrt{n\log n}))$ steps in the worst case (see [3] for the discussion of this and related topics).

In [14] an approach to the ISO based on the notion of a *cellular algebra* was developed. Let Mat_V be the full matrix algebra over $\mathbf{C}$ on a finite set V. A subalgebra of Mat_V is called cellular if it is closed under the Hadamard (componentwise) multiplication $\circ$, the Hermitian conjugation * and contains the matrix all of whose entries are equal to 1.[1] One of the most important examples of cellular algebras is the centralizer algebra $\mathcal{Z}(G, V)$ of a permutation group G on V, i.e., the set of all matrices of Mat_V stable with respect to the induced action

*Research supported by the Volkswagen-Stiftung Program on Computational Complexity.
†Research partially supported by the DFG Grant KA 67314-1, by ESPRITS Grants 7097 and ECUS 030, and by the Volkswagen-Stiftung.

of G on Mat_V. Conversely, we associate to each cellular algebra W its automorphism group $\mathrm{Aut}(W)$ which is by definition the group of all permutations of V preserving any matrix of W. This defines a cellular superalgebra $\mathrm{Sch}(W) = \mathcal{Z}(\mathrm{Aut}(W), V)$ of the algebra W with the same automorphism group, called the *Schurian closure* of W. We note that W does not necessarily coincide with $\mathrm{Sch}(W)$ (see [15]). If it does, then the algebra is called *Schurian*.

The idea of the cellular algebra approach to the ISO is the following. It is well-known that this problem is polynomial-time equivalent to the problem of finding the orbits of the automorphism group $\mathrm{Aut}(\Gamma)$ of a graph Γ (see [12]). However, these orbits can easily be derived from the algebra $\mathcal{Z}(\mathrm{Aut}(\Gamma), V)$ where V is the vertex set of Γ. The last algebra is in fact the Schurian closure of the smallest cellular algebra $W(\Gamma)$ containing the adjacency matrix of Γ. Hoping that the algebra $W(\Gamma)$ is always Schurian[2] in 1968 B. Weisfeiler and A. Lehman proposed some way to compute it. Their procedure is a special case of a more general algorithm called now the *Weisfeiler-Lehman algorithm* (see [14], Section C8), which given matrices $A_1, \ldots, A_s \in \mathrm{Mat}_V$ efficiently (in polynomial time) constructs their cellular closure $[A_1, \ldots, A_s]$, i.e., the smallest cellular algebra containing them. From the algebraic point of view, this algorithm constructs a sequence $L_0 \subset L_1 \subset \cdots \subset L_t = L_{t+1}$ of linear subspaces of Mat_V where $L_t = [A_1, \ldots, A_s]$, so that L_0 is linearly spanned by $A_1, \ldots, A_s$ and L_{i+1} is the smallest subspace of Mat_V containing $(L_i \cdot L_i) \circ (L_i \cdot L_i)^*$.

The Weisfeiler-Lehman algorithm gives a polynomial-time reduction of the ISO to the problem of constructing the Schurian closure of a cellular algebra. Here we face a common situation in mathematics: we want to construct some object but have in hand only an approximation to it. Certainly, it would be more convenient to deal with a sequence of some natural approximations giving eventually the object we are interested in. For this reason we introduce in this paper the notion of a *Schurian polynomial approximation scheme*.

Let us have a rule according to which given a cellular algebra $W \leq \mathrm{Mat}_V$ and a positive integer m a cellular algebra $W^{(m)} \leq \mathrm{Mat}_V$ can be constructed. We say that the operators $W \mapsto W^{(m)}$ ($m = 1, 2, \ldots$) define a Schurian polynomial approximation scheme if the following conditions are satisfied:

1. $W = W^{(1)} \leq \cdots \leq W^{(n)} = \cdots = \mathrm{Sch}(W)$;
2. $(W^{(m)})^{(l)} = W^{(m)}$ for all $l \in [m]$;
3. $W^{(m)}$ can be constructed in time $n^{O(m)}$

where n is the cardinality of V. Condition (1) obviously implies $\mathrm{Aut}(W^{(m)}) = \mathrm{Aut}(W)$ for all $m \geq 1$. Further, condition (3) means that the algebra $W^{(m)}$ is in a sense the trace of some m-dimensional object. All known to us Schurian polynomial approximation schemes are defined just in this way. On the other hand the last condition prevents a scheme from being degenerate.

In this paper we describe a special Schurian polynomial approximation scheme and study its main properties. The key notion of our approach is that of m-closure. To define it, given W and m denote by $\hat{W}^{(m)}$ the smallest cellular subalgebra of Mat_{V^m} containing the algebras $\mathcal{Z}(\mathrm{Sym}(V), V^m)$ and $W \otimes \cdots \otimes W$ (m times). Then $\bar{W}^{(m)}$ is by definition the cellular algebra being the restriction of $\hat{W}^{(m)}$ to V (included in V^m diagonalwise). We call $\bar{W}^{(m)}$ the *m-closure* of W.

Theorem 1.1 *The m-closure operators $W \mapsto \bar{W}^{(m)}$ $(m = 1, 2, \ldots)$ constructed above define a Schurian polynomial approximation scheme.*

Probably the book [14] was the first source where a series of constructions and procedures (deep stabilization) carrying into Schurian polynomial approximation schemes was considered. For example, a weaker analog of the algebra $\bar{W}^{(2)}$ is obtained from the dual graph described in Section O6.4 of this book. One more construction just outlined in Section O6.2 underlies the algorithm of [7] which given a cellular algebra $W \leq \mathrm{Mat}_V$ and a positive integer m produces a cellular algebra $B_m(W) \geq W$ with the same automorphism group by using a subalgebra of Mat_{V^m} canonically defined by W (for the exact definitions see Section 4 of this paper). This defines the Schurian polynomial approximation scheme $B_m : W \mapsto B_m(W)$ $(m = 1, 2, \ldots)$. The ideas of [14] also gave rise to the *m-dim W-L method* based on the m-dimensional stabilization procedure which refines a given initial coloring of V^m (see [5] and Section 4). It is worth noting that for $m = 2$ this procedure is the combinatorial analog of the Weisfeiler-Lehman algorithm for finding the cellular closure of a set of matrices (see above). The m-dim W-L method is generally used in isomorphism-like problems for graphs, the initial coloring of V^m chosen according to the isomorphism type of m-vertex labeled subgraphs (cf. [14], Section O6.3). Similarly, replacing a graph by a cellular algebra we come to the algorithms $A_m : W \mapsto A_m(W)$ $(m = 1, 2, \ldots)$ defining a Schurian polynomial approximation scheme (for the exact definition see Section 4).

In connection with the above discussion the natural question arises: what are the relations between the cellular algebras $A_m(W)$, $B_m(W)$ and $\bar{W}^{(m)}$? The next proposition gives a partial answer to the question.

