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Explicit Formulae for Some Kazhdan-Lusztig Polynomials

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Abstract. We consider the Kazhdan-Lusztig polynomials $P_{u,v}(q)$ indexed by permutations u, v having particular forms with regard to their monotonicity patterns. The main results are the following. First we obtain a simplified recurrence relation satisfied by $P_{u,v}(q)$ when the maximum value of $v \in S_n$ occurs in position $n-2$ or $n-1$. As a corollary we obtain the explicit expression for $P_{e,3\,4\,\ldots\,n\,1\,2}(q)$ (where e denotes the identity permutation), as a q-analogue of the Fibonacci number. This establishes a conjecture due to M. Haiman. Second, we obtain an explicit expression for $P_{e,3\,4\,\ldots\,(n-2)\,n\,(n-1)\,1\,2}(q)$. Our proofs rely on the recurrence relation satisfied by the Kazhdan-Lusztig polynomials when the indexing permutations are of the form under consideration, and on the fact that these classes of permutations lend themselves to the use of induction. We present several conjectures regarding the expression for $P_{u,v}(q)$ under hypotheses similar to those of the main results.

Keywords: Kazhdan-Lusztig polynomial, q-Fibonacci number, Bruhat order

1. Introduction

In their seminal 1979 paper [8], D. Kazhdan and G. Lusztig introduced a family of polynomials, $P_{u,v}(q)$, indexed by pairs of elements u, v of a Coxeter group. In the case when u and v are permutations, the polynomials have become known as the *Kazhdan-Lusztig polynomials for the symmetric group*. The interest in these polynomials and the broader circle of ideas of Kazhdan-Lusztig Theory to which they belong, is multifold. From the beginning of their study, the Kazhdan-Lusztig polynomials were shown to be related to Lie theory, and a significant body of literature treats their relation to the geometry of Schubert varieties, representation theory, and the (strong) Bruhat order of the symmetric group. The latter and a variety of properties of and conjectures concerning the Kazhdan-Lusztig polynomials make them interesting and challenging objects of study from the point of view of combinatorics. Developments along these lines include combinatorial derivations for the

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Kazhdan-Lusztig polynomials (see, e.g., the account in [7], section 7.12, [1, 4, 5]) as well as explicit formulas for Kazhdan-Lusztig polynomials for particular classes of permutations (see, e.g., [9, 10, 12, 14]).

The present paper contributes results in the direction of explicit formulae for the Kazhdan-Lusztig polynomials of the symmetric group when the indexing permutations are of particular forms. The main results, appearing in Section 3, are as follows. First we obtain a simplified recurrence relation satisfied by $P_{u,v}(q)$ for $u, v \in S_n$, when the maximum value of v occurs in position $n-2$ or $n-1$ (Theorem 3.1). As a corollary (Corollary 3.2), we obtain the explicit expression for $P_{e,3\,4\,\ldots\,n\,1\,2}(q)$ (where e denotes the identity permutation), as a q-analogue of the Fibonacci number. This establishes a conjecture due to M. Haiman (see [3, Conjecture 7.18]). Second, we obtain an explicit expression for $P_{e,3\,4\,\ldots\,(n-2)\,n\,(n-1)\,1\,2}(q)$ (Theorem 3.3). Our proofs rely on the recurrence relation satisfied by the Kazhdan-Lusztig polynomials when the indexing permutations are of the form under consideration, and on the fact that these classes of permutations lend themselves to the use of induction. Section 4 presents several conjectures regarding the expression for $P_{u,v}(q)$ under hypotheses similar to those of the main results.

As a starting point, in the next section we fix the notation and provide the necessary preliminaries concerning the Bruhat order. We also include a subset of facts about the Kazhdan-Lusztig polynomials which are used in proving the results of this paper.

