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Three-Class Association Schemes

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Abstract. We study (symmetric) three-class association schemes. The graphs with four distinct eigenvalues which are one of the relations of such a scheme are characterized. We also give an overview of most known constructions, and obtain necessary conditions for existence. A list of feasible parameter sets on at most 100 vertices is generated.

Keywords: association scheme, graph, eigenvalue

1. Introduction

In the theory of (algebraic) combinatorics association schemes play an important role. Association schemes may be seen as colorings of the edges of the complete graph satisfying nice regularity conditions, and they are used in coding theory, design theory, graph theory and group theory. Many chapters of books or complete books are devoted to association schemes (cf. [2, 10, 12, 34]).

The special case of two-class association schemes (colorings with two colors) is widely investigated (cf. [13, 62]), as these are equivalent to strongly regular graphs. Also the case of three-class association schemes is very special: there is more than just applying the general theory. However, there are not many papers about three-class association schemes in general. There is the early paper by Mathon [52], who gives many examples, and the thesis of Chang [19], who restricts to the imprimitive case. The special case of distance-regular graphs with diameter three has been paid more attention, and for more results on such graphs we refer to [10].

We shall discuss three-class association schemes, mainly starting from regular graphs with four distinct eigenvalues (cf. [23]), since for most of the (interesting) schemes indeed at least one of the relations is such a graph. However, most such graphs cannot be a relation in a three-class association scheme (cf. [26]). (It is even so that there are graphs that have the same spectrum as one of the relations of a three-class association scheme, which are themselves not a relation of a three-class association scheme, cf. [39]). We shall characterize the graphs with four distinct eigenvalues that are a relation of a three-class association scheme. We shall give several constructions, and obtain necessary number theoretic conditions for existence.

We start with a brief introduction to association schemes. For (more) basic results on association schemes and their proofs we refer to [10, 12]. At the end we shall classify the three-class association schemes into three classes, one which may be considered as degenerate, one in which all three relations are strongly regular, and one in which at least one of the relations is a graph with four distinct eigenvalues. This classification is used to generate all feasible parameter sets of (nondegenerate) three-class association schemes on at most 100 vertices, which are listed in the appendix.

2. Association schemes

Let V be a finite set of vertices. A d-class *association scheme* on V consists of a set of $d+1$ symmetric relations $\{R_0, R_1, \ldots, R_d\}$ on V, with identity relation $R_0 = \{(x, x) \mid x \in V\}$, such that any two vertices are in precisely one relation. Furthermore, there are *intersection numbers* p_{ij}^k such that for any $(x, y) \in R_k$, the number of vertices z such that $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k. If a pair of vertices is in relation R_i, then these vertices are called ith *associates*. If the union of some relations is a nontrivial equivalence relation, then the scheme is called *imprimitive*, otherwise it is called *primitive*.

Association schemes were introduced by Bose and Shimamoto [8]. Delsarte [27] applied association schemes to coding theory, and he used a slightly more general definition by not requiring symmetry for the relations, but for the total set of relations and for the intersection numbers. To study permutation groups, Higman (cf. [41]) introduced the even more general *coherent configurations*, for which the identity relation may be the union of some relations. In coherent configurations for which the identity relation is not one of its relations we must have at least 5 classes (6 relations).

There is a strong connection with group theory in the following way. If G is a permutation group acting on a vertex set V, then the *orbitals*, that is, the orbits of the action of G on V^2, form a coherent configuration. If G acts *generously transitive*, that is, for any two vertices there is a group element interchanging them, then we get an association scheme. If so, then we say the scheme is in the *group case*.

2.1. *The Bose-Mesner algebra*

The nontrivial relations can be considered as graphs, which in our case are undirected. One immediately sees that the respective graphs are regular with degree $n_i = p_{ii}^0$. For the corresponding adjacency matrices A_i the axioms of the scheme are equivalent to

$$\sum_{i=0}^{d} A_i = J, \qquad A_0 = I, \qquad A_i = A_i^T, \qquad A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k.$$

It follows that the adjacency matrices generate a $(d+1)$-dimensional commutative algebra $\mathbf{A}$ of symmetric matrices. This algebra was first studied by Bose and Mesner [7] and is called the *Bose-Mesner algebra* of the scheme. The corresponding algebra of a coherent configuration is called a *coherent algebra*, or by some authors a cellular algebra or cellular ring (with identity) (cf. [30]).

A very important property of the Bose-Mesner algebra is that it is not only closed under ordinary multiplication, but also under entrywise (Hadamard, Schur) multiplication $\circ$. In fact, any vector space of symmetric matrices that contains the identity matrix I and the all-one matrix J, and that is closed under ordinary and entrywise multiplication is the Bose-Mesner algebra of an association scheme (cf. [10, Theorem 2.6.1]).

2.2. *The spectrum of an association scheme*

Since the adjacency matrices of the scheme commute, they can be diagonalized simultaneously, that is, the whole space can be written as a direct sum of common eigenspaces. In fact, **A** has a unique basis of minimal *idempotents* E_i, $i = 0, \ldots, d$. These are matrices such that

$$E_i E_j = \delta_{ij} E_i \quad \text{and} \quad \sum_{i=0}^{d} E_i = I.$$

(The idempotents are projections on the eigenspaces.) Without loss of generality, we may take $E_0 = v^{-1} J$. Now let P and Q be matrices such that

$$A_j = \sum_{i=0}^{d} P_{ij} E_i \quad \text{and} \quad E_j = \frac{1}{v} \sum_{i=0}^{d} Q_{ij} A_j.$$

Thus $PQ = QP = vI$. It also follows that $A_j E_i = P_{ij} E_i$, so P_{ij} is an eigenvalue of A_j with multiplicity $m_i = \text{rank}(E_i)$. The matrices P and Q are called the *eigenmatrices* of the association scheme. The first row and column of these matrices are always given by $P_{i0} = Q_{i0} = 1$, $P_{0i} = n_i$ and $Q_{0i} = m_i$. Furthermore, P and Q are related by $m_i P_{ij} = n_j Q_{ji}$. Other important properties of the eigenmatrices are given by the *orthogonality relations*

$$\sum_{i=0}^{d} m_i P_{ij} P_{ik} = v n_j \delta_{jk} \quad \text{and} \quad \sum_{i=0}^{d} n_i Q_{ij} Q_{ik} = v m_j \delta_{jk}.$$

The *intersection matrices* L_i defined by $(L_i)_{kj} = p^k_{ij}$ also have eigenvalues P_{ji}. In fact, the columns of Q are eigenvectors of L_i. Moreover, the algebra generated by the intersection matrices is isomorphic to the Bose-Mesner algebra.

An association scheme is called *self-dual* if $P = Q$ for some ordering of the idempotents.

2.3. *The Krein parameters*

As the Bose-Mesner algebra is closed under entrywise multiplication, we can write

$$E_i \circ E_j = \frac{1}{v} \sum_{k=0}^{d} q^k_{ij} E_k$$

for some real numbers q_{ij}^k, called the *Krein parameters* or dual intersection numbers. We can compute these parameters from the eigenvalues of the scheme by the equation

$$q_{ij}^k = \frac{m_i m_j}{v} \sum_{l=0}^{d} \frac{P_{il} P_{jl} P_{kl}}{n_l^2}$$

The so-called *Krein conditions*, proven by Scott, state that the Krein parameters are nonnegative. Another restriction related to the Krein parameters is the so-called *absolute bound*, which states that for all i, j

$$\sum_{q_{ij}^k \neq 0} m_k \leq \begin{cases} m_i m_j & \text{if } i \neq j, \\ \frac{1}{2} m_i (m_i + 1) & \text{if } i = j. \end{cases}$$

2.4. *Distance-regular graphs and strongly regular graphs*

A *distance-regular graph* is a connected graph for which the distance relations (i.e., a pair of vertices is in R_i if their distance in the graph is i) form an association scheme. They were introduced by Biggs [5], and are widely investigated. As general reference we use [10]. It is well known that an imprimitive distance-regular graph is bipartite or antipodal. *Antipodal* means that the union of the distance d relation and the trivial relation is an equivalence relation.

A connected *strongly regular graph* is a distance-regular graph with diameter two. A graph G is strongly regular with parameters (v, k, λ, μ) if and only if it has v vertices, is regular of degree k (with $0 < k < v - 1$), any two adjacent vertices have λ common neighbours and any two nonadjacent vertices have μ common neighbours. The complement of G is also strongly regular, and in fact any 2-class association scheme is equivalent to a pair of complementary strongly regular graphs.

The property that one of the relations of a d-class association scheme forms a distance-regular graph with diameter d is equivalent to the scheme being *P-polynomial*, that is, the relations can be ordered such that the adjacency matrix A_i of relation R_i is a polynomial of degree i in A_1, for every i. In turn, this is equivalent to the conditions $p_{1i}^{i+1} > 0$ and $p_{1i}^k = 0$ for $k > i + 1$, $i = 0, \ldots, d - 1$. For a 3-class association scheme the conditions are equivalent to $p_{11}^3 = 0$, $p_{11}^2 > 0$ and $p_{12}^3 > 0$ for some ordering of the relations.

Dually we say that the scheme is *Q-polynomial* if the idempotents can be ordered such that the idempotent E_i is a polynomial of degree i in E_1 with respect to entrywise multiplication, for every i. Equivalent conditions are that $q_{1i}^{i+1} > 0$ and $q_{1i}^k = 0$ for $k > i + 1$, $i = 0, \ldots, d - 1$. In the case of a 3-class association scheme these conditions are equivalent to $q_{11}^3 = 0$, $q_{11}^2 > 0$ and $q_{12}^3 > 0$ for some ordering of the idempotents. (Here we say that the scheme has Q-polynomial ordering 123.)

In the case of distance-regular graphs, the relation corresponding to adjacency generates the whole corresponding association scheme. A similar thing often occurs if we have a 3-class association scheme. A scheme is said to be *generated* by one of its relations (or the corresponding graph) if this relation determines the other relations (immediately from the definition).

If one of the relations of a 3-class association scheme is a graph with four distinct eigenvalues, then the number of common neighbours of two nonadjacent vertices equals p_{11}^2 or p_{11}^3 (which are distinct, otherwise we have a strongly regular graph, which has only three distinct eigenvalues), and so we can see from this number whether two vertices are second or third associates. So the graph generates the whole scheme.

3. Examples

The d-class *Hamming scheme* $H(d, q)$ is defined on the ordered d-tuples on q symbols (words of length d over an alphabet with q letters), where two tuples are in relation R_i if they differ in i coordinates. The 3-class Hamming scheme is also known as the *cubic scheme*, as it was introduced by Raghavarao and Chandrasekhararao [61]. The Hamming scheme is characterized by its parameters unless $q = 4$, and then we also have the *Doob schemes*. For $d = 3$ there is one Doob scheme (cf. [10]).

The d-class *Johnson scheme* $J(n, d)$ is defined on the d-subsets of an n-set. Two d-subsets are in relation R_i if they intersect in $d - i$ elements. The 3-class version is also known as the *tetrahedral scheme*, and was first found as a generalization of the triangular graph by John [49]. The Johnson scheme is characterized by its parameters unless $d = 2$ and $n = 8$ (cf. [10]).

The *rectangular scheme* $R(m, n)$, introduced by Vartak [69], has as vertices the ordered pairs (i, j), with $i = 1, \ldots, m$, and $j = 1, \ldots, n$. For two distinct pairs we can have the following three situations. They agree in the first coordinate, or in the second coordinate, or in neither coordinate, and the relations are defined accordingly. Note that the graph of the third relation is the complement of the line graph of the complete bipartite graph $K_{m,n}$. The scheme is characterized by its parameters.

The Hamming scheme, the Johnson scheme and the rectangular scheme are all in the group case. Only the rectangular scheme does not define a distance-regular graph (unless m or n equals two). There are many more examples of distance-regular graphs with diameter three. In this paper we shall mainly focus on 3-class association schemes that are not such graphs, although, of course, the general results do apply. For more examples and specific results on distance-regular graphs we refer to [10]. The antipodal distance-regular graphs with diameter three form a special class, as they are antipodal covers of the complete graph. For more on such graphs, see [11, 16, 35, 50].

3.1. The disjoint union of strongly regular graphs

Take the disjoint union of, say n, strongly regular graphs, all with the same parameters and hence the same spectrum. Then this graph generates an imprimitive 3-class association scheme (the other relations are given by the disjoint union of the complements of the strongly regular graphs, and the complete n-partite graph).

Conversely, any association scheme with the same parameters must be obtained in the described way. Therefore, we may consider this case as degenerate, and it suffices to refer to the extensive literature (for example [13, 62]) on strongly regular graphs. The same remarks hold for the next construction.

3.2. *A product construction from strongly regular graphs*

If G is a strongly regular graph, then for any natural number n, the graph $G \otimes J_n$, defined by its adjacency matrix $A \otimes J_n$, where A is the adjacency matrix of G, generates an imprimitive 3-class association scheme (here the other relations are $\bar{G} \otimes J_n$ and a disjoint union of n-cliques).

It is easy to show that any 3-class association scheme with $p^2_{11} = n_1$ (or $p^3_{11} = n_1$) must be of this form.

3.3. *Pseudocyclic schemes*

A d-class association scheme is called *pseudocyclic* if all the nontrivial eigenvalues have the same multiplicities m. In this case, we also have all degrees equal to m.

If v is a prime power, and $v \equiv 1 \pmod 3$, we can define the 3-class *cyclotomic* association scheme Cycl(v) as follows. Let α be a primitive element of $GF(v)$. As vertices we take the elements of $GF(v)$. Two vertices will be ith associates if their difference equals α^{3t+i} for some t (or, if the discrete logarithm (base α) of their difference is congruent to i modulo 3), for $i = 1, 2, 3$.

A similar construction gives pseudocyclic d-class association schemes. Such schemes are used by Mathon [52] to construct antipodal distance-regular graphs with diameter three. The resulting graph has $d(v+1)$ vertices and we shall denote it by $d(P+1)$ if P is the original scheme. For $d = 2$, we get the so-called Taylor graphs (cf. [10]).

If v is not a prime power, then only three pseudocyclic 3-class association schemes are known. On 28 vertices Mathon [52] found one, and Hollmann [48] proved that there are precisely two. Furthermore, Hollmann [47] found one on 496 points.

3.4. *The block scheme of designs*

A quasi-symmetric design is a design in which the intersections of two blocks take two sizes x and y. The graph on the blocks of such a design with edges between blocks that intersect in x points is strongly regular, i.e., we have a 2-class association scheme.

Now, consider a block design with the property that the intersections of two blocks take three sizes. Then possibly the structure on the blocks with relations according to the intersection numbers is a 3-class association scheme. Delsarte [27] proved that if the design is a 4-design then we have a 3-class association scheme. Hobart [43] found several examples in her search for the more general coherent configurations of type $(2, 2; 4)$. She mentions the Witt designs 4-(11, 5, 1) and 5-(24, 8, 1) and their residuals, and the inversive planes of even order, that is, the 3-$(2^{2i}+1, 2^i+1, 1)$ designs. Of course, in any 3-design with $\lambda = 1$ the blocks can intersect only in 0, 1 or 2 points, but the corresponding relations do not always form a 3-class association scheme.

Hobart and Bridges [44] also constructed a unique 2-$(15, 5, 4)$ design with block intersections 0, 1 and 2, and it defines the distance-regular graph that is also obtained as the second subconstituent in the Hoffman-Singleton graph (see Section 5.1).

Beker and Haemers [3] proved that if one of the intersection numbers of a 2-(v, k, λ) design equals $k - r + \lambda$, where $r = \lambda(v-1)/(k-1)$ is the replication number of the design, and there are two other intersection numbers, then we have an imprimitive 3-class association scheme, that is generated by $G \otimes J_n$ for some strongly regular graph G (see Section 3.2).

3.5. Distance schemes and coset schemes of codes

Let C be a linear code with $e+1$ nonzero weights w_i. Take as vertices the codewords and let a pair of codewords be in relation R_i if their distance is w_i. It is a consequence of a result by Delsarte [27] (cf. [17]) that if the dual code $C^\perp$ is e-error-correcting, then these relations form an $(e+1)$-class association scheme. This scheme is called the *distance scheme* of the code. Moreover, it has a dual scheme, called the *coset scheme* which is defined on the cosets of $C^\perp$. Two cosets $x + C^\perp$ and $y + C^\perp$ are in relation R_i^* if the minimum weight in the coset $(x-y) + C^\perp$ equals i. Relation R_1^* is the coset graph of $C^\perp$, and is distance-regular.

A small example of a code with three nonzero weights is the binary zero-sum code of length 6, consisting of all 32 words of even weight. Its dual code consist of the zero word and the all-one word and certainly can correct 2 errors. Therefore, we have two dual 3-class association schemes on 32 vertices. The graph (in the distance scheme) defined by distance two in the code is a Taylor graph. The coset graph is the incidence graph of a symmetric 2-$(16, 6, 2)$ design. Larger examples are given by the (duals of the) binary Golay code [23, 12, 7] and its punctured [22, 12, 6] code and doubly punctured [21, 12, 5] code. For all three codes the dual codes have nonzero weights 8, 12 and 16, so these define 3-class association schemes on 2^{11}, 2^{10} and 2^9 vertices, respectively. Also the Kasami codes (which are binary BCH codes with minimum distance 5) give rise to 3-class association schemes (cf. [17]).