Theorem 1.2 *The Schurian polynomial approximation scheme defined by the m-closure operators dominates ones defined by A_m and B_m. More exactly,*

$$\bar{W}^{(m)} \geq A_m(W), \quad \bar{W}^{(m)} \geq B_m(W)$$

for all cellular algebras W and $m \geq 1$.

One of the most important problems concerning Schurian polynomial approximation schemes is a good estimation of the smallest $m \geq 1$ for which $W^{(m)} = \mathrm{Sch}(W)$. Note, that if such m was bounded by a constant then by condition (4) the ISO could be solved in polynomial time. We do not know whether this is true for ours or someone else's Schurian polynomial approximation scheme. However, in the first case we can give an upper bound for m in terms of the split number of a cellular algebra defined below.

The split number $s(W)$ of a cellular algebra W is by definition the smallest s for which there exist $v_1, \ldots, v_s \in V$ such that $W_{v_1,\ldots,v_s} = \mathrm{Mat}_V$ where $W_{v_1,\ldots,v_s}$ is the smallest cellular subalgebra of Mat_V containing W and all diagonal $\{0, 1\}$-matrix with exactly one nonzero element standing in row and column v_i. Clearly, $s(W) \leq n - 1$ for all W. Some non-trivial upper bounds for this number can be found in [2] and [6]. We also mention paper [10] where a similar invariant (called freedom degree) was defined for a permutation group.

Theorem 1.3 *Let W be a cellular algebra with $s(W) \leq m - 1$. Then* $\mathrm{Sch}(W) = \bar{W}^{(m)}$.

The idea of the proof is to study the cellular algebra $\hat{W}^{(m)} \le \mathrm{Mat}_{V^m}$ defined above. We show that this algebra contains (in a sense) all cellular algebras of the form $W_{v_1,\dots,v_{m-1}}$ with $v_i \in V$. By using this fact we find a faithful regular orbit of the componentwise action of the group $\mathrm{Aut}(W)$ on V^m (coinciding with an orbit of $\mathrm{Aut}(\hat{W}^{(m)})$). Comparing the cellular algebras being the restrictions of $\hat{W}^{(m)}$ to this orbit and the diagonal of V^m we conclude that the last algebra is Schurian. This means that so is $\bar{W}^{(m)}$.

As an easy corollary ($s(W) \le 1$) we get the following statement.

Corollary 1.4 *If a cellular algebra $W \le \mathrm{Mat}_V$ has no proper cellular superalgebras, then* $\bar{W}^{(2)} = \mathrm{Sch}(W)$.

The paper consists of six sections. The second one contains main definitions and some preliminary results concerning cellular algebras. In Section 3 we define the notion of m-closure and in detail study the properties of m-closed cellular algebras. As a result we get the proof of Theorem 1.1. Sections 4 and 5 are devoted to Theorems 1.2 and 1.3 respectively. In Section 6 we discuss some open problems.

Notation. As usual by $\mathbf{C}$ we denote the complex field.

Throughout the paper V denotes a finite set with $n = |V|$ elements. A subset of $V \times V$ is called a relation on V. For a relation R on V we define its support V_R to be the smallest set $U \subset V$ such that $R \subset U \times U$.

Under an equivalence E on V we always mean an ordinary equivalence on a subset of V (coinciding with V_E), the set of its equivalence classes is denoted by V/E.

The algebra of all complex matrices whose rows and columns are indexed by the elements of V is denoted by Mat_V, its unit element (the identity matrix) by I_V and the all one matrix by J_V. For $U \subset V$ the algebra Mat_U can be viewed in a natural way as a subalgebra of Mat_V.

The transpose of a matrix A is denoted by A^T, its Hermitian conjugate by A^*.

Each bijection $g : V \to V'$ ($v \mapsto v^g$) defines a natural algebra isomorphism from Mat_V onto $\mathrm{Mat}_{V'}$. The image of a matrix A under g is denoted by A^g.

The group of all permutations of V is denoted by $\mathrm{Sym}(V)$.

For integers l, m the set $\{l, l+1, \dots, m\}$ is denoted by $[l, m]$. If $l = 1$, we write $[m]$ instead of $[1, m]$.

2. Cellular algebras

By a *cellular algebra W* on V we mean a subalgebra of Mat_V containing the identity matrix I_V, the matrix J_V all of whose entries are equal to 1, and closed under the Hermitian conjugation and the Hadamard (componentwise) multiplication $\circ$. Below we give a combinatorial characterization of cellular algebras. It is convenient to view $\{0, 1\}$-matrices belonging to Mat_V as the adjacency matrices of relations on V. Throughout the paper we identify these matrices with the corresponding relations.

The next statement follows from Proposition E1 and Section C11 of [14].

Proposition 2.1 *A linear subspace $W \subset \mathrm{Mat}_V$ is a cellular algebra if and only if there exists a linear base $\mathcal{R} = \mathcal{R}(W)$ of W consisting of* $\{0, 1\}$*-matrices such that*

1. $\sum_{R\in\mathcal{R}} R = J_V$;
2. $R \in \mathcal{R} \Leftrightarrow R^T \in \mathcal{R}$;
3. *there exists a disjoint partition* $V = \bigcup_{i=1}^{s} V_i$ *of* V *into nonempty sets* V_i *such that*
   (a) $I_{V_i} \in \mathcal{R}$ *for all* i;
   (b) *for all* $R \in \mathcal{R}$ *there exist* $i, j \in [s]$ *such that* $R \subset V_i \times V_j$;
   (c) *the number of* 1*'s in the* u*th row* (*resp.* v*th column*) *of the matrix* $R \in \mathcal{R}$, $R \subset V_i \times V_j$, *does not depend on the choice of* $u \in V_i$ (*resp.* $v \in V_j$), *this number is denoted by* $d_{\mathrm{out}}(R)$ (*resp.* $d_{\mathrm{in}}(R)$);
4. *given* $R, S, T \in \mathcal{R}$ *the number*

$$p(u, v; S, T) = |\{w \in V : (u, w) \in S, (w, v) \in T\}|, \quad u, v \in V$$

does not depend on the choice of $(u, v) \in R$.

Remark 2.2 It is easily seen that the base $\mathcal{R}$ and the partition $V = \bigcup_{i=1}^{s} V_i$ are uniquely determined by W.

The linear base $\mathcal{R}$ of a cellular algebra W defined in Proposition 2.1 is called the *standard basis* of W and its elements the *basis matrices* or *basis relations*. Any subset $V_i \subset V$ (resp. a possibly empty union of V_i's) is called a *cell* (resp. a *cellular set*) of W. The set of all of them is denoted by $\mathrm{Cel}(W)$ (resp. $\mathrm{Cel}^*(W)$). Given a relation $R \in W$ its support V_R is, obviously, a cellular set of W.

Below we will use the following generalization of statement (4) of Proposition 2.1. Let $u, v \in V$ and $\tau = (R_1, \ldots, R_l) \in \mathcal{R}^l$. We say that $(v_0, \ldots, v_l) \in V^{l+1}$ is a (u, v)*-path of the type* τ if $v_0 = u$, $v_l = v$ and $(v_{i-1}, v_i) \in R_i$ for all $i \in [l]$. The number of all such paths will be denoted by $p(u, v; \tau)$.