2. Definitions, notation, and preliminaries

In this section we collect some definitions, notation and results that will be used in the rest of this paper. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$, and $\mathbf{Z}$ be the set of integers, and $\mathbf{R}$ be the field of real numbers; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, \ldots, a\}$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). Given $n, m \in \mathbf{P}$, $n \leq m$, we let $[n, m] \stackrel{\text{def}}{=} [m]\backslash[n-1]$. We write $S = \{a_1, \ldots, a_r\}_<$ to mean that $S = \{a_1, \ldots, a_r\}$ and $a_1 < \cdots < a_r$. The cardinality of a set A will be denoted by $|A|$. Given a polynomial $P(q)$ and $i \in \mathbf{N}$, we will denote by $[q^i](P(q))$ the coefficient of q^i in $P(q)$. Given $a \in \mathbf{R}$, we denote by $\lfloor a \rfloor$ the largest integer $\leq a$.

Given a set T we let $S(T)$ be the set of all bijections $\pi : T \to T$. In particular, $S_n \stackrel{\text{def}}{=} S([n])$ is the symmetric group on n elements. We denote by e the identity of S_n. If $T = \{t_1, \ldots, t_r\}_< \subseteq \mathbf{P}$ and $\sigma \in S(T)$, then we write $\sigma = \sigma_1 \cdots \sigma_r$ to mean that $\sigma(t_i) = \sigma_i$, for $i = 1, \ldots, r$. If $\sigma \in S_n$ then we also write σ in *disjoint cycle form* (see, e.g., [13], p. 17) and we will usually omit the 1-cycles of σ. For example, $\sigma = 365492187 \in S_9$ can be written alternatively as $\sigma = (9, 7, 1, 3, 5)(2, 6)$. Given $\sigma, \tau \in S_n$, we let $\sigma\tau \stackrel{\text{def}}{=} \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$. Given $\sigma \in S_n$, the *descent set of* σ is

$$D(\sigma) \stackrel{\text{def}}{=} \{i \in [n-1] : \sigma(i) > \sigma(i+1)\},$$

the *number of descents of* σ is

$$d(\sigma) = |D(\sigma)|,$$

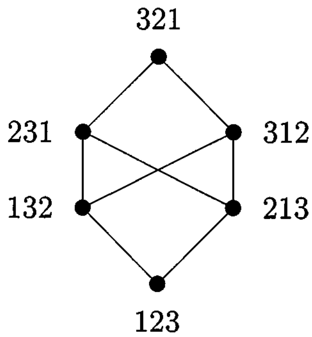

Figure 1. The Bruhat order on S_3.

and the *length of* σ is defined by the number of inversions:

$$l(\sigma) \stackrel{\text{def}}{=} \text{inv}(\sigma) \stackrel{\text{def}}{=} |\{(a, b) \in [n] \times [n] : a < b, \sigma(a) > \sigma(b)\}|.$$

For example, if $\sigma = 615243$ then $D(\sigma) = \{1, 3, 5\}$ and $l(\sigma) = 9$.

Throughout this paper we view S_n as a partially ordered set ordered by the (strong) *Bruhat order*. Recall (see, e.g., [11], Chapter 1) that this means that $\sigma \leq \tau$ if and only if there exist $r \in \mathbf{N}$ and $a_1, b_1, \ldots, a_r, b_r \in [n]$ such that $(a_r, b_r) \cdots (a_1, b_1)\sigma = \tau$ and $l((a_i, b_i) \cdots (a_1, b_1)\sigma) > l((a_{i-1}, b_{i-1}) \cdots (a_1, b_1)\sigma)$ for each $i = 1, \ldots, r$. For example, the Hasse diagram of the Bruhat order on S_3 is shown in figure 1.

The following characterization of the Bruhat order of S_n is due to Ehresmann [6], and will be used often in this work. We refer the reader to, e.g., [11], Chapter 1, for a proof. For $\sigma \in S_n$, and $j \in [n]$, let

$$\{\sigma^{j,1}, \ldots, \sigma^{j,j}\}_< \stackrel{\text{def}}{=} \{\sigma(1), \ldots, \sigma(j)\}. \tag{1}$$

Theorem 2.1 *Let $\sigma, \tau \in S_n$. Then $\sigma \leq \tau$ if and only if $\sigma^{j,i} \leq \tau^{j,i}$ for all $1 \leq i \leq j \leq n - 1$.*

For example, if $\sigma = 4123$ and $\tau = 2431$ then $(\sigma^{1,1}, \sigma^{2,1}, \sigma^{2,2}, \ldots, \sigma^{3,3}) = (4, 1, 4, 1, 2, 4)$ and $(\tau^{1,1}, \tau^{2,1}, \tau^{2,2}, \ldots, \tau^{3,3}) = (2, 2, 4, 2, 3, 4)$ and hence σ and τ are incomparable in Bruhat order.