3.6. Quadrics in projective geometries

Let Q be a nondegenerate quadric in $PG(3, q)$ with q odd (i.e., the set of isotropic points of the corresponding quadratic form Q). Let V be the set of projective points x such that $Q(x)$ is a nonzero square. Two distinct vertices are related according as the line through these points is a hyperbolic line (a secant, i.e., intersecting Q in two points), an elliptic line (a passant, i.e., disjoint from Q) or a tangent (i.e., intersecting Q in one point). These relations form a 3-class association scheme (cf. [10]). The number of vertices equals $q(q^2 - \varepsilon)/2$, where $\varepsilon = 1$ if Q is hyperbolic, and $\varepsilon = -1$ if Q is elliptic.

For q even, and $n \geq 3$, let Q be a nondegenerate quadric in $PG(n, q)$. Now, let V be the set of nonisotropic points (i.e., the points not on Q) distinct from the nucleus (for n odd there is no nucleus, for n even this is the unique point u such that $Q(u+v) = Q(u) + Q(v)$ for all v). The relations as defined above now form a 3-class association scheme (cf. [10]).

3.7. Merging classes

Sometimes we obtain a new association scheme by *merging* classes in a given association scheme. Merging means that a new relation is obtained as the union of some original relations, and then we say that the corresponding classes are merged. For example, take the 3-class association scheme with vertex set

$$V = \{(x_1, \{\{x_2, x_3, x_4\}, \{x_5, x_6, x_7\}\}) \mid \{x_i, i = 1, \ldots, 7\} = \{1, \ldots, 7\}\}.$$

Two vertices $(x_1, \{\{x_2, x_3, x_4\}, \{x_5, x_6, x_7\}\})$ and $(y_1, \{\{y_2, y_3, y_4\}, \{y_5, y_6, y_7\}\})$ are first associates if $x_1 = y_1$. If $x_1 \neq y_1$, then without loss of generality we may assume that $x_1 \in \{y_2, y_3, y_4\}$ and $y_1 \in \{x_2, x_3, x_4\}$. Now the vertices are second associates if $\{x_2, x_3, x_4\} \cap \{y_2, y_3, y_4\} = \emptyset$, otherwise they are third associates. This scheme was obtained by merging two classes in the 4-class association scheme that arose while letting the symmetric group S_7 act on V^2.

On the other hand, it can occur that merging two classes in a 3-class association scheme gives a 2-class association scheme. Of course, this occurs precisely if the remaining relation defines a strongly regular graph. If all three relations of a 3-class association scheme define strongly regular graphs, then we are in a very special situation. It means that by any merging we always get a new association scheme. After [36] we call schemes with this property *amorphic*. The amorphic 3-class association schemes are precisely the 3-class association schemes that are not generated by one of their relations.

4. Amorphic three-class association schemes

In the special case that all three relations are strongly regular graphs, we show that the parameters of the graphs are either all of Latin square type, or all of negative Latin square type. The proof is due to Higman [42]. The same results can be found in [36], where also all such schemes on at most 25 vertices can be found.

Theorem 4.1 *If all three relations of a* 3*-class association scheme are strongly regular graphs, then they either have parameters* $(n^2, l_i(n-1), n-2+(l_i-1)(l_i-2), l_i(l_i-1))$, $i = 1, 2, 3$ *or* $(n^2, l_i(n+1), -n-2+(l_i+1)(l_i+2), l_i(l_i+1))$, $i = 1, 2, 3$.

Proof: Suppose R_i is a strongly regular graph with degree n_i and eigenvalues n_i, r_i and s_i (we do not assume $r_i > s_i$). Without loss of generality, we may take

$$P = \begin{pmatrix} 1 & n_1 & n_2 & n_3 \\ 1 & r_1 & s_2 & s_3 \\ 1 & s_1 & r_2 & s_3 \\ 1 & s_1 & s_2 & r_3 \end{pmatrix}.$$

Since $PQ = vI$, we see that $1 + r_1 + s_2 + s_3 = 1 + s_1 + r_2 + s_3 = 1 + s_1 + s_2 + r_3 = 0$, and so

$$r_1 - s_1 = r_2 - s_2 = r_3 - s_3.$$

Furthermore, from the orthogonality relations we derive that

$$\frac{s_1}{n_1} = \frac{s_2}{n_2} = \frac{s_3}{n_3},$$

and we find that $P^2 = vI$, so $P = Q$, and so the scheme is self-dual. Now set $u = r_i - s_i$, then we find from the orthogonality relation

$$0 = 1 + \frac{r_1 s_1}{n_1} + \frac{r_2 s_2}{n_2} + \frac{s_3^2}{n_3} = 1 + \frac{s_1}{n_1}(u-1), \qquad \text{so } \frac{n_1}{s_1} = 1 - u.$$

Furthermore, we have that

$$\begin{aligned} \det P &= \det \begin{pmatrix} v & n_1 & n_2 & n_3 \\ 0 & r_1 & s_2 & s_3 \\ 0 & s_1 & r_2 & s_3 \\ 0 & s_1 & s_2 & r_3 \end{pmatrix} = \det \begin{pmatrix} v & n_1 & n_2 & n_3 \\ 0 & u & -u & 0 \\ 0 & 0 & u & -u \\ 0 & s_1 & s_2 & r_3 \end{pmatrix} \\ &= vu^2(s_1 + s_2 + s_3) = -vu^2, \end{aligned}$$

but on the other hand, $P^2 = vI$, so $(\det P)^2 = v^4$, and we find that $v = u^2$. This proves that the parameters of the relations are either all of Latin square type $(n^2, l_i(n-1), n-2+(l_i-1)(l_i-2), l_i(l_i-1))$ if $n = u > 0$, or all of negative Latin square type $(n^2, l_i(n+1), -n-2+(l_i+1)(l_i+2), l_i(l_i+1))$ if $n = -u > 0$. □

A large family of examples is given by the *Latin square schemes* $L_{i,j}(n)$. Suppose we have $m-2$ mutually orthogonal Latin squares, or equivalently an orthogonal array $\text{OA}(n, m)$, that is, an $m \times n^2$ matrix M such that for any two rows a, b we have that $\{(M_{ai}, M_{bi}) \mid i = 1, \ldots, n^2\} = \{(i, j) \mid i, j = 1, \ldots, n\}$. Now take as vertices $1, \ldots, n^2$. Let I_1 and I_2 be two disjoint nonempty subsets of $\{1, \ldots, m\}$ of sizes i and j, respectively. Now two distinct vertices v and w are lth associates if $M_{rv} = M_{rw}$ for some $r \in I_l$, for $l = 1, 2$, otherwise they are third associates.

Many constructions for $\text{OA}(n, m)$ are known (cf. [9]). For n a prime power, there are constructions of $\text{OA}(n, m)$ for every $m \leq n+1$, its maximal value. For $n = 6$, we have $m \leq 3$ (Euler's famous 36 officers problem), and for $n = 10$, currently only constructions for $m \leq 4$ are known. For $n \neq 4$, a Latin square scheme $L_{1,2}(n)$ is equivalent to the algebraic structure called a *loop* (cf. [59]). Two Latin square schemes are isomorphic if and only if the corresponding loops are *isotopic* (cf. [19]). From [20, incl. errata] we find that there are 22 nonisomorphic $L_{1,2}(6)$, 564 nonisomorphic $L_{1,2}(7)$ and 1,676,267 nonisomorphic $L_{1,2}(8)$.

The smallest examples of “schemes of negative Latin square type” are given by the cyclotomic scheme Cycl(16) on 16 vertices (see Section 3.3 for a definition), and another scheme with the same parameters (cf. [36]). Here all three relations are Clebsch graphs. The second feasible parameter set of negative Latin square type is on 49 vertices. Here all relations are strongly regular (49, 16, 3, 6) graphs, but such a graph does not exist, according to Bussemaker et al. [14].

In order to have an amorphic 3-class association scheme, we need a partition of the edges of the complete graph into three strongly regular graphs. On the other hand, this can be proven to be sufficient. This observation (cf. [36]) is very useful in the following examples. Let $q = p^{(e-1)t}$, where p and e are prime $(e > 2)$, p is primitive (mod e) and t is even. It was proven by van Lint and Schrijver [51] that the e-class cyclotomic scheme on the field $GF(q)$ (that is, let α be a primitive element of $GF(q)$, and let two vertices be ith associates if their difference equals α^{ej+i} for some j, for $i = 1, \ldots, e$) has the property that any union of classes gives a strongly regular graph. This implies that any partition of the classes into 3 sets gives a 3-class association scheme. van Lint and Schrijver also found several strongly regular graphs by merging classes in the 8-class cyclotomic scheme on 81 vertices. Using these we find a 3-class association scheme with degrees 30, 30 and 20, and at least two nonisomorphic 3-class association schemes with degrees 40, 20 and 20.

5. Regular graphs with four eigenvalues

A graph G which is one of the relations, say R_1, of a 3-class association scheme is regular with at most four distinct eigenvalues. Any two adjacent vertices have a constant number $\lambda = p^1_{11}$ of common neighbours, and any two nonadjacent vertices have $\mu = p^3_{11}$ or $\mu' = p^2_{11}$ common neighbours. If $\mu = \mu'$, then G is strongly regular, so it has at most three distinct eigenvalues (possibly it is disconnected). If $\mu \neq \mu'$, then G generates the scheme, as the other two relations are determined by the number of common neighbours. Then G must have four eigenvalues (and then G is connected) or be the disjoint union of some strongly regular graphs. If G has four eigenvalues, then the following theorem provides us with a handy tool to check whether it is one of the relations of a 3-class association scheme.

Theorem 5.1 *Let G be a connected regular graph with four distinct eigenvalues. Then G is one of the relations of a* 3*-class association scheme if and only if any two adjacent vertices have a constant number of common neighbours, and the number of common neighbours of any two nonadjacent vertices takes precisely two values.*

Proof: Suppose that G is regular of degree k, any two adjacent vertices in G have λ common neighbours, and that any two nonadjacent vertices have either μ or μ' common neighbours. Note that these requirements must necessarily hold in order for G to be one of the relations of a 3-class association scheme, and that $\mu \neq \mu'$, otherwise G is strongly regular, and so it has only three distinct eigenvalues.

Now let G have adjacency matrix A. To prove sufficiency we shall show that the adjacency algebra $\mathbf{A} = \langle A^2, A, I, J\rangle$, which is closed under ordinary matrix multiplication is also closed under entrywise multiplication $\circ$. Since $M \circ J = M$ for any matrix M, and any matrix $M \in \mathbf{A}$ has constant diagonal, so that $M \circ I \in \mathbf{A}$, we only need to show that $A \circ A$, $A^2 \circ A$ and $A^2 \circ A^2$ are in $\mathbf{A}$. Now $A \circ A = A$, $A^2 \circ A = \lambda A$, and

$$\begin{aligned} A^2 \circ A^2 &= k^2 I + \lambda^2 A + ((\mu + \mu')A^2 - \mu\mu' J) \circ (J - I - A) \\ &= (\mu + \mu')A^2 + (\lambda^2 - \lambda(\mu + \mu') + \mu\mu')A \\ &\quad + (k^2 - k(\mu + \mu') + \mu\mu')I - \mu\mu' J. \end{aligned}$$

So $\mathbf{A}$ is also closed under entrywise multiplication, and so G is one of the relations of a 3-class association scheme. □

If μ or μ' equals 0, then it follows that G is distance-regular with diameter three. We shall use the characterization of Theorem 5.1 in the following examples.

5.1. The second subconstituent of a strongly regular graph

The *second subconstituent* of a graph with respect to some vertex x is the induced graph on the vertices distinct from x, and that are not adjacent to x. For some strongly regular graphs the second subconstituent is a graph that generates a 3-class association scheme.

Suppose G is a strongly regular graph without triangles ($\lambda = 0$), with spectrum $\{[k]^1, [r]^f, [s]^g\}$. Then the second subconstituent $G_2(x)$ of G is a regular graph with spectrum $\{[k+r+s]^1, [r]^{f-k}, [r+s]^{k-1}, [s]^{g-k}\}$ (cf. [23]), so in general it is a connected regular graph with four distinct eigenvalues without triangles. So if the number of common neighbours of two nonadjacent vertices can take at most two values, then we have a 3-class association scheme. This is certainly the case if G is a strongly regular $(v, k, 0, \mu)$ graph with $\mu = 1$ or 2, as we shall see.

If $\mu = 1$ then it follows that in $G_2(x)$ two nonadjacent vertices can have either 0 or 1 common neighbours. For $k > 2$ the graph $G_2(x)$ has four distinct eigenvalues, so then it follows that this graph is distance-regular with diameter three. The distance three relation R_3 is the disjoint union of k cliques of size $k - 1$, which easily follows by computing the eigenvalues of $A_3 = J + (k-2)I - A - A^2$, where A is the adjacency matrix of $G_2(x)$. On the other hand, it follows that any distance-regular graph with such parameters can be constructed in this way, that is, given such a distance-regular graph, we can, using the structure of R_3, construct a strongly regular $(v, k, 0, 1)$ graph that has the distance-regular graph as second subconstituent (Take such a distance-regular graph, and order the cliques of the distance three relation. Extend the distance-regular graph with vertices ∞ and $i = 1, \ldots, k$, and with edges $\{\infty, i\}$ and $\{i, y\}$, y is a vertex of the ith clique, $i = 1, \ldots, k$, then we get a strongly regular $(1 + k^2, k, 0, 1)$ graph). In fact, it now follows from a result by Haemers [38, Corollary 5.4] that any graph with the same spectrum must be constructed in this way. The result by Haemers can also be shown using Corollary 5.6, which we shall prove later (see also [25]).

It is well known (cf. [62]) that strongly regular graphs with parameters $(v, k, 0, 1)$ can only exist for $k = 2, 3, 7$ or 57. For the first three cases there are unique graphs: the 5-cycle C_5, the Petersen graph and the Hoffman-Singleton graph. The case $k = 57$ is still undecided. The second subconstituent of the Petersen graph is the 6-cycle C_6. The more interesting case is the second subconstituent Ho-Si$_2(x)$ of the Hoffman-Singleton graph. It is unique, which follows from the uniqueness of the Hoffman-Singleton graph and the fact that its automorphism group acts transitively on its vertices.

If $\mu = 2$, then in $G_2(x)$ two nonadjacent vertices can have either 1 or 2 common neighbours (They have at least one common neighbour, since in G they cannot have two common neighbours that are both neighbours of x, as these two vertices then would have three common neighbours). For $k > 5$ the graph $G_2(x)$ has four distinct eigenvalues, so then we have a 3-class association scheme. Here we find for relation R_3 (two vertices are third associates if they have one common neighbour in $G_2(x)$) that $A_3 = 2J + (k-4)I - A - A^2$ with spectrum $\{[2k-4]^1, [k-4]^{k-1}, [-2]^{\frac{1}{2}k(k-3)}\}$, which is the spectrum of the triangular graph $T(k)$. Using this, it is also possible to prove that any association scheme with these parameters must be constructed as we did.

Consider the graph of the first relation of an association scheme with such parameters. It has degree $k-2$, no triangles, and any two nonadjacent vertices have either 1 or 2 common neighbours (corresponding to relations R_3 and R_2, respectively). Now the third relation has the spectrum of the triangular graph $T(k)$, and since this graph is uniquely determined by its spectrum (unless $k = 8$, but then there is no feasible parameter set: from the integrality of the multiplicities it follows that $k-1$ is a square), it follows that we can rename the vertices by the pairs $\{i, j\}$, $i, j = 1, \ldots, k$, such that two vertices are not adjacent and have one common neighbour if and only if the corresponding pairs intersect. Now we extend the graph with vertices ∞ and $i = 1, \ldots, k$, and with edges $\{\infty, i\}$ and $\{i, \{i, j\}\}$, $i, j = 1, \ldots, k$. Then it follows that this graph is strongly regular with parameters $(1 + \frac{1}{2}k(k+1), k, 0, 2)$. The only problem in proving this is that i and $\{j, h\}$ with $i \neq j, h$ have two common neighbours. By considering the original association scheme, we see that the number of vertices that are third associates with $\{i, j\}$ and first associates with $\{j, h\}$ equals $p^3_{31} = 2$. But such vertices are of the form $\{i, g\}$, which proves that $\mu = 2$. Thus we have proven the following proposition.

Proposition 5.2 *Let G be a strongly regular graph without triangles, and with $\mu = 1$ or 2, and degree k, with $k > 2$ if $\mu = 1$, and $k > 5$ if $\mu = 2$. Then the second subconstituent of G with respect to any vertex generates a 3-class association scheme. Furthermore, any scheme with the same parameters can be constructed in this way from a strongly regular graph with the same parameters as G.*

If $\mu = 2$, then the only known example for G with $k > 5$ is the Gewirtz graph, and since this graph is uniquely determined by its parameters, and it has a transitive automorphism group, the association scheme generated by its second subconstituent $\text{Gewirtz}_2(x)$ is uniquely determined by its parameters.