Lemma 2.3 (Path Proposition [14], Theorem C10) *Let* W *be a cellular algebra. Then given* $R \in \mathcal{R}(W)$ *the integer* $p(u, v; \tau)$ *does not depend on the choice of* $(u, v) \in R$.

The set of all cellular algebras on V is ordered by inclusion. The largest and the smallest elements of this set are respectively the full matrix algebra Mat_V and the simplex $\mathcal{Z}(\mathrm{Sym}(V), V)$, i.e., the algebra with the linear base $\{I_V, J_V\}$. For cellular algebras W and W' we write $W \le W'$ if W is a subalgebra of W'. If $A_1, \ldots, A_s \in \mathrm{Mat}_V$, then the intersection of all cellular algebras on V containing W and all the matrices A_i is also a cellular algebra on V. It is denoted by $W[A_1, \ldots, A_s]$. We use notation $[A_1, \ldots, A_s]$ if W is a simplex and $W_{v_1,\ldots,v_s}$ if $A_i = I_{v_i} = I_{\{v_i\}}$ with $v_i \in V$ for all i.

Two cellular algebras W on V and W' on V' are called *isomorphic* if $W^g = W'$ for some bijection $g : V \to V'$ called an *isomorphism* from W to W'. The group of all isomorphisms from W to itself contains a normal subgroup

$$\mathrm{Aut}(W) = \{g \in \mathrm{Sym}(V) : A^g = A \text{ for all } A \in W\}$$

called the *automorphism group* of W.

Following Section G3.1 of [14] let us define for cellular algebras the notion of tensor product. Let $W_1 \le \mathrm{Mat}_{V_1}$ and $W_2 \le \mathrm{Mat}_{V_2}$ be cellular algebras on V_1 and V_2. Obviously, the subalgebra $W_1 \otimes W_2$ of $\mathrm{Mat}_{V_1} \otimes \mathrm{Mat}_{V_2} = \mathrm{Mat}_{V_1 \times V_2}$ is closed under the Hadamard multiplication in $\mathrm{Mat}_{V_1\times V_2}$. It also contains the matrices $I_{V_1\times V_2} = I_{V_1} \otimes I_{V_2}$ and $J_{V_1\times V_2} = J_{V_1} \otimes J_{V_2}$. So $W_1 \otimes W_2$ is a cellular algebra on $V_1 \times V_2$ called the tensor product of W_1 and W_2. Clearly, $\mathcal{R}(W_1 \otimes W_2) = \mathcal{R}(W_1) \otimes \mathcal{R}(W_2)$ and $\mathrm{Aut}(W_1 \otimes W_2) = \mathrm{Aut}(W_1) \times \mathrm{Aut}(W_2)$.

A large class of cellular algebras comes from permutation groups as follows (see [14], Section F). Let G be a permutation group on V. Then its centralizer algebra

$$\mathcal{Z}(G, V) = \{A \in \mathrm{Mat}_V : A^g = A \text{ for all } g \in G\}$$

is a cellular algebra on V the standard basis of which consists of all orbits of the natural action of G on $V \times V$. For a cellular algebra W on V we set

$$\mathrm{Sch}(W) = \mathcal{Z}(\mathrm{Aut}(W), V).$$

Clearly, $W \leq \mathrm{Sch}(W)$ and $\mathrm{Aut}(W) = \mathrm{Aut}(\mathrm{Sch}(W))$. The algebra W is called *Schurian* if $W = \mathrm{Sch}(W)$. Certainly, $\mathrm{Sch}(W)$ is a Schurian algebra for all W. It follows from [1, 15] that there exist cellular algebras which are not Schurian.

Any isomorphism of cellular algebras obviously induces a bijection between the standard bases of them. The converse statement is not true. This motivates the following definition (cf. [14], Section E5). Cellular algebras W on V and W' on V' are called *weakly isomorphic* if there exists an algebra isomorphism $\varphi : W \to W'$ such that $\varphi(\mathcal{R}(W)) = \mathcal{R}(W')$. Any such φ is called a *weak isomorphism* from W to W'. The following statement describes the basic properties of weak isomorphisms.

Proposition 2.4 *Let $\varphi : W \to W'$ be a weak isomorphism. Then*
1. *$\varphi(A \circ B) = \varphi(A) \circ \varphi(B)$ and $\varphi(A^*) = \varphi(A)^*$ for all $A, B \in W$.*
2. *φ induces a natural bijection $X \mapsto X^\varphi$ from $\mathrm{Cel}^*(W)$ onto $\mathrm{Cel}^*(W')$ preserving cells such that $\varphi(I_X) = I_{X^\varphi}$. Moreover, $|X| = |X^\varphi|$ and, in particular, $|V| = |V'|$.*

Proof: The first part of statement (1) is trivial. The second follows from the observation that given $R \in \mathcal{R}(W)$, the matrix R^T is the only matrix of $\mathcal{R}(W)$ whose product by R is not orthogonal to I_V with respect to the Hadamard multiplication. Let $X \in \mathrm{Cel}^*(W)$. Then the equalities $I_X I_X = I_X \circ I_X = I_X$ imply that $\varphi(I_X)\varphi(I_X) = \varphi(I_X) \circ \varphi(I_X) = \varphi(I_X)$. So there exists $X' \subset V'$ such that $\varphi(I_X) = I_{X'}$. Since $I_{X'} \in W'$, we have $X' \in \mathrm{Cel}^*(W')$. Set $X^\varphi = X'$. Since $I_V = \sum_{X\in \mathrm{Cel}(W)} I_X$ and $\varphi(I_V) = I_{V'}$, the mapping $X \mapsto X^\varphi$ gives a bijection from $\mathrm{Cel}(W)$ to $\mathrm{Cel}(W')$, which proves the first part of statement (2). Note that $\varphi(J_V) = J_{V'}$. So $\varphi(J_X) = \varphi(I_X J_V I_X) = I_{X^\varphi} J_{V'} I_{X^\varphi} = J_{X^\varphi}$ for all $X \in \mathrm{Cel}^*(W)$ and the rest of statement (2) follows from the equality $J_X^2 = |X| J_X$. □

Remark 2.5 We note that the equality $\varphi(\mathcal{R}(W)) = \mathcal{R}(W')$ in the definition of a weak isomorphism can be replaced by the first equality of statement (1)

Let W be a cellular algebra on V and E be an equivalence on V. We say that E is an *equivalence* of W if it is the union of basis relations of W. A nonempty equivalence E of W is called *indecomposable* (in W) if E is not a disjoint union of two nonempty equivalences of W. Otherwise, it is called *decomposable*. Since the algebra W is closed with respect to the Hadamard multiplication, each equivalence of W can uniquely be represented as a disjoint union of indecomposable ones called the *indecomposable components* of E. It can be proved that the property to be an equivalence (resp. indecomposable equivalence) of a cellular algebra is preserved under weak isomorphisms.