The following result is fundamental in the theory of Kazhdan and Lusztig and is presented, e.g., in [7], §7.11, Eq. (23). Here it will serve as the definition of the *Kazhdan-Lusztig polynomials* $P_{u,v}(q)$ of S_n. It is interesting to note that no combinatorial proof of Theorem 2.2 is known.

Theorem 2.2 *There exists a unique family of polynomials $\{P_{u,v}(q)\}_{u,v \in S_n} \subseteq \mathbf{Z}[q]$ such that*:

i) $P_{u,v}(q) = 0$ *if* $u \not\leq v$;

ii) $P_{u,v}(q) = 1$ *if* $u = v$;
iii) *if* $u \leq v$ *and* $i \in D(v)$ *then*

$$P_{u,v}(q) = q^{1-c} P_{u(i,i+1),v(i,i+1)}(q) + q^c P_{u,v(i,i+1)}(q) - \sum_{\{z : i \in D(z)\}} q^{\frac{l(v)-l(z)}{2}} \mu(z, v(i, i+1)) P_{u,z}(q)$$

where for $u, w \in S_n$,

$$\mu(u, w) \stackrel{\text{def}}{=} \begin{cases} [q^{\frac{1}{2}(l(w)-l(u)-1)}](P_{u,w}(q)), & \text{if } u < w, \\ 0, & \text{otherwise}, \end{cases}$$

$c = 1$ *if* $i \in D(u)$, *and* $c = 0$ *otherwise.*

Two well-known simple but important consequences of Theorem 2.2 are the following (see, e.g., [7, Theorem 7.9, part (b), and Corollary 7.14]).

Proposition 2.3 *Let* $u, v \in S_n$ *such that* $u < v$. *Then* $deg(P_{u,v}(q)) \leq \frac{1}{2}(l(v) - l(u) - 1)$.

Proposition 2.4 *Let* $u, v \in S_n$ *such that* $u < v$. *If* $i \in D(v)$, *then*

$$P_{u,v}(q) = P_{u(i,i+1),v}(q).$$

So, for example, $P_{2\,1\,4\,7\,5\,6\,3,6\,1\,5\,7\,2\,4\,3}(q) = P_{1\,2\,4\,5\,7\,3\,6,6\,1\,5\,7\,2\,4\,3}(q)$, and

$$P_{u,n\,n-1\ldots 3\,2\,1}(q) = 1, \tag{2}$$

for all $u \in S_n$. Thus Proposition 2.4 shows that it is enough to compute Kazhdan-Lusztig polynomials $P_{u,v}(q)$ for pairs $u, v \in S_n$ such that $D(v) \subseteq D(u)$ (say).

The following result is an immediate consequence of Propositions 2.3 and 2.4.

Proposition 2.5 *Let* $z, w \in S_n$, $z \leq w$, *be such that* $\mu(z, w) \neq 0$ *and* $l(w) - l(z) > 1$. *Then* $D(z) \supseteq D(w)$.

The preceding proposition motivates the following definition. For $w \in S_n$ and $i \in [n]$ we let

$$E(w, i) \stackrel{\text{def}}{=} \{z \in S_n : z \leq w, D(z) \supseteq D(w) \cup \{i\},\ l(w) - l(z) > 1\} \cup \{z \in S_n : z \lhd w,\ D(z) \ni i\}, \tag{3}$$

where the notation $z \lhd w$ means that z *is covered by* w, that is, $z < w$ and if $z \leq t \leq w$ then $z = t$ or $t = w$.

Then by Proposition 2.5 we deduce from Theorem 2.2 the following result.