Payne [58] found that the second subconstituent of the collinearity graph of a generalized quadrangle with respect to a quasiregular point is a 3-class association scheme (or a strongly regular graph). Together with Hobart [45] he found conditions to embed the association scheme back in a generalized quadrangle. Note that the second subconstituent of a generalized quadrangle with respect to a point p is a regular graph with at most four distinct eigenvalues (cf. [23]). Furthermore, any two adjacent vertices have a constant number of common neighbours. The quasiregularity of the point p now implies that the number of common neighbours of two nonadjacent vertices can take only two values.

5.2. *Hoffman-cocliques in strongly regular graphs*

Let G be a k-regular graph on v vertices with smallest eigenvalue $\lambda_{\min}$. A *Hoffman-coclique* in G is a coclique whose size meets the Hoffman (upper) bound $c = v\lambda_{\min}/(\lambda_{\min} - k)$. If C is a Hoffman-coclique then every vertex not in C is adjacent to $-\lambda_{\min}$ vertices of C. If G is a strongly regular graph with parameters (v, k, λ, μ) and smallest eigenvalue s, then the adjacencies between C and its complement forms the incidence relation of a 2-$(c, -s, \mu)$ design D (which may be degenerate). Furthermore, the induced graph on the complement of C is a regular graph with at most four distinct eigenvalues (cf. [23]). A necessary condition for this graph to be one of the relations of a 3-class association scheme is that the design D has at most three distinct block intersection numbers. If it forms an association scheme then it is the block scheme of D (see Section 3.4).

An example is given by an ovoid in the generalized quadrangle $GQ(4, 4)$. An ovoid is a Hoffman-coclique in the collinearity graph of the generalized quadrangle. Here the corresponding design is an inversive plane, and the induced graph on the complement of the ovoid is the distance three graph of the distance-regular Doro graph.

5.3. *A characterization in terms of the spectrum*

Now suppose that G is a connected regular graph with spectrum $\{[k]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}, [\lambda_3]^{m_3}\}$ that is one of the relations of a 3-class association scheme. The degree $k = n_1$ is its largest eigenvalue, and also λ can be expressed in terms of the spectrum of the graph, since for a connected regular graph with four distinct eigenvalues the number of triangles through a vertex equals $\Delta = \mathrm{Trace}(A^3)/2v$ (cf. [23]), and so

$$\lambda = \frac{2\Delta}{k} = \frac{\mathrm{Trace}(A^3)}{vk} = \frac{1}{vk}\sum_{i=0}^{3} m_i\lambda_i^3.$$

In general, μ and μ' do not follow from the spectrum of G. For example, $GQ(2, 4)\backslash$spread and $H(3, 3)_3$ have the same spectrum, and are both graphs from association schemes, but they have distinct parameters (in fact, the first one is a distance-regular graph and the other is not). But in many cases the parameters of the scheme do follow from the spectrum, as they form the only nonnegative integral solution of the following system of equations.

If for every vertex x, the number of nonadjacent vertices that have μ' common neighbours with x equals n_2, and n_3 is the number of nonadjacent vertices that have μ common neighbours with x, then the parameters satisfy the following equations, which follow from easy counting arguments.

$$\begin{aligned}
n_2 + n_3 &= v - 1 - k,\\
n_2\mu' + n_3\mu &= k(k - 1 - \lambda),\\
n_2\binom{\mu'}{2} + n_3\binom{\mu}{2} &= \Xi - k\binom{\lambda}{2},
\end{aligned}$$

where

$$\Xi = \frac{1}{2}\left(\frac{1}{v}\sum_{i=0}^{3} m_i\lambda_i^4 - 2k^2 + k\right)$$

is the number of quadrangles through a vertex (cf. [23]). Here we allow the quadrangles to have diagonals. Since the number of triangles through an edge is constant, also the number of quadrangles through an edge is constant and equals $\xi = 2\Xi/k$ (cf. [23]). It follows that given the spectrum Σ of the graph and one extra parameter (for example μ), we can compute all other parameters of the association scheme. For n_3 this gives

$$\begin{aligned} n_3 &= h(\Sigma, \mu) \\ &= v - 1 - k - \frac{((v-1-k)\mu - k(k-1-\lambda))^2}{k\xi - k\lambda^2 + k(k-1) + (v-1-k)\mu^2 - 2\mu k(k-1-\lambda)}. \end{aligned}$$

The next theorem characterizes the regular graphs with four eigenvalues that generate a 3-class association scheme, as those graphs for which this number n_3 is what it should be. It is a generalization of a characterization of distance-regular graphs with diameter three among the graphs with four eigenvalues by Haemers and the author [25], and for its proof we refer to the author's thesis [24].

Theorem 5.3 *Let G be a connected regular graph on v vertices with four distinct eigenvalues, say with spectrum $\Sigma = \{[k]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}, [\lambda_3]^{m_3}\}$. Let p be the polynomial given by $p(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3) = x^3 + p_2x^2 + p_1x + p_0$ and let λ be given by $\lambda = (k^3 + m_1\lambda_1^3 + m_2\lambda_2^3 + m_3\lambda_3^3)/vk$. Then G is one of the relations of a 3-class association scheme if and only if there is a μ such that for every vertex x the number of nonadjacent vertices n_3, that have μ common neighbours with x equals*

$$\begin{aligned} g(\Sigma, \mu) = v - 1 - k \\ - \frac{k\left(k - 1 - \lambda - \frac{v-1-k}{k}\mu\right)^2}{(k-\lambda)(\lambda + p_2) - k - p_1 + p_0 - 2\mu(k-1-\lambda) + \frac{v-1-k}{k}\mu^2}. \end{aligned}$$

Obviously, for regular graphs with four eigenvalues that generate a 3-class association scheme, we have that $h(\Sigma, \mu) = g(\Sigma, \mu)$, since they both equal n_3. However, the equality holds for any feasible spectrum Σ of a regular graph with four eigenvalues and any μ. This can be proven using that

$$\begin{aligned} \lambda k + p_2 k + p_0 &= (k^3 + p_2k^2 + p_1k + p_0)/v, \quad \text{and} \\ \frac{1}{v}\sum_{i=0}^{3} m_i\lambda_i^4 + p_2\lambda k + p_1 k &= (k^4 + p_2k^3 + p_1k^2 + p_0k)/v, \end{aligned}$$

which follow by taking traces of the equations $p(A) = p(k)/v\,J$ and $Ap(A) = kp(k)/v\,J$, respectively.

For $\mu = 0$, in which case we have a distance-regular graph, the characterization was already obtained by Haemers and the author [25], as we mentioned before. Together with the previous remarks this gives the following.

Corollary 5.4 *Let G be a connected regular graph with four distinct eigenvalues, with k, λ and ξ (as functions of the spectrum) as before. Then G is a distance-regular graph (with diameter three) if and only if for every vertex the number of vertices k_2 at distance two equals*

$$k_2 = \frac{k(k-1-\lambda)^2}{\xi - \lambda^2 + k - 1}.$$

This settles a question by Haemers [38] on the characterization of distance-regular graphs with diameter three.

Added in proof: Fiol and Garriga [32] recently generalized this to all diameters.

If we have a 3-class association scheme, then $g(\Sigma, \mu)$ must be a nonnegative integer. On the other hand, if we have any graph with spectrum Σ and a μ such that $g(\Sigma, \mu)$ is a nonnegative integer, then for any vertex, we can bound the number of nonadjacent vertices that have μ common neighbours with this vertex. For the proof we again refer to [24].

Proposition 5.5 *With the hypothesis of the previous theorem, if $g(\Sigma, \mu)$ is a nonnegative integer, then $n_3 \leq g(\Sigma, \mu)$.*

Added in proof: It was recently proven by Fiol [31], that the condition, that $g(\Sigma, \mu)$ is a nonnegative integer can be dropped.

In the special case that H is cospectral with one of the relations of a 3-class association scheme, this gives the following.

Corollary 5.6 *Let G be a connected regular graph with four distinct eigenvalues that is one of the relations of a 3-class association scheme, such that the number of vertices nonadjacent to some vertex x, having μ common neighbours with x equals $n_3 > 0$. If H is a graph cospectral with G, then for any vertex x in H, the number of vertices that are not adjacent to x and have μ common neighbours with x is at most n_3, with equality for every vertex if and only if H is one of the relations of a 3-class association scheme with the same parameters as the scheme of G.*

5.4. Hoffman-colorings and systems of linked symmetric designs

Let G be a k-regular graph on v vertices with smallest eigenvalue $\lambda_{\min}$. A *Hoffman-coloring* in G is a partition of the vertices into Hoffman-cocliques, that is, cocliques meeting the Hoffman (upper) bound $c = v\lambda_{\min}/(\lambda_{\min} - k)$. It is well known that if C is a Hoffman-coclique, then every vertex not in C is adjacent to $-\lambda_{\min}$ vertices of C. A *spread* in G is a partition of the vertices into Hoffman-cliques, which is equivalent to a Hoffman-coloring in the complement of G. A *regular coloring* of a graph is a partition of the vertices into cocliques of equal size, say c, such that for some l, every vertex outside a coclique C of the coloring is adjacent to precisely l vertices of C. So regular colorings are generalizations of Hoffman-colorings. A graph with a regular coloring is regular, with degree $k = l(v/c - 1)$, and it also follows that it has an eigenvalue $\lambda = -l$. Now we find that $c = v\lambda/(\lambda - k)$, similar to the size of a coclique in a Hoffman-coloring. In the following we shall say that the regular coloring corresponds to eigenvalue λ.

Suppose G has a regular coloring. Then we define relations R_1 by adjacency in G, R_2 by nonadjacency in G and being in distinct cocliques of the coloring, and R_3 by nonadjacency in G and being in the same coclique of the coloring. It is easy to see that these relations form a 3-class association scheme if G is strongly regular (cf. [40]). A lot of Hoffman-colorings exist in the triangular graphs $T(n)$, for even n, as these (the schemes) are equivalent to *one-factorizations* of K_n. For $n = 4$ and 6, the one-factorizations of K_n are unique, there are six nonisomorphic ones for $n = 8$, and 396 for $n = 10$ (cf. [56]). Dinitz et al. [29] found that there are 526,915,620 nonisomorphic one-factorizations of K_{12}, and they estimated these numbers for $n = 14$, 16, and 18.

If the relations as defined above form an association scheme, then G can have at most four distinct eigenvalues. However, this is not sufficient, as the graph $L_2(3) \otimes J_2$ with spectrum $\{[8]^1, [2]^4, [0]^9, [-4]^4\}$ has a Hoffman-coloring, i.e., 3 disjoint cocliques of size 6, but the corresponding relations do not form an association scheme. It turns out that here the multiplicity of the eigenvalue $\lambda_3 = -4$ is too large. In fact, if the relations do form an association scheme, and we assume that the regular coloring corresponds to the eigenvalue λ_3, then it has eigenmatrix

$$P = \begin{pmatrix} 1 & k & v-k-c & c-1 \\ 1 & \lambda_1 & -\lambda_1 & -1 \\ 1 & \lambda_2 & -\lambda_2 & -1 \\ 1 & \lambda_3 & -\lambda_3 - c & c-1 \end{pmatrix},$$

with multiplicities 1, m_1, m_2, and m_3, respectively. Now it easily follows that $c(m_3 + 1) = v$, so that $m_3 = -k/\lambda_3$. On the other hand, this additional condition on m_3 is sufficient.

Theorem 5.7 *Let G be a connected k-regular graph on v vertices with four distinct eigenvalues. If G has a regular coloring corresponding to eigenvalue, say, λ_3, which has multiplicity $m_3 \leq -k/\lambda_3$, then the corresponding relations form an association scheme.*

Proof: Let A_1 be the adjacency matrix of G (and R_1), and A_3 the adjacency matrix corresponding to the regular coloring (R_3), so $A_3 = I_c \otimes J_{v/c} - I$, where c is the size of the cocliques. Since any vertex outside a coclique C of the coloring is adjacent to $-\lambda_3$ vertices of C, it follows that $A_1(A_3 + I) = -\lambda_3(J - (A_3 + I))$, and so $A_1 A_3 \in \langle I, J, A_1, A_3 \rangle$.

Let λ_1 and λ_2 be the remaining two eigenvalues of G, and let $B = (A_1 - \lambda_1 I)(A_1 - \lambda_2 I)$, then the nonzero eigenvalues of B are $(k - \lambda_1)(k - \lambda_2)$ with multiplicity 1, and $(\lambda_3 - \lambda_1)$

$(\lambda_3 - \lambda_2)$ with multiplicity m_3. If we let $E_0 = v^{-1}J$, and $E_3 = c^{-1}(A_3 + I) - v^{-1}J$, then

$$BE_0 = (k-\lambda_1)(k-\lambda_2)E_0 \quad \text{and} \quad BE_3 = (\lambda_3-\lambda_1)(\lambda_3-\lambda_2)E_3.$$

By use of $\text{rank}(E_0) = 1$, $\text{rank}(E_3) = v/c - 1 \geq m_3$, $E_0^2 = E_0$, $E_3^2 = E_3$, and $E_0E_3 = 0$, it follows that $B - (k-\lambda_1)(k-\lambda_2)E_0 - (\lambda_3-\lambda_1)(\lambda_3-\lambda_2)E_3 = 0$, as all its eigenvalues are zero. So $A_1^2 \in \langle I, J, A_1, A_3\rangle$, and it follows that this algebra is closed under multiplication. Hence we have an association scheme. □

A *system of l linked symmetric* 2-(v, k, λ) *designs* is a collection of sets V_i, $i = 1, \ldots, l+1$ and an incidence relation between each pair of sets forming a symmetric 2-(v, k, λ) design, such that for any i, j, h the number of $x \in V_i$ incident with both $y \in V_j$ and $z \in V_h$ depends only on whether y and z are incident.

Now take as vertices the union of all V_i, and define relations by being in the same subset V_i, being incident in the system of designs or being not incident in the system of designs. This defines a 3-class association scheme. The association scheme of $l-1$ linked designs (note that such a system is contained in the system of l linked designs) can also be considered as the block scheme of the 2-$(v, k, l\lambda)$ design that is obtained by taking as points the elements of the set V_1 and as blocks the elements of the remaining V_i, with the obvious incidence relation.

The only known nontrivial systems of linked designs have parameters $v = 2^{2m}$, $k = 2^{2m-1} - 2^{m-1}$, $\lambda = 2^{2m-2} - 2^{m-1}$, $l \leq 2^{2m-1} - 1$, $m > 1$ (and their complements) (see [18]). Mathon [53] determined all systems of linked 2-(16, 6, 2) designs.

The incidence graph of a system of linked designs is the graph of the relation defined by incidence. If G is a graph with four distinct eigenvalues, that is the incidence graph of a system of linked designs, then G has a regular coloring. The following theorem characterizes these graphs.

Theorem 5.8 *Let G be a connected k-regular graph on v vertices with four distinct eigenvalues. Suppose G has a regular coloring corresponding to, say, λ_3, with cocliques of size c such that the corresponding relations form an association scheme. Let m_1 and m_2 be the multiplicities of the remaining two eigenvalues λ_1 and λ_2, respectively, then $c - 1 \leq \min\{m_1, m_2\}$, with equality if and only if G is the incidence graph of a system of linked symmetric designs.*

Proof: Let $h = 1, 2$, and take

$$\begin{aligned} E &= \frac{v(v-k-c)}{m_h}E_h + \lambda_h J \\ &= (v-k-c+\lambda_h)I + \lambda_h \frac{v-c}{k}A_1 + \left(\lambda_h - \frac{v-k-c}{c-1}\right)A_3, \end{aligned}$$

then $\text{rank}(E) \leq m_h + 1$. Now partition E and A_1 according to the regular coloring, say

$E = (E_{ij})$, $A_1 = (A_{ij})$, $i, j = 0, \ldots, m_3$. Then it follows that if $i \neq j$, then

$$E_{ij} = \lambda_h \frac{v-c}{k} A_{ij} \quad \text{and} \quad E_{ii} = \frac{c(v-k-c)}{c-1} I + \left(\lambda_h - \frac{v-k-c}{c-1}\right) J.$$

Observe that it follows from $m_3 = -k/\lambda_3$ that $m_1\lambda_1 + m_2\lambda_2 = 0$, so $\lambda_h \neq 0$. So E_{ii} is nonsingular, so $c = \text{rank}(E_{ii}) \leq \text{rank}(E)$, which proves the inequality. In case of equality we have $\text{rank}(E_{00}) = \text{rank}(E)$, and then it follows that $E_{ij} = E_{i0} E_{00}^{-1} E_{0j}$. From this we derive that $A_{i0} A_{i0}^T = A_{i0} A_{0i} \in \langle I, J \rangle$, and since A_{i0} has constant row and column sums, we find that A_{i0} is the incidence matrix of a symmetric design. Furthermore, we find that $A_{i0} A_{0j} \in \langle A_{ij}, J \rangle$ for $i \neq j$, which proves that the designs are linked (cf. [18, Theorem 2]). □

For $l = 1$, a system of linked designs is just one design, and we get the incidence graph and corresponding incidence scheme of a symmetric 2-(v, k, λ) design. It is bipartite distance-regular. In fact, it is well known that any bipartite regular graph with four distinct eigenvalues is the incidence graph of a symmetric design (cf. [23]). This result now also follows from Theorem 5.8. In order to determine all nonisomorphic schemes given a certain parameter set of this form, we should mention that two dual (as well as complementary) designs generate the same association scheme. A general reference for designs is [4].