Let E be an equivalence of W. For each $U \in V/E$ the set $W_{E,U} = I_U W I_U$ can be viewed as a cellular algebra on U with the standard basis

$$\mathcal{R}(W_{E,U}) = \{I_U R I_U : R \in \mathcal{R}(W),\ R \subset E,\ I_U R I_U \neq 0\}. \tag{1}$$

Obviously, each basis relation of $W_{E,U}$ can uniquely be represented in the form $I_U R I_U$ with $R \in \mathcal{R}(W)$. If $E = J_U$ where $U \in \mathrm{Cel}^*(W)$, then the algebra $W_{E,U}$ is denoted by W_U and called the restriction of W to U.

Lemma 2.6 *If E is an indecomposable equivalence of W, then*
1. *the mapping $\varphi_{U,U'} : W_{E,U} \to W_{E,U'}$ such that $\varphi_{U,U'}(I_U A I_U) = I_{U'} A I_{U'}$, $A \in W$, is a well-defined weak isomorphism from $W_{E,U}$ to $W_{E,U'}$ for all $U, U' \in V/E$.*
2. $|U \cap X| = |U' \cap X| > 0$ *for all cell X of W, $X \subset V_E$, and all $U, U' \in V/E$.*

Proof: First we prove that

$$I_U R I_U \neq 0 \quad \text{for all } U \in V/E,\ R \in \mathcal{R}(W),\ R \subset E. \tag{2}$$

Indeed, if $I_U R I_U = 0$, then $V_R \cap U = \emptyset$. So E is the union of two nonempty equivalences of W : $I_{V_R} E I_{V_R}$ and $I_{V_E \setminus V_R} E I_{V_E \setminus V_R}$, which contradicts the indecomposability of E.

Let now $U, U' \in V/E$ and $\varphi = \varphi_{U,U'}$. Then formulas (1) and (2) imply that φ is a well-defined linear isomorphism from $W_{E,U}$ to $W_{E,U'}$. It is also an algebra isomorphism, since

$$\varphi(I_U R_1 I_U \cdot I_U R_2 I_U) = \varphi(I_U R_1 R_2 I_U) = I_{U'} R_1 R_2 I_{U'} = I_{U'} R_1 I_{U'} \cdot I_{U'} R_2 I_U'$$

for all $R_1, R_2 \in \mathcal{R}(W)$, $R_1, R_2 \subset E$. This proves statement (1). It follows from formula (1) that each cell of the algebra $W_{E,U}$ is of the form $U \cap X$ where $X \in \mathrm{Cel}(W)$, $X \subset V_E$. By statement (1) $\varphi = \varphi_{U,U'}$ is a weak isomorphism. Thus statement (2) is the consequence of statement (2) of Proposition 2.4, since $(U \cap X)^\varphi = U' \cap X$ by the definition of φ. $\square$

3. Extended algebras and *m*-closures

Let W be a cellular algebra on V. For each positive integer m set

$$\hat{W}^{(m)} = [\underbrace{W \otimes \cdots \otimes W}_{m}\ \mathcal{Z}(\mathrm{Sym}(V), V^m)]$$

with $\mathrm{Sym}(V)$ acting on V^m in a natural way: $(v_1, \dots, v_m)^g = (v_1^g, \dots, v_m^g)$, $g \in \mathrm{Sym}(V)$. We call the cellular algebra $\hat{W}^{(m)} \leq \mathrm{Mat}_{V^m}$ the *m-dimensional extended algebra* of W. Obviously, $\hat{W}^{(1)} = W$ and

$$\mathrm{Aut}\bigl(\hat{W}^{(m)}\bigr) = \{\underbrace{(g, \dots, g)}_{m} : g \in \mathrm{Aut}(W)\} \tag{3}$$

for all m.

Now we are going to describe some relations belonging to $\hat{W}^{(m)}$. To do this we define for an arbitrary $S \subset [m]^2$ a binary relation P_S on V^m by

$$(\bar{u}, \bar{v}) \in P_S \Leftrightarrow \forall (i, j) \in S : u_i = v_j \tag{4}$$

where $\bar{u} = (u_1, \dots, u_m)$, $\bar{v} = (v_1, \dots, v_m) \in V^m$. It follows that $P_S \in \mathcal{Z}(\mathrm{Sym}(V), V^m)$ for all $S \subset [m]^2$.

Examples. Let $M \subset [m]$.

1. Set

$$D_M = P_S \quad \text{where } S = J_M \cup I_{[m]\setminus M}. \tag{5}$$

Clearly, $D_M \subset I_{V^m}$ for all M, $D_\emptyset = I_{V^m}$ and $D_{[m]} = I_\Delta$ where

$$\Delta = \{(v, \dots, v) \in V^m : v \in V\}. \tag{6}$$

2. Set

$$E_M = P_S \quad \text{where } S = I_M. \tag{7}$$

Clearly, E_M is an equivalence on V^m for all M and $E_\emptyset = J_{V^m}$, $E_{[m]} = I_{V^m}$.

Below we will mainly use the relations D_M and E_M as well as matrices

$$\hat{A} = \underbrace{I_V \otimes \cdots \otimes I_V}_{m-1} \otimes A, \quad A \in W \tag{8}$$

also belonging to $\hat{W}^{(m)}$.

Each class U of the equivalence $E_{[m-1]}$ is of the form

$$U = U_{v_1,\dots,v_{m-1}} = \{(v_1, \dots, v_{m-1}, v) : v \in V\}$$

for some $v_i \in V$. Let us define a bijection ζ_U as follows:

$$\zeta_U : V \to U, \quad v \mapsto (v_1, \dots, v_{m-1}, v).$$

The following lemma describes the simplest properties of the map.

Lemma 3.1 *In the above notation the following statements hold*:
1. $A^{\zeta_U} = I_U \hat{A} I_U = I_U \hat{A} = \hat{A} I_U$ *for all* $A \in W$;
2. $(W_{v_1,\ldots,v_{m-1}})^{\zeta_U} \le \hat{W}_{E,U}$ *where* $\hat{W} = \hat{W}^{(m)}$ *and* $E = E_{[m-1]}$.

Proof: Statement (1) is trivial. It follows from it that $W^{\zeta_U} \le \hat{W}_{E,U}$. On the other hand,

$$\left(I_{v_i}\right)^{\zeta_U} = I_{(v_1,\ldots,v_{m-1},v_i)} = I_U D_{\{i,m\}} I_U \in \hat{W}_{E,U} \quad \text{for all } i \in [m-1]. \qquad \square$$

Thus $(W_{v_1,\ldots,v_{m-1}})^{\zeta_U} = W^{\zeta_U}[(I_{v_1})^{\zeta_U}, \ldots, (I_{v_{m-1}})^{\zeta_U}] \le \hat{W}_{E,U}$.