Corollary 2.6 *Let $u, v \in S_n$, $u \le v$, and $i \in D(v)$. Then*

$$P_{u,v}(q) = q^{1-c} P_{u(i,i+1),v(i,i+1)}(q) + q^c P_{u,v(i,i+1)}(q) - \sum_{z \in E(v(i,i+1),i)} \mu(z, v(i,i+1)) q^{\frac{1}{2}(l(v)-l(z))} P_{u,z}(q),$$

where $c = 1$ if $i \in D(u)$, and $c = 0$ otherwise.

Two other properties of the Kazhdan-Lusztig polynomials that we will use are the following.

Proposition 2.7 *Let $u, v \in S_n$. Then*

$$P_{u,v}(q) = P_{u^{-1},v^{-1}}(q) = P_{n+1-u(n)\ldots n+1-u(1),n+1-v(n)\ldots n+1-v(1)}(q). \tag{4}$$

The final result of this section will be applied repeatedly in the sequel, in a special case: if $u < v$ are permutations in S_n and n occurs in the same position in both u and v, then $P_{u,v}(q) = P_{u',v'}(q)$, where u', v' are obtained from u, v, respectively, by suppressing the value n.

Let $\sigma \in S_n$, and $i, j \in [n]$, $i \le j$. We define the *restriction* of σ to $[i, j]$ to be the unique permutation $\sigma_{[i,j]} \in S([i, j])$ such that

$$\sigma^{-1}\big(\sigma_{[i,j]}(i)\big) < \sigma^{-1}\big(\sigma_{[i,j]}(i+1)\big) < \cdots < \sigma^{-1}\big(\sigma_{[i,j]}(j)\big).$$

For example, if $\sigma = 7251634$ then $\sigma_{[3,5]} = 534$ (i.e, $\sigma_{[3,5]}(3) = 5$, $\sigma_{[3,5]}(4) = 3$, $\sigma_{[3,5]}(5) = 4$). Note that $\sigma_{[i,j]}$ is the identity permutation in $S([i, j])$ if and only if $\sigma^{-1}(i) < \sigma^{-1}(i+1) < \cdots < \sigma^{-1}(j)$.

Theorem 2.8 *Let $u, v \in S_n$, $u \le v$. Suppose that there exist $i \in [n]$ such that $u^{-1}([i]) = v^{-1}([i])$. Then*

$$P_{u,v}(q) = P_{u_{[i]},v_{[i]}}(q) P_{u_{[i+1,n]},v_{[i+1,n]}}(q). \tag{5}$$

Proofs of the equalities in (4) and (5) appear in [1, Corollaries 4.3 and 4.4], and in [2, Theorem 4.4], respectively.

3. Main results

Theorem 3.1 *Let $u, v \in S_n$, $u \le v$, be such that $v^{-1}(n) \in \{n-2, n-1\}$. Then*

$$P_{u,v}(q) = \begin{cases} q^{1-c} P_{u(i,i+1),v(i,i+1)}(q) + q^c\, P_{u,v(i,i+1)}(q) & \text{if } i+1 \notin D(v), \\ q^{1-c} P_{u(i,i+1),v(i,i+1)}(q) + q^c\, P_{u,v(i,i+1)}(q) - q P_{u,v(n,n-2,n-1)}(q), & \\ & \text{if } i+1 \in D(v), \end{cases}$$

where $i \stackrel{\text{def}}{=} v^{-1}(n)$, $c \stackrel{\text{def}}{=} 1$ if $i \in D(u)$, and $c \stackrel{\text{def}}{=} 0$ otherwise.

Proof: Let $i = v^{-1}(n)$. Then $i \in D(v)$ and Corollary 2.6 yields

$$P_{u,v}(q) = q^{1-c} P_{u(i,i+1),v(i,i+1)}(q) + q^c P_{u,v(i,i+1)}(q) - \sum_{z \in E} \mu(z, v(i, i+1)) q^{\frac{1}{2}(l(v)-l(z))} P_{u,z}(q), \tag{6}$$

where $c = 1$ if $i \in D(u)$ and $c = 0$ otherwise, and $E = E(v(i, i+1), i)$ as defined in 3. In particular, $z \in E$ satisfies $z \le v(i, i+1)$ and $z(i) > z(i+1)$.