Theorem 5.8 is the analogue of the following theorem by Haemers and Tonchev [40, Theorem 5.1] (here g is the multiplicity of the smallest eigenvalue).

Theorem 5.9 *If G is a primitive strongly regular graph with a Hoffman-coloring, then* $c - 1 \leq g - v/c + 1$, *with equality if and only if G is the incidence graph of a system of linked symmetric designs.*

6. Number theoretic conditions

Using the Hasse-Minkowski invariant of rational symmetric matrices, Bose and Connor [6] derived number theoretic conditions for the existence of so-called regular group divisible designs, which can be seen as extensions of the well-known Bruck-Ryser conditions for symmetric designs. Godsil and Hensel [35] applied the results of Bose and Connor to imprimitive distance-regular graphs with diameter three. In fact, we find that after slight adjustments of the results of Bose and Connor, they are also applicable to imprimitive 3-class association schemes. Also in the primitive case, Hasse-Minkowski theory can be useful, under the condition that one of the relations is a strongly regular graph, preferably one that is determined by its spectrum. If one of the relations is a lattice graph or a triangular graph, we can use results of Coster [21] or Coster and Haemers [22], respectively. These results are obtained by using the Grothendieck group, a technique similar to Hasse-Minkowski theory. The results are in a sense generalizations of [63] and [57], respectively, which are only applicable to designs. A general reference for applications of Hasse-Minkowski theory to designs is [60].

Consider an imprimitive 3-class association scheme, where one of the relations, say R_3, forms the disjoint union of m cliques of size n. Let A be the adjacency matrix of one of the other (nontrivial) relations, say R_1. Suppose that the graph defined by R_1 has degree k, any two adjacent vertices have λ common neighbours, any two nonadjacent vertices that are in the same clique of relation R_3 have μ common neighbours, and any two nonadjacent vertices from distinct cliques have μ' common neighbours. If $\delta = \frac{1}{2}(\mu' - \lambda)$, then A satisfies the equation

$$(A + \delta I)^2 = (k + \delta^2 - \mu)I + \mu' J + (\mu - \mu')I_m \otimes J_n$$

Since $A + \delta I$ is a symmetric rational matrix, it follows that the right-hand side of the equation is rationally congruent to the identity matrix. Note that the matrix has spectrum

$$\left\{[(k+\delta)^2]^1, [(k+\delta)^2 - mn\mu']^{m-1}, [k + \delta^2 - \mu]^{m(n-1)}\right\}$$

Now, the results of Bose and Connor generalize in an obvious way, and we obtain the following conditions. Here the Hilbert norm residue symbol $(a, b)_p$ is defined to be 1 if the equation $ax^2 + by^2 \equiv 1 \pmod{p^r}$ has a solution x, y, for every r, and otherwise it is defined to be -1.

Lemma 6.1 *If an imprimitive* 3-*class association scheme as given above exists*, *then*
(a) *if m is even*, *then $(k+\delta)^2 - mn\mu'$ is a rational square*, *and if $m \equiv 2 \pmod 4$ and n is even then $(k + \delta^2 - \mu, -1)_p = 1$ for all odd primes p.*
(b) *if m is odd*, *and n is even*, *then $k + \delta^2 - \mu$ is a rational square*, *and $((k+\delta)^2 - mn\mu', (-1)^{\frac{1}{2}(m-1)} n\mu')_p = 1$ for all odd primes p.*
(c) *if m and n are both odd*, *then $(k + \delta^2 - \mu, (-1)^{\frac{1}{2}(n-1)} n)_p ((k+\delta)^2 - mn\mu', (-1)^{\frac{1}{2}(m-1)} n\mu')_p = 1$ for all odd primes p.*

Actually, we know a little bit more, if $\mu \neq \mu'$, since then A has four distinct eigenvalues, and then it follows that at least one of $k + \delta^2 - \mu$ and $(k+\delta)^2 - mn\mu'$ is a rational square. Examples of parameter sets with $\mu \neq 0$ that are ruled out by these conditions are $(m, n, k, \lambda, \mu', \mu) = (10, 4, 18, 8, 8, 6)$, $(17, 5, 32, 12, 12, 8)$, $(22, 4, 42, 20, 20, 14)$.

7. Lists of small feasible parameter sets

In order to generate feasible parameter sets for 3-class association schemes we shall classify them into three sets:

1. At least one of the relations is a graph with four distinct eigenvalues;
2. At least one of the relations is the disjoint union of some (connected) strongly regular graphs having the same parameters;
3. All three relations are strongly regular graphs—The amorphic schemes.

These three cases cover all possibilities. Case 2 is degenerate (see Section 3.1). For the remaining two cases we generated all feasible parameter sets on at most 100 vertices. For Case 3 we used Theorem 4.1. For Case 1 we started from an algorithm to generate feasible spectra of graphs with four distinct eigenvalues (actually three algorithms for three types of spectra, cf. [26]; these generate the parameters $v, n_1, P_{11}, P_{21}, P_{31}, m_1, m_2, m_3$ and p^1_{11}), added the parameter $\mu = p^3_{11}$ and (using the results from Section 5.3 and Section 2) computed all other parameters, and checked them for necessary conditions (integrality conditions, Krein conditions, and the absolute bound).

Appendices

In the following appendices all possible parameter sets for 3-class association schemes on at most 100 vertices are listed, except for the more "degenerate" ones, i.e., the schemes generated by the disjoint union of strongly regular graphs, the schemes generated by $G \otimes J_n$, where G is a strongly regular graph, and the rectangular schemes $R(m, n)$, except for the very small schemes $R(2, 2)$, the 6-cycle C_6, and the Cube. The parameters of the more "degenerate" schemes are given below.

The number of vertices of the scheme is denoted by v. If the scheme is primitive, then this number is in **bold face**. The "spectrum" is given by the last three rows of P^T, and so the first row represents the spectrum of the first relation, and similarly for the second and third relation. In the first row of the spectrum, the multiplicities of the (eigenvalues of the) scheme are denoted in superscript. In Appendices A and D the multiplicities are omitted, since there the schemes are self-dual, so the multiplicities are equal to the degrees. L_1, L_2 and L_3 here denote the reduced intersection matrices, that is, the first row and column are omitted. # denotes the number of nonisomorphic schemes of that type. At the end of the line remarks are made. The more "degenerate" schemes would read as follows.

v	spectrum	L_1	L_2	L_3	#	
mn	$\{(m-1)(n-1),\ 1,\ 1-m,\ 1-n\}$	$(m-2)(n-2)$ $n-2$ $m-2$	$n-2$ 0 1	$m-2$ 1 0	1	$R(m,n)$
	$\{n-1,\ -1,\ -1,\ n-1\}$	$(m-1)(n-2)$ 0 $m-1$	0 $n-2$ 0	$m-1$ 0 0		If m xor n equals 2
	$\{m-1,\ -1,\ m-1,\ -1\}$	$(m-2)(n-1)$ $n-1$ 0	$n-1$ 0 0	0 0 $m-2$		then DRG, Q-polynomial
wn	$\{k,\ k^{n-1},\ r^{fn},\ s^{gn}\}$	λ $k-1-\lambda$ 0	$k-1-\lambda$ $w-2k+\lambda$ 0	0 0 $(n-1)w$	?	disjoint union of n
	$\{w-1-k,\ w-1-k,\ -1-r,\ -1-s\}$	μ $k-\mu$ 0	$k-\mu$ $w-2k-2+\mu$ 0	0 0 $(n-1)w$		SRG(w,k,λ,μ)
	$\{(n-1)w,\ -w,\ 0,\ 0\}$	0 0 k	0 0 $w-1-k$	k $w-1-k$ $(n-2)w$		
wn	$\{nk,\ nr^{f},\ 0^{(n-1)w},\ ns^{g}\}$	$n\lambda$ $n-1$ $n(k-1-\lambda)$	$n-1$ 0 0	$n(k-1-\lambda)$ 0 $n(w-2k+\lambda)$	?	SRG$(w,k,\lambda,\mu)\otimes J_n$
	$\{n-1,\ n-1,\ -1,\ n-1\}$	nk 0 0	0 $n-2$ 0	0 0 $n(w-1-k)$		
	$\{n(w-1-k),\ n(-1-r),\ 0,\ n(-1-s)\}$	$n\mu$ 0 $n(k-\mu)$	0 0 $n-1$	$n(k-\mu)$ $n-1$ $n(w-2k-2+\mu)$		

Appendix A

The amorphic schemes—all relations are strongly regular; excluded here are the rectangular schemes $R(m, m)$, except $R(2, 2)$.

v	spectrum	L_1	L_2	L_3	#	
4	{1, 1, -1, -1}	0 0 0	0 0 1	0 1 0	1	$L_{1,1}(2) \simeq R(2,2)$
	{1, -1, 1, -1}	0 0 1	0 0 0	1 0 0		
	{1, -1, -1, 1}	0 1 0	1 0 0	0 0 0		
16	{6, 2, -2, -2}	2 2 1	2 2 2	1 2 0	4	$L_{1,2}(4)$ [36]
	{6, -2, 2, -2}	2 2 2	2 2 1	2 1 0		
	{3, -1, -1, 3}	2 4 0	4 2 0	0 0 2		
16	{5, -3, 1, 1}	0 2 2	2 2 1	2 1 2	2	Cycl(16) [36]
	{5, 1, -3, 1}	2 2 1	2 0 2	1 2 2		
	{5, 1, 1, -3}	2 1 2	1 2 2	2 2 0		

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v	spectrum	L_1			L_2			L_3			#	
25	{12, 2, -3, -3}	5	4	2	4	2	2	2	2	0	2	$L_{1,2}(5)$
	{ 8, -2, 3, -2}	6	3	3	3	3	1	3	1	0		
	{ 4, -1, -1, 4}	6	6	0	6	2	0	0	0	3		
25	{ 8, 3, -2, -2}	3	2	2	2	2	4	2	4	2	2	$L_{2,2}(5)$, Cycl(25) [36]
	{ 8, -2, 3, -2}	2	2	4	2	3	2	4	2	2		
	{ 8, -2, -2, 3}	2	4	2	4	2	2	2	2	3		
36	{20, 2, -4, -4}	10	6	3	6	2	2	3	2	0	22	$L_{1,2}(6)$ [19, 59]
	{10, -2, 4, -2}	12	4	4	4	4	1	4	1	0		
	{ 5, -1, -1, 5}	12	8	0	8	2	0	0	0	4		
36	{15, 3, -3, -3}	6	6	2	6	6	3	2	3	0	0	SRG(36,20,10,12)\spread
	{15, -3, 3, -3}	6	6	3	6	6	2	3	2	0		Bussemaker, Haemers, Spence
	{ 5, -1, -1, 5}	6	9	0	9	6	0	0	0	4		
36	{15, 3, -3, -3}	6	4	4	4	2	4	4	4	2	0	$L_{2,2}(6)$
	{10, -2, 4, -2}	6	3	6	3	4	2	6	2	2		
	{10, -2, -2, 4}	6	6	3	6	2	2	3	2	4		
49	{30, 2, -5, -5}	17	8	4	8	2	2	4	2	0	564	$L_{1,2}(7)$ [20, errata]
	{12, -2, 5, -2}	20	5	5	5	5	1	5	1	0		
	{ 6, -1, -1, 6}	20	10	0	10	2	0	0	0	5		
49	{24, 3, -4, -4}	11	9	3	9	6	3	3	3	0	≥1	$L_{1,3}(7)$
	{18, -3, 4, -3}	12	8	4	8	7	2	4	2	0		
	{ 6, -1, -1, 6}	12	12	0	12	6	0	0	0	5		
49	{24, 3, -4, -4}	11	6	6	6	2	4	6	4	2	≥1	$L_{2,2}(7)$
	{12, -2, 5, -2}	12	4	8	4	5	2	8	2	2		
	{12, -2, -2, 5}	12	8	4	8	2	2	4	2	5		
49	{18, 4, -3, -3}	7	6	4	6	6	6	4	6	2	≥1	$L_{2,3}(7)$
	{18, -3, 4, -3}	6	6	6	6	7	4	6	4	2		
	{12, -2, -2, 5}	6	9	3	9	6	3	3	3	5		
49	{16, -5, 2, 2}	3	6	6	6	6	4	6	4	6	0	SRG does not exist [14]
	{16, 2, -5, 2}	6	6	4	6	3	6	4	6	6		
	{16, 2, 2, -5}	6	4	6	4	6	6	6	6	3		
64	{42, 2, -6, -6}	26	10	5	10	2	2	5	2	0	1676267	$L_{1,2}(8)$ [20]
	{14, -2, 6, -2}	30	6	6	6	6	1	6	1	0		
	{ 7, -1, -1, 7}	30	12	0	12	2	0	0	0	6		
64	{35, 3, -5, -5}	18	12	4	12	6	3	4	3	0	≥1	$L_{1,3}(8)$
	{21, -3, 5, -3}	20	10	5	10	8	2	5	2	0		
	{ 7, -1, -1, 7}	20	15	0	15	6	0	0	0	6		
64	{28, 4, -4, -4}	12	12	3	12	12	4	3	4	0	≥1	$L_{1,4}(8)$
	{28, -4, 4, -4}	12	12	4	12	12	3	4	3	0		
	{ 7, -1, -1, 7}	12	16	0	16	12	0	0	0	6		
64	{35, 3, -5, -5}	18	8	8	8	2	4	8	4	2	≥1	$L_{2,2}(8)$
	{14, -2, 6, -2}	20	5	10	5	6	2	10	2	2		
	{14, -2, -2, 6}	20	10	5	10	2	2	5	2	6		

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v	spectrum	L_1	L_2	L_3	#	
64	{28, 4, -4, -4}	12 9 6	9 6 6	6 6 2	≥1	$L_{2,3}(8)$
	{21, -3, 5, -3}	12 8 8	8 8 4	8 4 2		
	{14, -2, -2, 6}	12 12 4	12 6 3	4 3 6		
64	{27, -5, 3, 3}	10 8 8	8 6 4	8 4 6	≥1	de Caen, Van Dam
	{18, 2, -6, 2}	12 9 6	9 2 6	6 6 6		
	{18, 2, 2, -6}	12 6 9	6 6 6	9 6 2		
64	{21, 5, -3, -3}	8 6 6	6 6 9	6 9 6	≥1	$L_{3,3}(8)$, Cycl(64)
	{21, -3, 5, -3}	6 6 9	6 8 6	9 6 6		
	{21, -3, -3, 5}	6 9 6	9 6 6	6 6 8		
81	{56, 2, -7, -7}	37 12 6	12 2 2	6 2 0	≥1	$L_{1,2}(9)$
	{16, -2, 7, -2}	42 7 7	7 7 1	7 1 0		
	{ 8, -1, -1, 8}	42 14 0	14 2 0	0 0 7		
81	{48, 3, -6, -6}	27 15 5	15 6 3	5 3 0	≥1	$L_{1,3}(9)$
	{24, -3, 6, -3}	30 12 6	12 9 2	6 2 0		
	{ 8, -1, -1, 8}	30 18 0	18 6 0	0 0 7		
81	{40, 4, -5, -5}	19 16 4	16 12 4	4 4 0	≥1	$L_{1,4}(9)$
	{32, -4, 5, -4}	20 15 5	15 13 3	5 3 0		
	{ 8, -1, -1, 8}	20 20 0	20 12 0	0 0 7		
81	{48, 3, -6, -6}	27 10 10	10 2 4	10 4 2	≥1	$L_{2,2}(9)$
	{16, -2, 7, -2}	30 6 12	6 7 2	12 2 2		
	{16, -2, -2, 7}	30 12 6	12 2 2	6 2 7		
81	{40, 4, -5, -5}	19 12 8	12 6 6	8 6 2	≥1	$L_{2,3}(9)$
	{24, -3, 6, -3}	20 10 10	10 9 4	10 4 2		
	{16, -2, -2, 7}	20 15 5	15 6 3	5 3 7		
81	{32, 5, -4, -4}	13 12 6	12 12 8	6 8 2	≥1	$L_{2,4}(9)$
	{32, -4, 5, -4}	12 12 8	12 13 6	8 6 2		
	{16, -2, -2, 7}	12 16 4	16 12 4	4 4 7		
81	{40, -5, 4, 4}	19 10 10	10 6 4	10 4 6	≥2	Van Lint-Schrijver
	{20, 2, -7, 2}	20 12 8	12 1 6	8 6 6		
	{20, 2, 2, -7}	20 8 12	8 6 6	12 6 1		
81	{30, -6, 3, 3}	9 12 8	12 12 6	8 6 6	≥1	Van Lint-Schrijver
	{30, 3, -6, 3}	12 12 6	12 9 8	6 8 6		
	{20, 2, 2, -7}	12 9 9	9 12 9	9 9 1		
81	{32, 5, -4, -4}	13 9 9	9 6 9	9 9 6	≥1	$L_{3,3}(9)$
	{24, -3, 6, -3}	12 8 12	8 9 6	12 6 6		
	{24, -3, -3, 6}	12 12 8	12 6 6	8 6 9		
100	{72, 2, -8, -8}	50 14 7	14 2 2	7 2 0	≥1	$L_{1,2}(10)$
	{18, -2, 8, -2}	56 8 8	8 8 1	8 1 0		
	{ 9, -1, -1, 9}	56 16 0	16 2 0	0 0 8		
100	{63, 3, -7, -7}	38 18 6	18 6 3	6 3 0	≥1	$L_{1,3}(10)$
	{27, -3, 7, -3}	42 14 7	14 10 2	7 2 0		
	{ 9, -1, -1, 9}	42 21 0	21 6 0	0 0 8		