For $l \in [m]$ let us define another map

$$\delta_l^m : V^l \to V^m, \quad (v_1, \ldots, v_l) \mapsto (v_1, \ldots, v_l, \ldots, v_l). \tag{9}$$

It is easy to see that δ_l^m is an injection and $\delta_l^m(V^l)$ (coinciding with the support of $D_{[l,m]}$) is a cellular set of $\hat{W}^{(m)}$.

The important feature of the cellular algebra $\hat{W}^{(m)}$ is the possibility to extend the algebra W without changing its automorphism group. To show it set

$$\bar{W}^{(m)} = \left(\left(\hat{W}^{(m)}\right)_\Delta\right)^{\delta^{-1}}.$$

where $\delta = \delta_1^m : V \to V^m$ is the injection (9) and $\Delta = \delta(V)$ is the cellular set (6). Clearly, $\bar{W}^{(m)} \ge W$ and $\mathrm{Aut}(\bar{W}^{(m)}) = \mathrm{Aut}(W)$ (see (3)). We say that W is *m-closed* if $W = \bar{W}^{(m)}$. Each algebra is certainly 1-closed. However it is not the case for $m \ge 2$. In fact we will show later that a non-Schurian cellular algebra cannot be m-closed for all $m \ge 2$.

Below we list some properties of the operators $W \mapsto \hat{W}^{(m)}$, $W \mapsto \bar{W}^{(m)}$.

Lemma 3.2 *For all cellular algebras W, W_1, W_2 on V and positive integer m*
1. $W_1 \le W_2$ *implies* $\hat{W}_1^{(m)} \le \hat{W}_2^{(m)}$ *and* $\bar{W}_1^{(m)} \le \bar{W}_2^{(m)}$;
2. $\widehat{(W_1 \cap W_2)}^{(m)} \le \hat{W}_1^{(m)} \cap \hat{W}_2^{(m)}$, $\overline{(W_1 \cap W_2)}^{(m)} \le \bar{W}_1^{(m)} \cap \bar{W}_2^{(m)}$;
3. *the intersection of m-closed cellular algebras is m-closed*;
4. $(\hat{W}^{(l)})^{\delta_l^m} \le (\hat{W}^{(m)})_X$ *for all* $l \in [m]$ *where* $X = \delta_l^m(V^l)$;
5. $\bar{W}^{(m)}$ *is l-closed for all* $l \in [m]$.

Proof: Statement (1) is clear. (2) follows from (1). If $\bar{W}_1^{(m)} = W_1$ and $\bar{W}_2^{(m)} = W_2$, then $\overline{(W_1 \cap W_2)}^{(m)} \le W_1 \cap W_2$ by (2). Since the inverse inclusion is obvious, we have (3). Further, our definitions imply that

$$(\mathcal{Z}(\mathrm{Sym}(V), V^l))^{\delta_l^m} \subset \mathcal{Z}(\mathrm{Sym}(V), V^m), (\underbrace{W \otimes \cdots \otimes W}_{l})^{\delta_l^m} \subset I_X(\underbrace{W \otimes \cdots \otimes W}_{m})I_X.$$

where $X = \delta_l^m(V^l)$. As far as X is a cellular set of $\hat{W}^{(m)}$, (4) follows.

It follows from statement (4) and the equality $\delta_l^m \circ \delta_1^l = \delta_1^m$ (see (9)) that $\bar{W'}^{(l)} \le \bar{W'}^{(m)}$ for all W'. Applying it to $W' = \bar{W}^{(m)}$ we see that it suffices to prove statement (5) for $l = m$. We will check that the m-dimensional extended algebras of W and $\bar{W}^{(m)}$ coincide. Clearly, the second contains the first. To prove the inverse inclusion set $R_j = P_{S_j}$ where $S_j = \{(i, j) : i \in [m]\}$, $j \in [m]$ (see (4)). A straightforward calculation in $\mathrm{Mat}_{V^m} = \mathrm{Mat}_V \otimes \cdots \otimes \mathrm{Mat}_V$ shows that for all $j \in [m]$

$$R_j^T A^\delta R_j = \underbrace{J_V \otimes \ldots \otimes J_V}_{j-1} \otimes A \otimes \underbrace{J_V \otimes \ldots \otimes J_V}_{m-j}, \quad A \in \mathrm{Mat}_V$$

where δ is the map (9). Since the Hadamard multiplication in $\mathrm{Mat}_V \otimes \cdots \otimes \mathrm{Mat}_V$ can be done factorwise,

$$A_1 \otimes \cdots \otimes A_m = \left(R_1^T A_1^\delta R_1\right) \circ \cdots \circ \left(R_m^T A_m^\delta R_m\right) \quad \text{for all } A_1, \ldots, A_m \in \mathrm{Mat}_V.$$

Thus $\bar{W}^{(m)} \otimes \cdots \otimes \bar{W}^{(m)} \subset \hat{W}^{(m)}$ by the definition of $\bar{W}^{(m)}$. □

It follows from statement (5) of Lemma 3.2 that the cellular algebra $\bar{W}^{(m)}$ is m-closed. We call it the *m-closure* of W.

The following proposition describes some relations between the notions of m-closure and Schurian closure $\mathrm{Sch}(W)$ of a cellular algebra W. It shows that in a sense $\bar{W}^{(m)}$ can be interpreted as an approximation to $\mathrm{Sch}(W)$.

Proposition 3.3 *For each cellular algebra W on V the following statements hold*:
1. $\mathrm{Aut}(\bar{W}^{(m)}) = \mathrm{Aut}(W)$ *for all* $m \ge 1$;
2. $W = \bar{W}^{(1)} \le \cdots \le \bar{W}^{(n)} = \cdots = \mathrm{Sch}(W)$;
3. $\overline{(\bar{W}^{(m)})}^{(l)} = \bar{W}^{(m)}$ *for all* $l \in [m]$.

Proof: Statement (1) is clear. Let us prove (2). The inclusion $\bar{W}^{(l)} \le \bar{W}^{(m)}$ for $l \le m$ is contained in the proof of statement (5) of Lemma 3.2. The equality $\bar{W}^{(m)} = \mathrm{Sch}(W)$ for $m \ge n$ follows from Theorem 1.3, since, obviously, $s(W) \le n - 1$ for all W. (Note that Theorem 1.3 is proved in Section 5 independently of this assertion.) Finally, (3) coincides with statement (5) of Lemma 3.2. □

Proposition 3.4 *Given a cellular algebra W on V and a positive integer m the standard bases of the cellular algebras $\hat{W}^{(m)}$ and $\bar{W}^{(m)}$ can be constructed in time $n^{O(m)}$.*

Proof: Since the standard bases of $W \otimes \cdots \otimes W$ (m times) and $\mathcal{Z}(\mathrm{Sym}(V), V^m)$ can be found in time $n^{O(m)}$, the standard basis of $\hat{W}^{(m)}$ (and so of $\bar{W}^{(m)}$) can be found within the same time due to the Weisfeiler-Lehman algorithm for constructing the cellular closure of a set of matrices (see Section 1). □

Remark 3.5 The time analysis of the Weisfeiler-Lehman algorithm done in [13] gives an $O(mn^{5m} \log n)$ upper bound for the time of constructing the algebras $\hat{W}^{(m)}$ and $\bar{W}^{(m)}$. The algorithm from [4] enables us to reduce it to $O(mn^{3m} \log n)$.