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v	spectrum	L_1	L_2	L_3	#	
100	{54, 4, -6, -6}	28 20 5	20 12 4	5 4 0	?	$L_{1,4}(10)$, SRG(100,63,38,42)\spread
	{36, -4, 6, -4}	30 18 6	18 14 3	6 3 0		
	{ 9, -1, -1, 9}	30 24 0	24 12 0	0 0 8		
100	{45, 5, -5, -5}	20 20 4	20 20 5	4 5 0	?	$L_{1,5}(10)$, SRG(100,54,28,30)\spread
	{45, -5, 5, -5}	20 20 5	20 20 4	5 4 0		
	{ 9, -1, -1, 9}	20 25 0	25 20 0	0 0 8		
100	{63, 3, -7, -7}	38 12 12	12 2 4	12 4 2	≥1	$L_{2,2}(10)$
	{18, -2, 8, -2}	42 7 14	7 8 2	14 2 2		
	{18, -2, -2, 8}	42 14 7	14 2 2	7 2 8		
100	{54, 4, -6, -6}	28 15 10	15 6 6	10 6 2	?	$L_{2,3}(10)$
	{27, -3, 7, -3}	30 12 12	12 10 4	12 4 2		
	{18, -2, -2, 8}	30 18 6	18 6 3	6 3 8		
100	{45, 5, -5, -5}	20 16 8	16 12 8	8 8 2	?	$L_{2,4}(10)$
	{36, -4, 6, -4}	20 15 10	15 14 6	10 6 2		
	{18, -2, -2, 8}	20 20 5	20 12 4	5 4 8		
100	{55, -5, 5, 5}	30 12 12	12 6 4	12 4 6	?	
	{22, 2, -8, 2}	30 15 10	15 0 6	10 6 6		
	{22, 2, 2, -8}	30 10 15	10 6 6	15 6 0		
100	{44, -6, 4, 4}	18 15 10	15 12 6	10 6 6	?	
	{33, 3, -7, 3}	20 16 8	16 8 8	8 8 6		
	{22, 2, 2, -8}	20 12 12	12 12 9	12 9 0		
100	{45, 5, -5, -5}	20 12 12	12 6 9	12 9 6	?	$L_{3,3}(10)$
	{27, -3, 7, -3}	20 10 15	10 10 6	15 6 6		
	{27, -3, -3, 7}	20 15 10	15 6 6	10 6 10		
100	{36, 6, -4, -4}	14 12 9	12 12 12	9 12 6	?	$L_{3,4}(10)$
	{36, -4, 6, -4}	12 12 12	12 14 9	12 9 6		
	{27, -3, -3, 7}	12 16 8	16 12 8	8 8 10		
100	{33, -7, 3, 3}	8 12 12	12 12 9	12 9 12	?	
	{33, 3, -7, 3}	12 12 9	12 8 12	9 12 12		
	{33, 3, 3, -7}	12 9 12	9 12 12	12 12 8		

Appendix B

Four integral eigenvalues; excluded here are association schemes generated by SRG$\otimes J_n$, and the rectangular schemes $R(m, n)$, except the 6-cycle C_6 and the Cube.

v	spectrum	L_1	L_2	L_3	#	
6	{2, 1^2, -1^2, -2^1}	0 1 0	1 0 1	0 1 0	1	$C_6 \simeq R(3,2)$
	{2, -1, -1, 2}	1 0 1	0 1 0	1 0 0		DRG
	{1, -1, 1, -1}	0 2 0	2 0 0	0 0 0		Q-123

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v	spectrum	L_1	L_2	L_3	#	
8	$\{3, 1^3, -1^3, -3^1\}$	0 2 0	2 0 1	0 1 0	1	Cube $\simeq R(4,2)$
	$\{3, -1, -1, 3\}$	2 0 1	0 2 0	1 0 0		DRG
	$\{1, -1, 1, -1\}$	0 3 0	3 0 0	0 0 0		Q-123
15	$\{4, 2^5, -1^4, -2^5\}$	1 2 0	2 4 2	0 2 0	1	L(Petersen)
	$\{8, -2, -2, 2\}$	1 2 1	2 4 1	1 1 0		DRG, R_2 SRG
	$\{2, -1, 2, -1\}$	0 4 0	4 4 0	0 0 1		
20	$\{9, 3^5, -1^9, -3^5\}$	4 4 0	4 4 1	0 1 0	1	$J(6,3)$
	$\{9, -3, -1, 3\}$	4 4 1	4 4 0	1 0 0		$R_1 \simeq R_2$ DRG
	$\{1, -1, 1, -1\}$	0 9 0	9 0 0	0 0 0		Q-123, Q-321
27	$\{6, 3^6, 0^{12}, -3^8\}$	1 4 0	4 4 4	0 4 4	1	$H(3,3)$
	$\{12, 0, -3, 3\}$	2 2 2	2 5 4	2 4 2		DRG
	$\{8, -4, 2, -1\}$	0 3 3	3 6 3	3 3 1		Q-123
27	$\{8, 2^{12}, -1^8, -4^6\}$	1 6 0	6 8 2	0 2 0	2	GQ(2,4)\spread
	$\{16, -2, -2, 4\}$	3 4 1	4 10 1	1 1 0		R_1 DRG, R_2 SRG
	$\{2, -1, 2, -1\}$	0 8 0	8 8 0	0 0 1		
28	$\{12, 2^{14}, -2^6, -4^7\}$	4 6 1	6 4 2	1 2 0	56	$T(8)^c$\spread, Chang^c\spread [68]
	$\{12, -2, -2, 4\}$	6 4 2	4 6 1	2 1 0		R_2 SRG
	$\{3, -1, 3, -1\}$	4 8 0	8 4 0	0 0 2		
30	$\{7, 2^{14}, -2^{14}, -7^1\}$	0 6 0	6 0 8	0 8 0	4	$IG(15,7,3)$
	$\{14, -1, -1, 14\}$	3 0 4	0 13 0	4 0 4		R_1 and R_3 DRG
	$\{8, -2, 2, -8\}$	0 7 0	7 0 7	0 7 0		Q-123, Q-213
32	$\{6, 2^{15}, -2^{15}, -6^1\}$	0 5 0	5 0 10	0 10 0	3	$IG(16,6,2)$
	$\{15, -1, -1, 15\}$	2 0 4	0 14 0	4 0 6		R_1 and R_3 DRG
	$\{10, -2, 2, -10\}$	0 6 0	6 0 9	0 9 0		Q-123, Q-213
32	$\{15, 3^{10}, -1^{15}, -5^6\}$	6 8 0	8 6 1	0 1 0	1	2(GQ(2,2)+1)
	$\{15, -3, -1, 5\}$	8 6 1	6 8 0	1 0 0		R_1 and R_2 DRG
	$\{1, -1, 1, -1\}$	0 15 0	15 0 0	0 0 0		Q-123, Q-321
35	$\{12, 5^6, 0^{14}, -3^{14}\}$	5 6 0	6 9 3	0 3 1	1	$J(7,3)$
	$\{18, -3, -3, 3\}$	4 6 2	6 9 2	2 2 0		R_1 and R_3 DRG, R_2 SRG
	$\{4, -3, 2, -1\}$	0 9 3	9 9 0	3 0 0		Q-123
35	$\{12, 3^{14}, -2^6, -3^{14}\}$	4 6 1	6 9 3	1 3 0	≥1	SRG(35,16,6,8)\spread [40]
	$\{18, -3, -3, 3\}$	4 6 2	6 9 2	2 2 0		R_2 SRG
	$\{4, -1, 4, -1\}$	3 9 0	9 9 0	0 0 3		
35	$\{12, 4^{10}, -2^{20}, -3^4\}$	5 2 4	2 0 4	4 4 8	0	SRG(35,18,9,9)\spread [40]
	$\{6, -1, -1, 6\}$	4 0 8	0 5 0	8 0 8		R_3 SRG
	$\{16, -4, 2, -4\}$	3 3 6	3 0 3	6 3 6		
36	$\{5, 2^{16}, -1^{10}, -3^9\}$	0 4 0	4 8 8	0 8 2	1	Sylvester, block scheme of residual of 4-(11,5,1)
	$\{20, -1, -4, 4\}$	1 2 2	2 11 6	2 6 2		DRG, $R_3 \simeq L_2(6)$
	$\{10, -2, 4, -2\}$	0 4 1	4 12 4	1 4 4		

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v	spectrum	L_1	L_2	L_3	#	
40	{ 9, 3^{15}, -1^{9}, -3^{15}}	2 6 0	6 18 3	0 3 0	≥1	GQ(3,3)\spread
	{27, -3, -3, 3}	2 6 1	6 18 2	1 2 0		DRG, R_2 SRG
	{ 3, -1, 3, -1}	0 9 0	9 18 0	0 0 2		
40	{18, 4^{12}, -2^{24}, -6^{3}}	8 5 4	5 0 4	4 4 4	0	SRG(40,27,18,18) \spread [40]
	{ 9, -1, -1, 9}	10 0 8	0 8 0	8 0 4		R_3 SRG
	{12, -4, 2, -4}	6 6 6	6 0 3	6 3 2		
42	{ 5, 2^{20}, -2^{20}, -5^{1}}	0 4 0	4 0 16	0 16 0	1	*IG*(21,5,1)
	{20, -1, -1, 20}	1 0 4	0 19 0	4 0 12		R_1 and R_3 DRG
	{16, -2, 2, -16}	0 5 0	5 0 15	0 15 0		*Q*-123, *Q*-213
42	{ 6, 2^{21}, -1^{6}, -3^{14}}	0 5 0	5 20 5	0 5 0	1	Ho-Si_2(*x*), block scheme of 2-(15,5,4)
	{30, -2, -5, 3}	1 4 1	4 21 4	1 4 0		DRG
	{ 5, -1, 5, -1}	0 6 0	6 24 0	0 0 4		
45	{ 8, 2^{25}, -2^{9}, -4^{10}}	0 5 2	5 5 10	2 10 4	1	$Gewirtz_2$(*x*)
	{20, -1, -5, 5}	2 2 4	2 9 8	4 8 4		R_3 SRG
	{16, -2, 6, -2}	1 5 2	5 10 5	2 5 8		
45	{ 8, 4^{12}, -1^{8}, -2^{24}}	3 4 0	4 24 4	0 4 0	0	SRG(45,12,3,3)\spread [10, p. 152]
	{32, -4, -4, 2}	1 6 1	6 22 3	1 3 0		DRG, R_2 SRG
	{ 4, -1, 4, -1}	0 8 0	8 24 0	0 0 3		
45	{16, 4^{15}, -2^{20}, -4^{9}}	6 6 3	6 4 6	3 6 3	?	
	{16, -2, -2, 6}	6 4 6	4 8 3	6 3 3		$R_2 \simeq T(10)$, R_3 SRG
	{12, -3, 3, -3}	4 8 4	8 4 4	4 4 3		
45	{24, 3^{20}, -3^{20}, -6^{4}}	12 5 6	5 0 3	6 3 3	≥2	$GQ(4,2)^c$\spread
	{ 8, -1, -1, 8}	15 0 9	0 7 0	9 0 3		R_3 SRG
	{12, -3, 3, -3}	12 6 6	6 0 2	6 2 3		
45	{24, 2^{27}, -3^{8}, -6^{9}}	11 10 2	10 4 2	2 2 0	396	$T(10)^c$\spread [56]
	{16, -2, -2, 6}	15 6 3	6 8 1	3 1 0		R_2 SRG
	{ 4, -1, 4, -1}	12 12 0	12 4 0	0 0 3		
48	{12, 2^{30}, -4^{15}, -6^{2}}	1 5 5	5 0 10	5 10 5	3	system of 2 linked 2-(16,6,2) designs [53]
	{15, -1, -1, 15}	4 0 8	0 14 0	8 0 12		
	{20, -2, 4, -10}	3 6 3	6 0 9	3 9 7		*Q*-213
48	{15, 5^{12}, -1^{15}, -3^{20}}	6 8 0	8 20 2	0 2 0	0	[35, Lemma 3.5]
	{30, -5, -2, 3}	4 10 1	10 18 1	1 1 0		DRG
	{ 2, -1, 2, -1}	0 15 0	15 15 0	0 0 1		
51	{16, 4^{17}, -1^{16}, -4^{17}}	5 10 0	10 20 2	0 2 0	≥1	3(Cycl(16)+1)
	{32, -4, -2, 4}	5 10 1	10 20 1	1 1 0		DRG
	{ 2, -1, 2, -1}	0 16 0	16 16 0	0 0 1		
52	{25, 5^{13}, -1^{25}, -5^{13}}	12 12 0	12 12 1	0 1 0	4	Taylor [15, 67]
	{25, -5, -1, 5}	12 12 1	12 12 0	1 0 0		R_1 and R_2 DRG
	{ 1, -1, 1, -1}	0 25 0	25 0 0	0 0 0		*Q*-123, *Q*-321
56	{15, 7^{7}, 1^{20}, -3^{28}}	6 8 0	8 16 6	0 6 4	1	*J*(8,3)
	{30, -2, -5, 3}	4 8 3	8 15 6	3 6 1		DRG
	{10, -6, 3, -1}	0 9 6	9 18 3	6 3 0		*Q*-123

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v	spectrum	L_1	L_2	L_3	#	
56	$\{27, 3^{21}, -1^{27}, -9^{7}\}$	10 16 0	16 10 1	0 1 0	1	2(Schlafli+1)
	$\{27, -3, -1, 9\}$	16 10 1	10 16 0	1 0 0		R_1 and R_2 DRG
	$\{1, -1, 1, -1\}$	0 27 0	27 0 0	0 0 0		Q-123, Q-321
60	$\{15, 3^{25}, 0^{16}, -5^{18}\}$	2 8 4	8 8 8	4 8 8	≥1	hyperbolic quadric in $PG(3,5)$
	$\{24, 0, -6, 4\}$	5 5 5	5 8 10	5 10 5		
	$\{20, -4, 5, 0\}$	3 6 6	6 12 6	6 6 7		
60	$\{21, 3^{32}, -4^{24}, -7^{3}\}$	6 6 8	6 0 8	8 8 8	?	
	$\{14, -1, -1, 14\}$	9 0 12	0 13 0	12 0 12		
	$\{24, -3, 4, -8\}$	7 7 7	7 0 7	7 7 9		
63	$\{6, 3^{21}, -1^{27}, -3^{14}\}$	1 4 0	4 4 16	0 16 16	2	GH(2,2)
	$\{24, 0, -4, 6\}$	1 1 4	1 10 12	4 12 16		DRG, R_3 SRG
	$\{32, -4, 4, -4\}$	0 3 3	3 9 12	3 12 16		
63	$\{24, 4^{27}, -3^{8}, -4^{27}\}$	9 12 2	12 16 4	2 4 0	≥1	SRG(63,30,13,15) \spread [40]
	$\{32, -4, -4, 4\}$	9 12 3	12 16 3	3 3 0		R_2 SRG
	$\{6, -1, 6, -1\}$	8 16 0	16 16 0	0 0 5		
63	$\{24, 5^{21}, -3^{35}, -4^{6}\}$	10 3 10	3 0 5	10 5 15	?	SRG(63,32,16,16) \spread
	$\{8, -1, -1, 8\}$	9 0 15	0 7 0	15 0 15		R_3 SRG
	$\{30, -5, 3, -5\}$	8 4 12	4 0 4	12 4 13		
64	$\{7, 3^{21}, -1^{35}, -5^{7}\}$	0 6 0	6 0 15	0 15 20	1	Folded 7-cube
	$\{21, 1, -3, 9\}$	2 0 5	0 10 10	5 10 20		R_1 and R_2 DRG, R_3 SRG
	$\{35, -5, 3, -5\}$	0 3 4	3 6 12	4 12 18		Q-123, Q-312
64	$\{9, 5^{9}, 1^{27}, -3^{27}\}$	2 6 0	6 12 9	0 9 18	2	$H(3,4)$, Doob
	$\{27, 3, -5, 3\}$	2 4 3	4 10 12	3 12 12		DRG, R_2 SRG
	$\{27, -9, 3, -1\}$	0 3 6	3 12 12	6 12 8		Q-123
64	$\{14, 2^{42}, -2^{7}, -6^{14}\}$	0 12 1	12 24 6	1 6 0	≥1	de Caen, Van Dam
	$\{42, -2, -6, 6\}$	4 8 2	8 28 5	2 5 0		
	$\{7, -1, 7, -1\}$	2 12 0	12 30 0	0 0 6		
64	$\{15, 3^{30}, -1^{15}, -5^{18}\}$	2 12 0	12 30 3	0 3 0	≥5	SRG(64,18,2,6) \spread [50]
	$\{45, -3, -3, 5\}$	4 10 1	10 32 2	1 2 0		DRG, R_2 SRG
	$\{3, -1, 3, -1\}$	0 15 0	15 30 0	0 0 2		
64	$\{18, 6^{15}, -2^{45}, -6^{3}\}$	7 5 5	5 0 10	5 10 15	0	linked designs
	$\{15, -1, -1, 15\}$	6 0 12	0 14 0	12 0 18		
	$\{30, -6, 2, -10\}$	3 6 9	6 0 9	9 9 11		Q-123
64	$\{30, 6^{15}, -2^{45}, -10^{3}\}$	14 9 6	9 0 6	6 6 6	12	SRG(64,45,32,30) \spread, 3 linked 2-(16,6,2) designs [53]
	$\{15, -1, -1, 15\}$	18 0 12	0 14 0	12 0 6		R_3 SRG
	$\{18, -6, 2, -6\}$	10 10 10	10 0 5	10 5 2		Q-123
65	$\{10, 5^{13}, 0^{26}, -3^{25}\}$	3 6 0	6 12 12	0 12 12	1	Locally Petersen
	$\{30, 0, -5, 4\}$	2 4 4	4 13 12	4 12 8		DRG
	$\{24, -6, 4, -2\}$	0 5 5	5 15 10	5 10 8		

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v	spectrum	L_1	L_2	L_3	#	
66	$\{15, 2^{44}, -3^{11}, -7^{10}\}$	0 10 4	10 8 12	4 12 4	≥1	block scheme 4-(11,5,1) design
	{30, -1, -6, 8}	5 4 6	4 15 10	6 10 4		$R_3 \simeq T(12)$
	{20, -2, 8, -2}	3 9 3	9 15 6	3 6 10		Q-312
66	$\{20, 2^{44}, -2^{10}, -8^{11}\}$	2 16 1	16 20 4	1 4 0	?	
	{40, -2, -4, 8}	8 10 2	10 26 3	2 3 0		
	{ 5, -1, 5, -1}	4 16 0	16 24 0	0 0 4		
66	$\{40, 2^{44}, -4^{10}, -8^{11}\}$	22 14 3	14 4 2	3 2 0	526915620	$T(12)^c$\spread [29]
	{20, -2, -2, 8}	28 8 4	8 10 1	4 1 0		R_2 SRG
	{ 5, -1, 5, -1}	24 16 0	16 4 0	0 0 4		
68	$\{12, 4^{17}, 0^{34}, -5^{16}\}$	1 10 0	10 20 10	0 10 5	1	Doro, block scheme of 3-(17,5,1)
	{40, 0, -4, 6}	3 6 3	6 24 9	3 9 3		DRG
	{15, -5, 3, -2}	0 8 4	8 24 8	4 8 2		
70	$\{17, 3^{34}, -3^{34}, -17^{1}\}$	0 16 0	16 0 18	0 18 0	≥53387	$IG(35,17,8)$ [68]
	{34, -1, -1, 34}	8 0 9	0 33 0	9 0 9		R_1 and R_3 DRG
	{18, -3, 3, -18}	0 17 0	17 0 17	0 17 0		Q-123, Q-213
70	$\{18, 2^{49}, -3^{6}, -7^{14}\}$	1 14 2	14 21 7	2 7 0	≥1	Merging example
	{42, -2, -7, 7}	6 9 3	9 26 6	3 6 0		
	{ 9, -1, 9, -1}	4 14 0	14 28 0	0 0 8		
70	$\{18, 7^{14}, -2^{49}, -3^{6}\}$	8 2 7	2 0 7	7 7 28	?	SRG(70,27,12,9) \spread
	{ 9, -1, -1, 9}	4 0 14	0 8 0	14 0 28		R_3 SRG
	{42, -7, 2, -7}	3 3 12	3 0 6	12 6 23		
70	$\{36, 3^{40}, -4^{9}, -6^{20}\}$	17 15 3	15 9 3	3 3 0	≥1	SRG(70,42,23,28) \spread [40]
	{27, -3, -3, 6}	20 12 4	12 12 2	4 2 0		R_2 SRG
	{ 6, -1, 6, -1}	18 18 0	18 9 0	0 0 5		
72	$\{15, 3^{35}, -3^{35}, -15^{1}\}$	0 14 0	14 0 21	0 21 0	≥25634	$IG(36,15,6)$ [66, 67]
	{35, -1, -1, 35}	6 0 9	0 34 0	9 0 12		R_1 and R_3 DRG
	{21, -3, 3, -21}	0 15 0	15 0 20	0 20 0		Q-123, Q-213
72	$\{35, 5^{21}, -1^{35}, -7^{15}\}$	16 18 0	18 16 1	0 1 0	≥227	Taylor [67]
	{35, -5, -1, 7}	18 16 1	16 18 0	1 0 0		R_1 and R_2 DRG
	{ 1, -1, 1, -1}	0 35 0	35 0 0	0 0 0		Q-123, Q-321
75	$\{24, 6^{20}, -1^{24}, -4^{30}\}$	9 14 0	14 32 2	0 2 0	?	
	{48, -6, -2, 4}	7 16 1	16 30 1	1 1 0		DRG
	{ 2, -1, 2, -1}	0 24 0	24 24 0	0 0 1		
75	$\{28, 3^{42}, -2^{14}, -7^{18}\}$	8 18 1	18 21 3	1 3 0	?	SRG(75,32,10,16) \spread
	{42, -3, -3, 7}	12 14 2	14 25 2	2 2 0		R_2 SRG
	{ 4, -1, 4, -1}	7 21 0	21 21 0	0 0 3		

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v	spectrum	L_1	L_2	L_3	#	
75	$\{28, 8^{14}, -2^{56}, -7^{4}\}$	13 6 8	6 0 8	8 8 16	0	SRG(75,42,25,21) \spread [40]
	{14, -1, -1, 14}	12 0 16	0 13 0	16 0 16		R_3 SRG, linked designs
	{32, -8, 2, -8}	7 7 14	7 0 7	14 7 10		*Q*-123
76	$\{18, 3^{38}, -1^{18}, -6^{19}\}$	2 15 0	15 36 3	0 3 0	0	SRG(76,21,2,7) \spread [40]
	{54, -3, -3, 6}	5 12 1	12 39 2	1 2 0		DRG, R_2 SRG (does not exist)
	{ 3, -1, 3, -1}	0 18 0	18 36 0	0 0 2		
78	$\{25, 5^{26}, -1^{25}, -5^{26}\}$	8 16 0	16 32 2	0 2 0	≥1	3(Cycl(25)+1)
	{50, -5, -2, 5}	8 16 1	16 32 1	1 1 0		DRG
	{ 2, -1, 2, -1}	0 25 0	25 25 0	0 0 1		
80	$\{13, 3^{39}, -3^{39}, -13^{1}\}$	0 12 0	12 0 27	0 27 0	≥930	*IG*(40,13,4) [68]
	{39, -1, -1, 39}	4 0 9	0 38 0	9 0 18		R_1 and R_3 SRG
	{27, -3, 3, -27}	0 13 0	13 0 26	0 26 0		*Q*-123, *Q*-213
80	$\{24, 2^{60}, -6^{4}, -8^{15}\}$	3 15 5	15 15 10	5 10 0	1	system of 4 linked 2-(16,6,2) designs [53]
	{40, -2, -10, 8}	9 9 6	9 21 9	6 9 0		
	{15, -1, 15, -1}	8 16 0	16 24 0	0 0 14		*Q*-312
81	$\{10, 7^{10}, 1^{20}, -2^{50}\}$	5 4 0	4 6 10	0 10 40	0	$\lambda_3 = -2$
	{20, 2, -7, 2}	2 3 5	3 1 15	5 15 30		DRG, R_2 SRG
	{50, -10, 5, -1}	0 2 8	2 6 12	8 12 29		*Q*-123
81	$\{16, 4^{32}, -2^{32}, -5^{16}\}$	3 8 4	8 8 16	4 16 12	?	
	{32, -1, -4, 8}	4 4 8	4 15 12	8 12 12		R_3 SRG
	{32, -4, 5, -4}	2 8 6	8 12 12	6 12 13		
81	$\{20, 5^{20}, 2^{20}, -4^{40}\}$	5 6 8	6 6 8	8 8 24	?	
	{20, 2, -7, 2}	6 6 8	6 1 12	8 12 20		R_2 SRG (unique)
	{40, -8, 4, 1}	4 4 12	4 6 10	12 10 17		
81	$\{24, 3^{48}, -3^{8}, -6^{24}\}$	5 16 2	16 26 6	2 6 0	?	
	{48, -3, -6, 6}	8 13 3	13 29 5	3 5 0		
	{ 8, -1, 8, -1}	6 18 0	18 30 0	0 0 7		
84	$\{18, 9^{8}, 2^{27}, -3^{48}\}$	7 10 0	10 25 10	0 10 10	1	*J*(9,3)
	{45, 0, -7, 3}	4 10 4	10 22 12	4 12 4		DRG
	{20, -10, 4, -1}	0 9 9	9 27 9	9 9 1		*Q*-123
84	$\{20, 4^{35}, -1^{20}, -5^{28}\}$	4 15 0	15 42 3	0 3 0	?	
	{60, -4, -3, 5}	5 14 1	14 43 2	1 2 0		DRG
	{ 3, -1, 3, -1}	0 20 0	20 40 0	0 0 2		
85	$\{16, 4^{34}, -1^{16}, -4^{34}\}$	3 12 0	12 48 4	0 4 0	≥2	GQ(4,4)\spread
	{64, -4, -4, 4}	3 12 1	12 48 3	1 3 0		DRG, R_2 SRG
	{ 4, -1, 4, -1}	0 16 0	16 48 0	0 0 3		
85	$\{48, 5^{30}, -3^{50}, -12^{4}\}$	26 11 10	11 0 5	10 5 5	?	SRG(85,64,48,48)\spread
	{16, -1, -1, 16}	33 0 15	0 15 0	15 0 5		R_3 SRG
	{20, -5, 3, -5}	24 12 12	12 0 4	12 4 3		

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v	spectrum	L_1	L_2	L_3	#	
88	{12, 4^{22}, 1^{32}, -4^{33}}	1 10 0	10 40 10	0 10 5	?	
	{60, 0, -6, 4}	2 8 2	8 40 11	2 11 2		DRG
	{15, -5, 4, -1}	0 8 4	8 44 8	4 8 2		
90	{12, 3^{44}, -3^{44}, -12^{1}}	0 11 0	11 0 33	0 33 0	≥2285	*IG*(45,12,3) [55]
	{44, -1, -1, 44}	3 0 9	0 43 0	9 0 24		R_1 and R_3 DRG
	{33, -3, 3, -33}	0 12 0	12 0 32	0 32 0		*Q*-123, *Q*-213
90	{44, 4^{33}, -1^{44}, -11^{12}}	18 25 0	25 18 1	0 1 0	0	Taylor
	{44, -4, -1, 11}	25 18 1	18 25 0	1 0 0		R_1 and R_2 DRG
	{ 1, -1, 1, -1}	0 44 0	44 0 0	0 0 0		*Q*-123, *Q*-321
91	{20, 7^{12}, 0^{65}, -8^{13}}	3 12 4	12 6 12	4 12 24	?	
	{30, 4, -3, 9}	8 4 8	4 13 12	8 12 20		
	{40, -12, 2, -2}	2 6 12	6 9 15	12 15 12		*Q*-123
91	{60, 2^{65}, -5^{12}, -10^{13}}	37 18 4	18 4 2	4 2 0	$\sim 1.13 * 10^{18}$	$T(14)^{c}$\spread [29]
	{24, -2, -2, 10}	45 10 5	10 12 1	5 1 0		R_2 SRG
	{ 6, -1, 6, -1}	40 20 0	20 4 0	0 0 5		
95	{36, 3^{57}, -2^{18}, -9^{19}}	10 24 1	24 27 3	1 3 0	?	SRG(95,40,12,20) \spread
	{54, -3, -3, 9}	16 18 2	18 33 2	2 2 0		R_2 SRG
	{ 4, -1, 4, -1}	9 27 0	27 27 0	0 0 3		
96	{15, 5^{30}, -1^{15}, -3^{50}}	4 10 0	10 60 5	0 5 0	≥1	GQ(5,3)\spread
	{75, -5, -5, 3}	2 12 1	12 58 4	1 4 0		DRG, R_2 SRG
	{ 5, -1, 5, -1}	0 15 0	15 60 0	0 0 4		
96	{15, 7^{18}, -1^{45}, -3^{32}}	6 8 0	8 36 16	0 16 4	0	[10, p. 6]
	{60, -4, -4, 6}	2 9 4	9 38 12	4 12 4		DRG, R_2 and R_3 SRG
	{20, -4, 4, -4}	0 12 3	12 36 12	3 12 4		
96	{19, 7^{19}, -1^{57}, -5^{19}}	6 12 0	12 30 15	0 15 4	0	Neumaier [10, corrections and additions]
	{57, -3, -3, 9}	4 10 5	10 36 10	5 10 4		DRG, R_2 and R_3 SRG
	{19, -5, 3, -5}	0 15 4	15 30 12	4 12 2		*Q*-123
96	{25, 5^{20}, 1^{50}, -7^{25}}	4 8 12	8 4 8	12 8 30	?	
	{20, 4, -4, 4}	10 5 10	5 4 10	10 10 30		R_2 and R_3 SRG
	{50, -10, 2, 2}	6 4 15	4 4 12	15 12 22		
96	{30, 2^{75}, -6^{5}, -10^{15}}	4 20 5	20 20 10	5 10 0	1	system of 5 linked 2-(16,6,2) designs [53]
	{50, -2, -10, 10}	12 12 6	12 28 9	6 9 0		
	{15, -1, 15, -1}	10 20 0	20 30 0	0 0 14		*Q*-312
96	{30, 4^{48}, -2^{15}, -6^{32}}	8 20 1	20 36 4	1 4 0	?	SRG(96,35,10,14) \spread
	{60, -4, -4, 6}	10 18 2	18 38 3	2 3 0		R_2 SRG
	{ 5, -1, 5, -1}	6 24 0	24 36 0	0 0 4		
96	{30, 6^{30}, -2^{45}, -6^{20}}	10 15 4	15 18 12	4 12 4	?	
	{45, -3, -3, 9}	10 12 8	12 24 8	8 8 4		R_2 and R_3 SRG
	{20, -4, 4, -4}	6 18 6	18 18 9	6 9 4		

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v	spectrum	L_1	L_2	L_3	#	
96	{30, 10^{15}, -2^{75}, -6^{5}}	14 5 10	5 0 10	10 10 30	0	SRG(96,45,24,18) \spread [40]
	{15, -1, -1, 15}	10 0 20	0 14 0	20 0 30		R_3 SRG, linked designs
	{50, -10, 2, -10}	6 6 18	6 0 9	18 9 22		*Q*-123
96	{60, 4^{45}, -4^{45}, -12^{5}}	36 11 12	11 0 4	12 4 4	≥1	GQ(5,3)^c\spread
	{15, -1, -1, 15}	44 0 16	0 14 0	16 0 4		R_3 SRG
	{20, -4, 4, -4}	36 12 12	12 0 3	12 3 4		
96	{38, 6^{19}, 2^{38}, -6^{38}}	14 9 14	9 4 6	14 6 18	?	
	{19, 3, -5, 3}	18 8 12	8 2 8	12 8 18		R_2 and R_3 SRG
	{38, -10, 2, 2}	14 6 18	6 4 9	18 9 10		
96	{45, 3^{60}, -3^{15}, -9^{20}}	18 24 2	24 18 3	2 3 0	?	SRG(96,50,22,30)\spread
	{45, -3, -3, 9}	24 18 3	18 24 2	3 2 0		R_2 SRG
	{ 5, -1, 5, -1}	18 27 0	27 18 0	0 0 4		
96	{45, 7^{27}, -3^{63}, -9^{5}}	22 8 14	8 0 7	14 7 14	?	SRG(96,60,38,36)\spread
	{15, -1, -1, 15}	24 0 21	0 14 0	21 0 14		R_3 SRG
	{35, -7, 3, -7}	18 9 18	9 0 6	18 6 10		
99	{28, 3^{63}, -5^{21}, -8^{14}}	5 10 12	10 3 15	12 15 15	?	
	{28, -1, -5, 10}	10 3 15	3 12 12	15 12 15		R_3 SRG
	{42, -3, 9, -3}	8 10 10	10 8 10	10 10 21		*Q*-312
99	{28, 6^{21}, 1^{44}, -6^{33}}	7 6 14	6 2 6	14 6 36	?	
	{14, 3, -4, 3}	12 4 12	4 1 8	12 8 36		R_2 and R_3 SRG
	{56, -10, 2, 2}	7 3 18	3 2 9	18 9 28		
99	{32, 8^{22}, -1^{32}, -4^{44}}	13 18 0	18 44 2	0 2 0	?	
	{64, -8, -2, 4}	9 22 1	22 40 1	1 1 0		DRG
	{ 2, -1, 2, -1}	0 32 0	32 32 0	0 0 1		
99	{40, 5^{44}, -4^{10}, -5^{44}}	16 20 3	20 25 5	3 5 0	≥1	SRG(99,48,22,24) \spread [40]
	{50, -5, -5, 5}	16 20 4	20 25 4	4 4 0		R_2 SRG
	{ 8, -1, 8, -1}	15 25 0	25 25 0	0 0 7		
99	{40, 6^{36}, -4^{54}, -5^{8}}	17 4 18	4 0 6	18 6 24	?	SRG(99,50,25,25)\spread
	{10, -1, -1, 10}	16 0 24	0 9 0	24 0 24		R_3 SRG
	{48, -6, 4, -6}	15 5 20	5 0 5	20 5 22		
100	{18, 3^{56}, -2^{18}, -6^{25}}	1 14 2	14 35 14	2 14 2	?	
	{63, -2, -7, 7}	4 10 4	10 40 12	4 12 2		$R_3 \simeq L_2(10)$
	{18, -2, 8, -2}	2 14 2	14 42 7	2 7 8		
100	{22, 7^{16}, 2^{33}, -4^{50}}	6 9 6	9 12 12	6 12 26	?	
	{33, 3, -7, 3}	6 8 8	8 8 16	8 16 20		R_2 SRG
	{44, -11, 4, 0}	3 6 13	6 12 15	13 15 15		
100	{22, 8^{20}, -2^{55}, -3^{24}}	9 6 6	6 9 18	6 18 20	?	
	{33, -3, -3, 8}	4 6 12	6 14 12	12 12 20		R_2 and R_3 SRG
	{44, -6, 4, -6}	3 9 10	9 9 15	10 15 18		
100	{48, 4^{50}, -2^{24}, -8^{25}}	21 25 1	25 21 2	1 2 0	?	
	{48, -4, -2, 8}	25 21 2	21 25 1	2 1 0		
	{ 3, -1, 3, -1}	16 32 0	32 16 0	0 0 2		
100	{49, 7^{25}, -1^{49}, -7^{25}}	24 24 0	24 24 1	0 1 0	≥18	2(*P*(49)+1) [15, 67]
	{49, -7, -1, 7}	24 24 1	24 24 0	1 0 0		R_1 and R_2 DRG
	{ 1, -1, 1, -1}	0 49 0	49 0 0	0 0 0		*Q*-123, *Q*-321