Propositions 3.3 and 3.4 show that the operators $W \mapsto \bar{W}^{(m)}$ ($m = 1, 2, \ldots$) define a Schurian polynomial approximation scheme (see Section 1). This proves Theorem 1.1.

We complete the section by a statement being of use later. For each $R \subset V \times V$ set

$$X_R = \{(u, \ldots, u, v) \in V^m : (u, v) \in R\}.$$

Proposition 3.6 *Let W be a cellular algebra on V and $m \geq 2$. Then*
1. $\forall R \subset V \times V : R \in \mathcal{R}(\bar{W}) \Leftrightarrow X_R \in \mathrm{Cel}(\hat{W})$;
2. $\forall X \in \mathrm{Cel}(\hat{W})\, \forall i, j \in [m]\, \exists R \in \mathcal{R}(\bar{W}) : ((v_1, \ldots, v_m) \in X \Rightarrow (v_i, v_j) \in R)$

where $\bar{W} = \bar{W}^{(m)}$ and $\hat{W} = \hat{W}^{(m)}$.

Proof: Below we write $v_1 \cdots v_m$ instead of $(v_1, \ldots, v_m)$. Let us prove statement (1). Assume that $R \in \mathcal{R}(\bar{W})$. Choose $(u, v) \in R$ and denote by S_1, T, S_2 the basis relations of $\hat{W}$ containing the pairs $(u^m, u^{m-1}v)$, $(u^{m-1}v, u^{m-1}v)$ and $(u^{m-1}v, v^m)$ respectively. Clearly, $p(u^m, v^m; \tau) = 1$ where $\tau = (S_1, T, S_2)$. By the Path Proposition (Lemma 2.3) the equality holds for all $(u')^m, (v')^m$ with $(u', v') \in R$. So $T = I_{X_R}$, whence $X_R \in \mathrm{Cel}(\hat{W})$. Conversely, let $X_R \in \mathrm{Cel}(\hat{W})$. Choose $u^{m-1}v \in X_R$ and denote by S_1', R', S_2' the basis relations of $\hat{W}$ containing the pairs $(u^{m-1}v, u^m)$, (u^m, v^m) and $(v^m, u^{m-1}v)$ respectively. Clearly, $p(u^{m-1}v, u^{m-1}v; \tau') = 1$ where $\tau' = (S_1', R', S_2')$. By the Path Proposition the equality holds for all points of X_R. It follows that $R' = R^\delta$ where δ is defined in (9). That is $R \in \mathcal{R}(\bar{W})$.

To prove statement (2) we assume without loss of generality that $i = m - 1$, $j = m$. Let $X \in \mathrm{Cel}(\hat{W})$. Choose $\bar{v} = v_1 \cdots v_m \in X$ and denote by R the basis relation of $\bar{W}$ containing the pair (v_{m-1}, v_m). By statement (1) we have $X_R \in \mathrm{Cel}(\hat{W})$. Set

$$S = (X_R \times X) \cap E_{\{m-1,m\}}$$

where $E_{\{m-1,m\}}$ is defined in (7). Clearly, $S \in \mathcal{R}(\hat{W})$, $d_{\mathrm{in}}(S) = 1$ and $(v_{m-1}^{m-1} v_m, \bar{v}) \in S$. So for any $\bar{v}' \in X$ there exists $\bar{u}' \in X_R$ such that $(\bar{u}', \bar{v}') \in S$. If $\bar{v}' = v_1' \cdots v_m'$, then $\bar{u}' = (v_{m-1}')^{m-1} v_m'$, whence $(v_{m-1}', v_m') \in R$. □

4. High dimensional Weisfeiler-Lehman procedures

In this section we prove Propositions 4.1 and 4.2 from which Theorem 1.2 follows.

A map f from V^m onto $[d]$ is called a *coloring* of V^m. Any set $f^{-1}(i) \subset V^m$ is called a *color class* of f. Let $m \geq 2$. Denote by $\mathcal{R}_f$ the partition of $V \times V$ into the classes of the form

$$R_{u,v} = \{(u', v') \in V \times V : f(u', \ldots, u', v') = f(u, \ldots, u, v)\}, \quad u, v \in V.$$

Conversely, given a partition $\mathcal{R}$ of $V \times V$ let us define a coloring $f_{\mathcal{R}}$ of V^m by

$$f_{\mathcal{R}}(\bar{v}) = f_{\mathcal{R}}(\bar{v}') \Leftrightarrow \forall R \in \mathcal{R}\, \forall i, j \in [m] : ((v_i, v_j) \in R \Leftrightarrow (v_i', v_j') \in R).$$

In this notation for a cellular algebra W on V we set

$$A_1(W) = W, \quad A_m(W) = [\mathcal{R}_f], \; m \geq 2$$

where f is the coloring of V^m derived from $f_0 = f_{\mathcal{R}(W)}$ by the following procedure.

4.1. m-dim stabilization

Input: a coloring f_0 of V^m.
Output: a coloring f of V^m.

Step 1. Set $l = 0$.
Step 2. For each $\bar{v} \in V^m$ find a formal sum $S(\bar{v}) = \sum_{u \in V} f_l(\bar{v}/u)$ where

$$\bar{v}/u = (\bar{v}_{1,u}, \ldots, \bar{v}_{m,u}) \quad \text{with} \quad \bar{v}_{i,u} = (v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_m)$$

and

$$f_l(\bar{v}/u) = (f_l(\bar{v}_{1,u}), \ldots, f_l(\bar{v}_{m,u})).$$

Step 3. Find a coloring f_{l+1} of V^m such that

$$f_{l+1}(\bar{v}) = f_{l+1}(\bar{v}') \Leftrightarrow (f_l(\bar{v}) = f_l(\bar{v}'), \; S(\bar{v}) = S(\bar{v}')).$$

If the numbers of color classes of f_l and f_{l+1} are different, then $l := l + 1$ and go to Step 2. Otherwise set $f = f_l$.

Proposition 4.1 *Let W be a cellular algebra on V. Then $\bar{W}^{(m)} \geq A_m(W)$.*

Proof: We will show by induction on l that each color class of f_l is a union of the cells of the algebra $\hat{W}^{(m)}$. Then given $R \in \mathcal{R}(\bar{W}^{(m)})$, by statement (1) of Proposition 3.6 $f(\bar{v}) = f(\bar{v}')$ for all $\bar{v}, \bar{v}' \in X_R$ and we are done.