Appendix C

Two integral eigenvalues; excluded here are association schemes generated by SRG$\otimes J_n$.

v	spectrum	L_1	L_2	L_3	#	
12	{ 5, -1^{5}, 2.236^{3}, -2.236^{3}}	2 2 0	2 2 1	0 1 0	1	Icosahedron
	{ 5, -1, -2.236, 2.236}	2 2 1	2 2 0	1 0 0		$R_1 \simeq R_2$ DRG
	{ 1, 1, -1.000, -1.000}	0 5 0	5 0 0	0 0 0		Q-213, Q-312
14	{ 3, -3^{1}, 1.414^{6}, -1.414^{6}}	0 2 0	2 0 4	0 4 0	1	IG(7,3,1)
	{ 6, 6, -1.000, -1.000}	1 0 2	0 5 0	2 0 2		R_1 and R_3 DRG
	{ 4, -4, -1.414, 1.414}	0 3 0	3 0 3	0 3 0		Q-231, Q-321
21	{ 4, -2^{8}, 2.414^{6}, -0.414^{6}}	1 2 0	2 2 4	0 4 4	1	$L(IG(7,3,1))$
	{ 8, 2, -0.586, -3.414}	1 1 2	1 2 4	2 4 2		DRG
	{ 8, -1, -2.828, 2.828}	0 2 2	2 4 2	2 2 3		
22	{ 5, -5^{1}, 1.732^{10}, -1.732^{10}}	0 4 0	4 0 6	0 6 0	1	IG(11,5,2)
	{10, 10, -1.000, -1.000}	2 0 3	0 9 0	3 0 3		R_1 and R_3 DRG
	{ 6, -6, -1.732, 1.732}	0 5 0	5 0 5	0 5 0		Q-231, Q-321
24	{ 7, -1^{7}, 2.646^{8}, -2.646^{8}}	2 4 0	4 8 2	0 2 0	1	Klein
	{14, -2, -2.646, 2.646}	2 4 1	4 8 1	1 1 0		DRG
	{ 2, 2, -1.000, -1.000}	0 7 0	7 7 0	0 0 1		
26	{ 4, -4^{1}, 1.732^{12}, -1.732^{12}}	0 3 0	3 0 9	0 9 0	1	IG(13,4,1)
	{12, 12, -1.000, -1.000}	1 0 3	0 11 0	3 0 6		R_1 and R_3 DRG
	{ 9, -9, -1.732, 1.732}	0 4 0	4 0 8	0 8 0		Q-231, Q-321
28	{13, -1^{13}, 3.606^{7}, -3.606^{7}}	6 6 0	6 6 1	0 1 0	1	Taylor
	{13, -1, -3.606, 3.606}	6 6 1	6 6 0	1 0 0		$R_1 \simeq R_2$ DRG
	{ 1, 1, -1.000, -1.000}	0 13 0	13 0 0	0 0 0		Q-213, Q-312
33	{10, -1^{10}, 3.162^{11}, -3.162^{11}}	3 6 0	6 12 2	0 2 0	0	Hasse-Minkowski
	{20, -2, -3.162, 3.162}	3 6 1	6 12 1	1 1 0		DRG
	{ 2, 2, -1.000, -1.000}	0 10 0	10 10 0	0 0 1		
35	{ 6, -1^{6}, 2.449^{14}, -2.449^{14}}	1 4 0	4 16 4	0 4 0	0	Hasse-Minkowski, PG(2,6)
	{24, -4, -2.449, 2.449}	1 4 1	4 16 3	1 3 0		DRG
	{ 4, 4, -1.000, -1.000}	0 6 0	6 18 0	0 0 3		
36	{17, -1^{17}, 4.123^{9}, -4.123^{9}}	8 8 0	8 8 1	0 1 0	1	2(P(17)+1)
	{17, -1, -4.123, 4.123}	8 8 1	8 8 0	1 0 0		$R_1 \simeq R_2$ DRG
	{ 1, 1, -1.000, -1.000}	0 17 0	17 0 0	0 0 0		Q-213, Q-312
38	{ 9, -9^{1}, 2.236^{18}, -2.236^{18}}	0 8 0	8 0 10	0 10 0	6	IG(19,9,4)
	{18, -18, -1.000, -1.000}	4 0 5	0 17 0	5 0 5		R_1 and R_3 DRG
	{10, -10, -2.236, 2.236}	0 9 0	9 0 9	0 9 0		Q-231, Q-321
40	{ 9, 1^{15}, 2.162^{12}, -4.162^{12}}	0 4 4	4 4 4	4 4 10	?	
	{12, -4, 2.000, 2.000}	3 3 3	3 2 6	3 6 9		R_2 SRG
	{18, 2, -5.162, 1.162}	2 2 5	2 4 6	5 6 6		
40	{18, -2^{9}, 3.464^{15}, -3.464^{15}}	8 8 1	8 8 2	1 2 0	0	Hasse-Minkowski
	{18, -2, -3.464, 3.464}	8 8 2	8 8 1	2 1 0		
	{ 3, 3, -1.000, -1.000}	6 12 0	12 6 0	0 0 2		

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v	spectrum	L_1	L_2	L_3	#	
42	{13, -1^{13}, 3.606^{14}, -3.606^{14}}	4 8 0	8 16 2	0 2 0	≥1	3(Cycl(13)+1)
	{26, -2, -3.606, 3.606}	4 8 1	8 16 1	1 1 0		DRG
	{ 2, 2, -1.000, -1.000}	0 13 0	13 13 0	0 0 1		
44	{ 7, -7^{1}, 2.236^{21}, -2.236^{21}}	0 6 0	6 0 15	0 15 0	0	*IG*(22,7,2)
	{21, 21, -1.000, -1.000}	2 0 5	0 20 0	5 0 10		R_1 and R_3 DRG
	{15, -15, -2.236, 2.236}	0 7 0	7 0 14	0 14 0		*Q*-231, *Q*-321
44	{21, -1^{21}, 4.583^{11}, -4.583^{11}}	10 10 0	10 10 1	0 1 0	0	Hasse-Minkowski
	{21, -1, -4.583, 4.583}	10 10 1	10 10 0	1 0 0		$R_1 \simeq R_2$ DRG
	{ 1, 1, -1.000, -1.000}	0 21 0	21 0 0	0 0 0		*Q*-213, *Q*-312
45	{16, -2^{20}, 4.873^{12}, -2.873^{12}}	7 5 3	5 5 6	3 6 3	?	
	{16, -2, -2.873, 4.873}	5 5 6	5 7 3	6 3 3		R_3 SRG
	{12, 3, -3.000, -3.000}	4 8 4	8 4 4	4 4 3		
46	{11, -11^{1}, 2.449^{22}, -2.449^{22}}	0 10 0	10 0 12	0 12 0	582	*IG*(23,11,5) [68]
	{22, 22, -1.000, -1.000}	5 0 6	0 21 0	6 0 6		R_1 and R_3 DRG
	{12, -12, -2.449, 2.449}	0 11 0	11 0 11	0 11 0		*Q*-231, *Q*-231
50	{ 9, -9^{1}, 2.449^{24}, -2.449^{24}}	0 8 0	8 0 16	0 16 0	50	*IG*(25,9,3) [28]
	{24, 24, -1.000, -1.000}	3 0 6	0 23 0	6 0 10		R_1 and R_3 DRG
	{16, -16, -2.449, 2.449}	0 9 0	9 0 15	0 15 0		*Q*-231, *Q*-321
52	{ 6, -2^{27}, 3.732^{12}, 0.268^{12}}	2 3 0	3 6 9	0 9 18	1	*L*(*IG*(13,4,1))
	{18, 2, 0.464, -6.464}	1 2 3	2 3 12	3 12 12		DRG
	{27, -1, -5.196, 5.196}	0 2 4	2 8 8	4 8 14		
54	{13, -13^{1}, 2.646^{26}, -2.646^{26}}	0 12 0	12 0 14	0 14 0	105041	*IG*(27,13,6) [68]
	{26, 26, -1.000, -1.000}	6 0 7	0 25 0	7 0 7		R_1 and R_3 DRG
	{14, -14, -2.646, 2.646}	0 13 0	13 0 13	0 13 0		*Q*-231, *Q*-321
55	{18, -4^{10}, 3.854^{22}, -2.854^{22}}	6 6 5	6 4 8	5 8 5	?	
	{18, 7, -2.000, -2.000}	6 4 8	4 9 4	8 4 6		$R_1 \simeq T(11)$
	{18, -4, -2.854, 3.854}	5 8 5	8 4 6	5 6 6		
56	{ 5, -3^{15}, 2.414^{20}, -0.414^{20}}	0 4 0	4 4 12	0 12 18	0	Fon-der-Flaass [33]
	{20, 4, 0.828, -4.828}	1 1 3	1 6 12	3 12 15		DRG
	{30, -2, -4.243, 4.243}	0 2 3	2 8 10	3 10 16		
57	{ 6, -3^{20}, 2.618^{18}, 0.382^{18}}	0 5 0	5 15 10	0 10 10	≥1	Perkel
	{30, 3, 0.854, -5.854}	1 3 2	3 14 12	2 12 6		DRG
	{20, -1, -4.472, 4.472}	0 3 3	3 18 9	3 9 7		
58	{ 8, -8^{1}, 2.449^{28}, -2.449^{28}}	0 7 0	7 0 21	0 21 0	0	*IG*(29,8,2)
	{28, 28, -1.000, -1.000}	2 0 6	0 27 0	6 0 15		R_1 and R_3 DRG
	{21, -21, -2.449, 2.449}	0 8 0	8 0 20	0 20 0		*Q*-231, *Q*-321
60	{11, -1^{11}, 3.317^{24}, -3.317^{24}}	2 8 0	8 32 4	0 4 0	≥1	Mathon
	{44, -4, -3.317, 3.317}	2 8 1	8 32 3	1 3 0		DRG
	{ 4, 4, -1.000, -1.000}	0 11 0	11 33 0	0 0 3		
60	{19, -1^{19}, 4.359^{20}, -4.359^{20}}	6 12 0	12 24 2	0 2 0	≥1	3(Cycl(19)+1)
	{38, -2, -4.359, 4.359}	6 12 1	12 24 1	1 1 0		DRG
	{ 2, 2, -1.000, -1.000}	0 19 0	19 19 0	0 0 1		
60	{29, -1^{29}, 5.385^{15}, -5.385^{15}}	14 14 0	14 14 1	0 1 0	6	2(*P*(29)+1) [67]
	{29, -1, -5.385, 5.385}	14 14 1	14 14 0	0 0 0		R_1 and R_2 DRG
	{ 1, 1, -1.000, -1.000}	0 29 0	29 0 0	0 0 0		*Q*-213, *Q*-312

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v	spectrum	L_1	L_2	L_3	#	
62	{ 6, -6^{1}, 2.236^{30}, -2.236^{30}}	0 5 0	5 0 25	0 25 0	1	*IG*(31,6,1)
	{30, 30, -1.000, -1.000}	1 0 5	0 29 0	5 0 20		R_1 and R_3 DRG
	{25, -25, -2.236, 2.236}	0 6 0	6 0 24	0 24 0		Q-231, Q-321
62	{10, -10^{1}, 2.646^{30}, -2.646^{30}}	0 9 0	9 0 21	0 21 0	82	*IG*(31,10,3) [64, 65]
	{30, 30, -1.000, -1.000}	3 0 7	0 29 0	7 0 14		R_1 and R_3 DRG
	{21, -21, -2.646, 2.646}	0 10 0	10 0 20	0 20 0		Q-231, Q-321
62	{15, -15^{1}, 2.828^{30}, -2.828^{30}}	0 14 0	14 0 16	0 16 0	≥633446	*IG*(31,15,7) [66]
	{30, 30, -1.000, -1.000}	7 0 8	0 29 0	8 0 8		R_1 and R_3 DRG
	{16, -16, -2.828, 2.828}	0 15 0	15 0 15	0 15 0		Q-231, Q-321
63	{ 8, -1^{8}, 2.828^{27}, -2.828^{27}}	1 6 0	6 36 6	0 6 0	1	*PG*(2,8)
	{48, -6, -2.828, 2.828}	1 6 1	6 36 5	1 5 0		DRG
	{ 6, 6, -1.000, -1.000}	0 8 0	8 40 0	0 0 5		
64	{14, -2^{7}, 3.464^{28}, -3.464^{28}}	3 9 1	9 27 6	1 6 0	?	
	{42, -6, -3.464, 3.464}	3 9 2	9 27 5	2 5 0		
	{ 7, 7, -1.000, -1.000}	2 12 0	12 30 0	0 0 6		
64	{30, -2^{15}, 4.472^{24}, -4.472^{24}}	14 14 1	14 14 2	1 2 0	?	
	{30, -2, -4.472, 4.472}	14 14 2	14 14 1	2 1 0		
	{ 3, 3, -1.000, -1.000}	10 20 0	20 10 0	0 0 2		
68	{12, -12^{1}, 2.828^{33}, -2.828^{33}}	0 11 0	11 0 22	0 22 0	0	*IG*(34,12,4)
	{33, 33, -1.000, -1.000}	4 0 8	0 32 0	8 0 14		R_1 and R_3 DRG
	{22, -22, -2.828, 2.828}	0 12 0	12 0 21	0 21 0		Q-231, Q-321
68	{33, -1^{33}, 5.745^{17}, -5.745^{17}}	16 16 0	16 16 1	0 1 0	0	Taylor, Hasse-Minkowski
	{33, -1, -5.745, 5.745}	16 16 1	16 16 0	1 0 0		R_1 and R_2 DRG
	{ 1, 1, -1.000, -1.000}	0 33 0	33 0 0	0 0 0		Q-213, Q-312
69	{22, -1^{22}, 4.690^{23}, -4.690^{23}}	7 14 0	14 28 2	0 2 0	0	Hasse-Minkowski
	{44, -2, -4.690, 4.690}	7 14 1	14 28 1	1 1 0		DRG
	{ 2, 2, -1.000, -1.000}	0 22 0	22 22 0	0 0 1		
72	{17, -1^{17}, 4.123^{27}, -4.123^{27}}	4 12 0	12 36 3	0 3 0	≥1	Mathon
	{51, -3, -4.123, 4.123}	4 12 1	12 36 2	1 2 0		DRG
	{ 3, 3, -1.000, -1.000}	0 17 0	17 34 0	0 0 2		
74	{ 9, -9^{1}, 2.646^{36}, -2.646^{36}}	0 8 0	8 0 28	0 28 0	3	*IG*(37,9,2) [1]
	{36, 36, -1.000, -1.000}	2 0 7	0 35 0	7 0 21		R_1 and R_3 DRG
	{28, -28, -2.646, 2.646}	0 9 0	9 0 27	0 27 0		Q-231, Q-321
76	{37, -1^{37}, 6.083^{19}, -6.083^{19}}	18 18 0	18 18 1	0 1 0	≥11	Taylor [15, 67]
	{37, -1, -6.083, 6.083}	18 18 1	18 18 0	1 0 0		R_1 and R_2 DRG
	{ 1, 1, -1.000, -1.000}	0 37 0	37 0 0	0 0 0		Q-213, Q-312
78	{19, -19^{1}, 3.162^{38}, -3.162^{38}}	0 18 0	18 0 20	0 20 0	≥19	*IG*(39,19,9) [66]
	{38, 38, -1.000, -1.000}	9 0 10	0 37 0	10 0 10		R_1 and R_3 DRG
	{20, -20, -3.162, 3.162}	0 19 0	19 0 19	0 19 0		Q-231, Q-321
81	{ 8, -1^{32}, 3.854^{24}, -2.854^{24}}	2 5 0	5 15 20	0 20 12	0	[10, Prop. 1.2.1]
	{40, -5, -0.854, 5.854}	1 3 4	3 20 16	4 16 12		DRG, R_3 SRG
	{32, 5, -4.000, -4.000}	0 5 3	5 20 15	3 15 13		

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v	spectrum	L_1	L_2	L_3	#	
81	$\{10, 1^{20}, 2.854^{30}, -3.854^{30}\}$	0 4 5	4 6 10	5 10 35	?	
	$\{20, -7, 2.000, 2.000\}$	2 3 5	3 1 15	5 15 30		R_2 SRG (unique)
	$\{50, 5, -5.854, 0.854\}$	1 2 7	2 6 12	7 12 30		
81	$\{16, -2^{32}, 5.243^{24}, -3.243^{24}\}$	5 6 4	6 10 16	4 16 12	?	
	$\{32, -4, -2.243, 6.243\}$	3 5 8	5 14 12	8 12 12		R_3 SRG
	$\{32, 5, -4.000, -4.000\}$	2 8 6	8 12 12	6 12 13		
81	$\{20, 2^{20}, 3.243^{30}, -5.243^{30}\}$	3 6 10	6 6 8	10 8 22	?	
	$\{20, -7, 2.000, 2.000\}$	6 6 8	6 1 12	8 12 20		R_2 SRG (unique)
	$\{40, 4, -6.243, 2.243\}$	5 4 11	4 6 10	11 10 18		
81	$\{24, -3^{8}, 4.243^{36}, -4.243^{36}\}$	7 14 2	14 28 6	2 6 0	?	
	$\{48, -6, -4.243, 4.243\}$	7 14 3	14 28 5	3 5 0		
	$\{8, 8, -1.000, -1.000\}$	6 18 0	18 30 0	0 0 7		
81	$\{28, 1^{56}, 3.374^{12}, -10.374^{12}\}$	4 12 11	12 6 6	11 6 11	?	
	$\{24, -3, 6.000, 6.000\}$	14 7 7	7 9 7	7 7 14		R_2 SRG
	$\{28, 1, -10.374, 3.374\}$	11 6 11	6 6 12	11 12 4		
82	$\{16, -16^{1}, 3.162^{40}, -3.162^{40}\}$	0 15 0	15 0 25	0 25 0	≥56000	*IG*(41,16,6) [66]
	$\{40, 40, -1.000, -1.000\}$	6 0 10	0 39 0	10 0 15		R_1 and R_3 DRG
	$\{25, -25, -3.162, 3.162\}$	0 16 0	16 0 24	0 24 0		*Q*-231, *Q*-321
84	$\{13, -1^{13}, 3.606^{35}, -3.606^{35}\}$	2 10 0	10 50 5	0 5 0	≥1	Mathon
	$\{65, -5, -3.606, 3.606\}$	2 10 1	10 50 4	1 4 0		DRG
	$\{5, 5, -1.000, -1.000\}$	0 13 0	13 52 0	0 0 4		
84	$\{41, -1^{41}, 6.403^{21}, -6.403^{21}\}$	20 20 0	20 20 1	0 1 0	≥18	Taylor [15,67]
	$\{41, -1, -6.403, 6.403\}$	20 20 1	20 20 0	1 0 0		R_1 and R_2 DRG
	$\{1, 1, -1.000, -1.000\}$	0 41 0	41 0 0	0 0 0		*Q*-213, *Q*-312
85	$\{12, -5^{16}, 3.449^{34}, -1.449^{34}\}$	1 6 4	6 2 16	4 16 28	?	
	$\{24, 7, 0.449, -4.449\}$	3 1 8	1 8 14	8 14 26		
	$\{48, -3, -4.899, 4.899\}$	1 4 7	4 7 13	7 13 27		
85	$\{32, -2^{50}, 8.325^{17}, -4.325^{17}\}$	15 11 5	11 11 10	5 10 5	?	
	$\{32, -2, -4.325, 8.325\}$	11 11 10	11 15 5	10 5 5		R_3 SRG
	$\{20, 3, -5.000, -5.000\}$	8 16 8	16 8 8	8 8 3		
85	$\{32, -2^{16}, 4.899^{34}, -4.899^{34}\}$	12 18 1	18 27 3	1 3 0	0	Hasse-Minkowski
	$\{48, -3, -4.899, 4.899\}$	12 18 2	18 27 2	2 2 0		
	$\{4, 4, -1.000, -1.000\}$	8 24 0	24 24 0	0 0 3		
86	$\{7, -7^{1}, 2.449^{42}, -2.449^{42}\}$	0 6 0	6 0 36	0 36 0	0	*IG*(43,7,1)
	$\{42, 42, -1.000, -1.000\}$	1 0 6	0 41 0	6 0 30		R_1 and R_3 DRG
	$\{36, -36, -2.449, 2.449\}$	0 7 0	7 0 35	0 35 0		*Q*-231, *Q*-321
86	$\{15, -15^{1}, 3.162^{42}, -3.162^{42}\}$	0 14 0	14 0 28	0 28 0	0	*IG*(43,15,5)
	$\{42, 42, -1.000, -1.000\}$	5 0 10	0 41 0	10 0 18		R_1 and R_3 DRG
	$\{28, -28, -3.162, 3.162\}$	0 15 0	15 0 27	0 27 0		*Q*-231, *Q*-321
86	$\{21, -21^{1}, 3.317^{42}, -3.317^{42}\}$	0 20 0	20 0 22	0 22 0	≥14	*IG*(43,21,10) [68]
	$\{42, 42, -1.000, -1.000\}$	10 0 11	0 41 0	11 0 11		R_1 and R_3 DRG
	$\{22, -22, -3.317, 3.317\}$	0 21 0	21 0 21	0 21 0		*Q*-231, *Q*-321

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v	spectrum	L_1	L_2	L_3	#	
87	{28, -1^{28}, 5.292^{29}, -5.292^{29}}	9 18 0	18 36 2	0 2 0	≥1	3(PseudoCycl(28)+1)
	{56, -2, -5.292, 5.292}	9 18 1	18 36 1	1 1 0		DRG
	{ 2, 2, -1.000, -1.000}	0 28 0	28 28 0	0 0 1		
88	{42, -2^{21}, 5.292^{33}, -5.292^{33}}	20 20 1	20 20 2	1 2 0	0	Hasse-Minkowski
	{42, -2, -5.292, 5.292}	20 20 2	20 20 1	2 1 0		
	{ 3, 3, -1.000, -1.000}	14 28 0	28 14 0	0 0 2		
92	{10, -10^{1}, 2.828^{45}, -2.828^{45}}	0 9 0	9 0 36	0 36 0	0	*IG*(46,10,2)
	{45, 45, -1.000, -1.000}	2 0 8	0 44 0	8 0 28		R_1 and R_3 DRG
	{36, -36, -2.828, 2.828}	0 10 0	10 0 35	0 35 0		*Q*-231, *Q*-321
92	{45, -1^{45}, 6.708^{23}, -6.708^{23}}	22 22 0	22 22 1	0 1 0	≥80	Taylor [15, 67]
	{45, -1, -6.708, 6.708}	22 22 1	22 22 0	1 0 0		R_1 and R_2 DRG
	{ 1, 1, -1.000, -1.000}	0 45 0	45 0 0	0 0 0		*Q*-213, *Q*-312
93	{20, 5^{32}, -1.586^{30}, -4.414^{30}}	5 8 6	8 8 16	6 16 18	?	
	{32, -1, -5.657, 5.657}	5 5 10	5 11 15	10 15 15		
	{40, -5, -6.243, -2.243}	3 8 9	8 12 12	9 12 18		
94	{23, -23^{1}, 3.464^{46}, -3.464^{46}}	0 22 0	22 0 24	0 24 0	≥1	*IG*(47,23,11)
	{46, 46, -1.000, -1.000}	11 0 12	0 45 0	12 0 12		R_1 and R_3 DRG
	{24, -24, -3.464, 3.464}	0 23 0	23 0 23	0 23 0		*Q*-231, *Q*-321
96	{19, -5^{19}, 4.464^{38}, -2.464^{38}}	4 12 2	12 30 15	2 15 2	?	
	{57, 9, -3.000, -3.000}	4 10 5	10 36 10	5 10 4		R_2 SRG
	{19, -5, -2.464, 4.464}	2 15 2	15 30 12	2 12 4		
96	{30, -6^{5}, 4.472^{45}, -4.472^{45}}	9 5 15	5 0 10	15 10 25	?	
	{15, 15, -1.000, -1.000}	10 0 20	0 14 0	20 0 30		
	{50, -10, -4.472, 4.472}	9 6 15	6 0 9	15 9 25		
96	{30, -2^{63}, 9.708^{16}, -3.708^{16}}	14 8 7	8 8 14	7 14 14	?	
	{30, -2, -3.708, 9.708}	8 8 14	8 14 7	14 7 14		R_3 SRG
	{35, 3, -7.000, -7.000}	6 12 12	12 6 12	12 12 10		
96	{31, -1^{31}, 5.568^{32}, -5.568^{32}}	10 20 0	20 40 2	0 2 0	≥1	3(Cycl(31)+1)
	{62, -2, -5.568, 5.568}	10 20 1	20 40 1	1 1 0		DRG
	{ 2, 2, -1.000, -1.000}	0 31 0	31 31 0	0 0 1		
96	{38, -2^{57}, 8.928^{19}, -4.928^{19}}	18 14 5	14 14 10	5 10 4	?	
	{38, -2, -4.928, 8.928}	14 14 10	14 18 5	10 5 4		R_3 SRG
	{19, 3, -5.000, -5.000}	10 20 8	20 10 8	8 8 2		
98	{16, -16^{1}, 3.317^{48}, -3.317^{48}}	0 15 0	15 0 33	0 33 0	≥6073	*IG*(49,16,5) [54]
	{48, 48, -1.000, -1.000}	5 0 11	0 47 0	11 0 22		R_1 and R_3 DRG
	{33, -33, -3.317, 3.317}	0 16 0	16 0 32	0 32 0		*Q*-231, *Q*-321
99	{10, -1^{10}, 3.162^{44}, -3.162^{44}}	1 8 0	8 64 8	0 8 0	0	*PG*(2,10)
	{80, -8, -3.162, 3.162}	1 8 1	8 64 7	1 7 0		DRG
	{ 8, 8, -1.000, -1.000}	0 10 0	10 70 0	0 0 7		
99	{42, -2^{54}, 8.374^{22}, -5.374^{22}}	20 17 4	17 17 8	4 8 2	?	
	{42, -2, -5.374, 8.374}	17 17 8	17 20 4	8 4 2		R_3 SRG
	{14, 3, -4.000, -4.000}	12 24 6	24 12 6	6 6 1		

Appendix D

One integral eigenvalue.

v	spectrum	L_1	L_2	L_3	#	
7	{ 2, 1.247, -0.445, -1.802}	0 1 0	1 0 1	0 1 1	1	C_7
	{ 2, -0.445, -1.802, 1.247}	1 0 1	0 0 1	1 1 0		$R_1 \simeq R_2 \simeq R_3$ DRG
	{ 2, -1.802, 1.247, -0.445}	0 1 1	1 1 0	1 0 0		Q-123, Q-231, Q-312
13	{ 4, 1.377, 0.274, -2.651}	0 2 1	2 1 1	1 1 2	1	Cycl(13)
	{ 4, 0.274, -2.651, 1.377}	2 1 1	1 0 2	1 2 1		
	{ 4, -2.651, 1.377, 0.274}	1 1 2	1 2 1	2 1 0		
19	{ 6, 2.507, -1.222, -2.285}	2 2 1	2 1 3	1 3 2	1	Cycl(19)
	{ 6, -1.222, -2.285, 2.507}	2 1 3	1 2 2	3 2 1		
	{ 6, -2.285, 2.507, -1.222}	1 3 2	3 2 1	2 1 2		
28	{ 9, 2.604, -0.110, -3.494}	2 4 2	4 2 3	2 3 4	2	Mathon, Hollmann
	{ 9, -0.110, -3.494, 2.604}	4 2 3	2 2 4	3 4 2		
	{ 9, -3.494, 2.604, -0.110}	2 3 4	3 4 2	4 2 2		
31	{10, 3.084, -0.787, -3.297}	3 4 2	4 2 4	2 4 4	≥1	Cycl(31)
	{10, -0.787, -3.297, 3.084}	4 2 4	2 3 4	4 4 2		
	{10, -3.297, 3.084, -0.787}	2 4 4	4 4 2	4 2 3		
37	{12, 2.187, 1.158, -4.345}	2 5 4	5 4 3	4 3 5	≥1	Cycl(37)
	{12, 1.158, -4.345, 2.187}	5 4 3	4 2 5	3 5 4		
	{12, -4.345, 2.187, 1.158}	4 3 5	3 5 4	5 4 2		
43	{14, 2.888, 0.615, -4.503}	3 6 4	6 4 4	4 4 6	≥1	Cycl(43)
	{14, 0.615, -4.503, 2.888}	6 4 4	4 3 6	4 6 4		
	{14, -4.503, 2.888, 0.615}	4 4 6	4 6 4	6 4 3		
49	{16, 4.296, -2.137, -3.159}	6 5 4	5 4 7	4 7 5	≥1	Cycl(49)
	{16, -2.137, -3.159, 4.296}	5 4 7	4 6 5	7 5 4		
	{16, -3.159, 4.296, -2.137}	4 7 5	7 5 4	5 4 6		
52	{17, 4.302, -1.548, -3.754}	6 6 4	6 4 7	4 7 6	?	
	{17, -1.548, -3.754, 4.302}	6 4 7	4 6 6	7 6 4		
	{17, -3.754, 4.302, -1.548}	4 7 6	7 6 4	6 4 6		
61	{20, 4.230, -0.445, -4.786}	6 8 5	8 5 7	5 7 8	≥1	Cycl(61)
	{20, -0.445, -4.786, 4.230}	8 5 7	5 6 8	7 8 5		
	{20, -4.786, 4.230, -0.445}	5 7 8	7 8 5	8 5 6		
67	{22, 4.085, 0.230, -5.316}	6 9 6	9 6 7	6 7 9	≥1	Cycl(67)
	{22, 0.230, -5.316, 4.085}	9 6 7	6 6 9	7 9 6		
	{22, -5.316, 4.085, 0.230}	6 7 9	7 9 6	9 6 6		
73	{24, 4.950, -1.132, -4.818}	8 9 6	9 6 9	6 9 9	≥1	Cycl(73)
	{24, -1.132, -4.818, 4.950}	9 6 9	6 8 9	9 9 6		
	{24, -4.818, 4.950, -1.132}	6 9 9	9 9 6	9 6 8		
76	{25, 3.570, 1.444, -6.014}	6 10 8	10 8 7	8 7 10	?	
	{25, 1.444, -6.014, 3.570}	10 8 7	8 6 10	7 10 8		
	{25, -6.014, 3.570, 1.444}	8 7 10	7 10 8	10 8 6		
79	{26, 3.122, 2.108, -6.230}	6 10 9	10 9 7	9 7 10	≥1	Cycl(79)
	{26, 2.108, -6.230, 3.122}	10 9 7	9 6 10	7 10 9		
	{26, -6.230, 3.122, 2.108}	9 7 10	7 10 9	10 9 6		

(*Continued on next page.*)

(*Continued*).

v	spectrum	L_1	L_2	L_3	#	
91	{30, 4.412, 0.960, -6.373}	8 12 9	12 9 9	9 9 12	?	
	{30, 0.960, -6.373, 4.412}	12 9 9	9 8 12	9 12 9		
	{30, -6.373, 4.412, 0.960}	9 9 12	9 12 9	12 9 8		
91	{30, 5.909, -2.404, -4.506}	11 10 8	10 8 12	8 12 10	?	
	{30, -2.404, -4.506, 5.909}	10 8 12	8 11 10	12 10 8		
	{30, -4.506, 5.909, -2.404}	8 12 10	12 10 8	10 8 11		
97	{32, 6.207, -3.098, -4.109}	12 10 9	10 9 13	9 13 10	$\geq$1	Cycl(97)
	{32, -3.098, -4.109, 6.207}	10 9 13	9 12 10	13 10 9		
	{32, -4.109, 6.207, -3.098}	9 13 10	13 10 9	10 9 12		

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