By statement (2) of Proposition 3.6 and the fact that $W \leq \bar{W}^{(m)}$ the above claim is true for $l = 0$. Suppose it is true for all $k < l$. Let $\bar{v} \in V^m$. For each $u \in V$ set

$$P_u(\bar{v}) = (\bar{v}, \bar{v}_{1,u}, \ldots, \bar{v}_{m,u}, \bar{v}).$$

It is easy to see that the path $P_u(\bar{v})$ from $\bar{v}$ to itself is of the type $\tau = (R_0, \ldots, R_m)$ for some basis relations $R_i \subset P_{S_i}$, (see (4)) where

$$S_i = \{(j, j), (i, i+1) \in [m]^2 : j \neq i, \, j \neq i+1\}, \quad i \in [0, m].$$

Moreover, any $(\bar{v}, \bar{v})$-path of the type τ coincides with $P_u(\bar{v})$ for some $u \in V$.

Let $\bar{v}, \bar{v}' \in V^m$ belong to the same cell of $\hat{W}^{(m)}$. Then by the induction hypothesis $f_{l-1}(\bar{v}) = f_{l-1}(\bar{v}')$. Besides by the Path Proposition (Lemma 2.3) $p(\bar{v}, \bar{v}; \tau) = p(\bar{v}', \bar{v}'; \tau)$.

If $P_u(\bar{v})$ and $P_{u'}(\bar{v}')$ are of the type τ, then $\bar{v}_{i,u}$ and $\bar{v}'_{i,u'}$ belong to the same cell of $\hat{W}^{(m)}$ for all i. So by the induction hypothesis $f_{l-1}(\bar{v}/u) = f_{l-1}(\bar{v}'/u')$. Thus $S_{l-1}(\bar{v}) = S_{l-1}(\bar{v}')$ and consequently $f_l(\bar{v}) = f_l(\bar{v}')$. □

Another implementation of the m-dimensional procedure was described in [7]. We are going to prove that this procedure constructs a cellular subalgebra of the m-closure.

For $i \geq 1$ set

$$\mathcal{A}_{V,i} = E_{[i-1]} \circ \mathrm{Mat}_{V^i} = \sum_{(v_1,\ldots,v_{i-1}) \in V^{i-1}} I_{v_1,\ldots,v_{i-1}} \otimes \mathrm{Mat}_V$$

where $I_{v_1,\ldots,v_{i-1}} = I_{v_1} \otimes \cdots \otimes I_{v_{i-1}}$. Clearly, $\mathcal{A}_{V,i}$ is a subalgebra of Mat_{V^i} closed under the Hadamard multiplication and the Hermitian conjugation. Let us define a linear map

$$\pi_i : \mathcal{A}_{V,i+1} \to \mathcal{A}_{V,i}, \quad i \geq 1,$$

by

$$\pi_i \left(\sum_{(v_1,\ldots,v_i)} I_{v_1,\ldots,v_i} \otimes A_{v_1,\ldots,v_i} \right) = \sum_{(v_1,\ldots,v_{i-1})} I_{v_1,\ldots,v_{i-1}} \otimes \sum_{v_i \in V} A_{v_1,\ldots,v_i}. \tag{10}$$

In these terms the procedure from [7] can be described as follows.

4.2. Procedure B_m $(m \geq 1)$

Input: a cellular algebra W on V.
Output: a cellular algebra $B_m(W) \geq W$.

Step 1. Construct the set $\mathcal{R}_m = \{\hat{R} : R \in \mathcal{R}(W)\} \subset \mathcal{A}_{V,m}$ and the cellular algebra

$$W(m) = \left[\mathcal{R}_m, D_{\{1,m\}}, \ldots, D_{\{m-1,m\}}\right]$$

where $\hat{R}$ and $D_{\{i,m\}}$ are as in (8) and (5) respectively.
Step 2. For $i = m-1, \ldots, 1$ find sucsessively the linear spaces

$$W(i) = \pi_i(W(i+1)) \subset \mathcal{A}_{V,i}.$$

Set $W' = [W(1)]$.
Step 3. If $W' \neq W$, then $W{:=}W'$ and go to Step 1. Otherwise, set $B_m(W) = W'$.

Proposition 4.2 *Let W be a cellular algebra on V and $m \geq 1$. Then $B_m(W) \leq \bar{W}^{(m)}$.*

Proof: For $i \in [m]$ set

$$W_i = E_{[i-1]} \circ \left(D_{[i,m]} \hat{W}^{(m)} D_{[i,m]}\right).$$

Then $W_i \subset \varphi_i(\mathcal{A}_{V,i})$ where $\varphi_i : \mathcal{A}_{V,i} \to \mathcal{A}_{V,m}$ is the linear map induced by the injection $\delta_i^m : V^i \to V^m$ defined in (9). We will prove that

$$\pi_i'(W_{i+1}) \subset W_i \quad \text{for all } i \in [m-1]$$

where $\pi_i' = \varphi_i \pi_i \varphi_{i+1}^{-1}$ and π_i is defined by (10).

A straightforward check shows that

$$\pi_i'(A) = D_{[i,m]} E_{[m]\setminus\{i\}} A E_{[m]\setminus\{i\}} D_{[i,m]}, \quad A \in \varphi_{i+1}(\mathcal{A}_{V,i+1}).$$

So $\pi_i'(W_{i+1}) \subset W_i$ for all i. By the definition of $W(m)$ at Step 1 $W(m) \subset W_m$. Therefore,

$$W(1) = \pi_1 \cdots \pi_{m-1}(W(m)) \subset \varphi_1^{-1} \pi_1' \cdots \pi_{m-1}'(W_m) \subset \varphi_1^{-1}(W_1) = \bar{W}^{(m)},$$

which completes the proof. □

5. Proof of Theorem 1.3

In this section we prove Theorem 1.3. Given W with $s(W) \le m-1$ we will show that the algebra $\bar{W} = \bar{W}^{(m)}$ is Schurian.

By the hypothesis of the theorem $W_{v_1,\ldots,v_{m-1}} = \mathrm{Mat}_V$ for some $(v_1, \ldots, v_{m-1}) \in V^{m-1}$. Denote by E the indecomposable component (in $\hat{W} = \hat{W}^{(m)}$) of the equivalence $E_{[m-1]}$ for which $U = U_{v_1,\ldots,v_{m-1}}$ is one of the classes. By statement (2) of Lemma 3.1 we have $\hat{W}_{E,U} \ge (W_{v_1,\ldots,v_{m-1}})^{\zeta_U} = \mathrm{Mat}_U$, whence $\hat{W}_{E,U} = \mathrm{Mat}_U$. By statement (1) of Lemma 2.6 and statement (2) of Proposition 2.4

$$\hat{W}_{E,U'} = \mathrm{Mat}_{U'} \quad \text{for all } U' \in V^m/E. \tag{11}$$

Statement (2) of Lemma 2.6 implies that

$$(V^m)_E = \bigcup_{i=1}^{s} X_i \tag{12}$$

where $X_i \in \mathrm{Cel}(\hat{W})$ with $U' \cap X_i \ne \emptyset$ for all $U' \in V^m/E$. It follows from (11) that $|U' \cap X_i| = 1$ for all U' and i. In particular, $s = n$.

For any $U' \in V^m/E$ let $\varphi_{U,U'} : \hat{W}_{E,U} \to \hat{W}_{E,U'}$ be the weak isomorphism from statement (1) of Lemma 2.6 (with $\hat{W}$ instead of W). By (11) $\varphi_{U,U'}$ is induced by a bijection $g_{U,U'}$: $U \to U'$, i.e., $\varphi_{U,U'}(A) = A^{g_{U,U'}}$ for all $A \in \hat{W}_{E,U}$. Set

$$h_{U'} = \zeta_U g_{U,U'} \zeta_{U'}^{-1}, \quad U' \in V^m/E \tag{13}$$

Clearly, $h_{U'} \in \mathrm{Sym}(V)$ for all U'. Moreover, by Lemma 3.1 and the definition of the isomorphism $\varphi_{U,U'}$ we have

$$A^{h_{U'}} = A^{\zeta_U g_{U,U'} \zeta_{U'}^{-1}} = (I_U \hat{A} I_U)^{g_{U,U'} \zeta_{U'}^{-1}} = (\varphi_{U,U'}(I_U \hat{A} I_U))^{\zeta_{U'}^{-1}} = (I_{U'} \hat{A} I_{U'})^{\zeta_{U'}^{-1}} = A$$

for all $A \in W$ where $\hat{A} = I_V \otimes \cdots \otimes I_V \otimes A$ (see (8)). Thus

$$h_{U'} \in \mathrm{Aut}(W) \quad \text{for all } U' \in V^m/E. \tag{14}$$

We are to show that given $R \in \mathcal{R}(\bar{W})$ and $(u, v), (u', v') \in R$ there exists $U' \in V^m/E$ such that

$$\left(u^{h_{U'}}, v^{h_{U'}}\right) = (u', v'). \tag{15}$$

Then it will imply by (14) that $\mathrm{Aut}(W)$ acts transitively on each basis relation of $\bar{W}$, i.e., the cellular algebra $\bar{W}$ is Schurian.

Let $R \in \mathcal{R}(\bar{W})$ and $(u, v), (u', v') \in R$. Consider the following path

$$(u, \ldots, u, v) \to (v_1, \ldots, v_{m-1}, u) \to (v_1, \ldots, v_{m-1}, v) \to (u, \ldots, u, v).$$

Denote its type by (R_0, R_1, R_2) where $R_i \in \mathcal{R}(\hat{W})$, $i = 0, 1, 2$. Clearly (see (4)),

$$R_0 \subset P_{\{(m-1,m)\}}, \quad R_1 \subset E, \quad R_2 \subset P_{\{(m,m)\}}. \tag{16}$$

By statement (1) of Proposition 3.6 the points $(u, \ldots, u, v)$ and $(u', \ldots, u', v')$ belong to the same cell of $\hat{W}$. So by the Path Proposition there exists a path from $(u', \ldots, u', v')$ to itself of the type (R_0, R_1, R_2). By (16) it is of the form

$$(u', \ldots, u', v') \to (v'_1, \ldots, v'_{m-1}, u') \to (v'_1, \ldots, v'_{m-1}, v') \to (u', \ldots, u', v')$$

for some $(v'_1, \ldots, v'_{m-1}) \in V^{m-1}$, and $U' = U_{v'_1,\ldots,v'_{m-1}}$ is a class modulo E. To complete the proof it suffices to check that $u^{h_{U'}} = u'$ and $v^{h_{U'}} = v'$. We prove only the first equality, since the second one is proved similarly.

Since $R_1 \in \mathcal{R}(\hat{W})$, the points $(v_1, \ldots, v_{m-1}, u)$ and $(v'_1, \ldots, v'_{m-1}, u')$ belong to the same cell of $\hat{W}$. From $R_1 \subset E$ it follows that the cell coincides with X_i for some i. Since $|U \cap X_i| = |U' \cap X_i| = 1$ (see above) we have

$$U \cap X_i = \{(v_1, \ldots, v_{m-1}, u)\}, \quad U' \cap X_i = \{(v'_1, \ldots, v'_{m-1}, u')\}.$$

By the definition of $g_{U,U'}$ (see also Lemma 2.6) we see that $(U \cap X_i)^{g_{U,U'}} = U' \cap X_i$. So

$$(v_1, \ldots, v_{m-1}, u)^{g_{U,U'}} = (v'_1, \ldots, v'_{m-1}, u').$$

On the other hand, by the definition of $h_{U'}$ (see (13))

$$(v_1, \ldots, v_{m-1}, w)^{g_{U,U'}} = \left(v'_1, \ldots, v'_{m-1}, w^{h_{U'}}\right)$$

for all $w \in V$. Therefore $u^{h_{U'}} = u'$. Theorem is proved. $\square$

6. Concluding remarks and open problems

There is a lot of problems concerning Schurian polynomial approximation schemes. We concentrate here only on two of them.

1. Let $S : W \mapsto S_m(W)$ and $T : W \mapsto T_m(W)$ $(m = 1, 2 \ldots)$ be two Schurian polynomial approximation schemes. We say that S is reducible to T if there exists a linearly bounded function $f : \mathbf{N} \to \mathbf{N}$ where $\mathbf{N} = \{1, 2, \ldots\}$ such that $S_m(W) \leq T_{f(m)}(W)$ for all cellular algebras W and all m. S and T are called equivalent if each of them is reducible to the other. Theorem 1.2 shows that the schemes A and B (see Section 4) are reducible to the scheme defined by the m-closure operators.

Problem 6.1 *Are all the three schemes equivalent*?[3]

2. From the algorithmic point of view the Schurian polynomial approximation scheme defined by the m-closure operators is based on finding the cellular closure of a set of matrices. This problem can efficiently (in polynomial time) be solved by the standard Weisfeiler-Lehman algorithm.

Problem 6.2 *Is the above problem in* **NC**? *In other words*, *can the cellular closure of an* $n \times n$*-matrix be found by* $n^{O(1)}$ *parallel computers in time* $(\log n)^{O(1)}$?

(For the exact definition of **NC** and related concepts see [11].) The main difficulty here is that the cellular closure is defined by means of two binary operations (the ordinary matrix multiplication and the Hadamard one) which do not commute with each other. Note, that for each of them the problem of constructing the closure with respect to it is in **NC**.

Notes

1. Throughout the paper we assume that the unity of a cellular algebra coincides with the identity matrix of Mat_V. In this case cellular algebras coincide with coherent algebras introduced in [9].
2. A counterexample was revealed in [1].
3. It was proved in the Electronic Journal of Combinatorics 6 (1999), #R18 that the scheme A is equivalent to the scheme defined by the m-closure operators.

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