Suppose first that $i = n-2$. Then $z \le v(n-2, n-1)$ and Theorem 2.1 imply $z^{-1}(n) \in \{n-1, n\}$. Together with $z(i) > z(i+1)$, this forces $z^{-1}(n) = n$ and therefore that $n-1 \notin D(z)$. But $n-1 \in D(v(n-2, n-1))$, so Proposition 2.5 implies that $\mu(z, v(n-2, n-1)) = 0$ if $l(v(n-2, n-1)) - l(z) > 1$, and we can restrict our attention to $z \in E$ such that $z \lhd v(n-2, n-1)$. Since $z^{-1}(n) = n$, the only candidate for $z \lhd v(n-2, n-1)$ which belongs to E is $z = v(n, n-2, n-1)$. This is indeed in E if and only if $n-1 \in D(v)$. Therefore

$$\sum_{z \in E} \mu(z, v(n-2, n-1)) q^{\frac{1}{2}(l(v)-l(z))} P_{u,z}(q) = \begin{cases} 0, & \text{if } n-1 \notin D(v), \\ q\, P_{u,v(n,n-2,n-1)}(q), & \text{if } n-1 \in D(v), \end{cases}$$

and the desired result follows.

Suppose now that $i = n-1$. As before, $z \le v(n-1, n)$ and Theorem 2.1 imply $z^{-1}(n) = n$, but now we have $i = n-1 \notin D(z)$. Therefore $E = \emptyset$. Also, $i+1 = n \notin D(v)$, by the definition of the descent set. So the result again follows. □

For $n \in \mathbf{P}$ consider the q-analogue $F_n(q)$ of the Fibonacci number F_n defined by

$$F_n(q) \stackrel{\text{def}}{=} F_{n-1}(q) + qF_{n-2}(q),$$

where $F_n(q) \stackrel{\text{def}}{=} 0$ if $n < 0$ and $F_0(q) \stackrel{\text{def}}{=} 1$. It is an easy exercise to verify that for $n \ge 0$,

$$F_n(q) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} q^i.$$

Corollary 3.2 *Let $n \ge 3$. Then*

$$P_{e,3\,4\ldots n\,1\,2}(q) = F_{n-2}(q).$$

Proof: We proceed by induction on $n \ge 3$, the result being easily verified for $n = 3, 4$. From Theorem 3.1 (the instance $i = n-2$, $c = 0$, $i+1 \notin D(v)$) we obtain that

$$P_{e,3\,4\ldots n\,1\,2}(q) = q P_{(n-2,n-1),3\,4\ldots n-1\,1\,n\,2}(q) + P_{e,3\,4\ldots n-1\,1\,n\,2}(q).$$

By applying Proposition 2.4 (in turn, with $i = n-3$ and $i = n-1$), then Theorem 2.8 (in turn, with $i = n-1$, and with $i = n-2$), and then the induction hypothesis, we have

$$\begin{aligned} P_{(n-2,n-1),3\,4\ldots n-1\,1\,n\,2}(q) &= P_{1\,2\ldots n-4\,n-1\,n-3\,n\,n-2,3\,4\ldots n-1\,1\,n\,2}(q) \\ &= P_{1\,2\ldots n-3\,n-2,3\,4\ldots n-2\,1\,2}(q) \\ &= F_{n-4}(q), \end{aligned}$$

and similarly

$$\begin{aligned} P_{e,3\,4\,\ldots n-1\,1\,n\,2}(q) &= P_{1\,2\ldots n-2\,n\,n-1,3\,4\ldots n-1\,1\,n\,2}(q) \\ &= P_{1\,2\ldots n-2\,n-1,3\,4\ldots n-1\,1\,2}(q) \\ &= F_{n-3}(q) \end{aligned}$$

and the result follows. □

So, for example, $P_{1234567,3456712}(q) = 1 + 4q + 3q^2$. Corollary 3.2, by Proposition 2.7, verifies a conjecture of M. Haiman ([3, Conjecture 7.18]).

It should be noted that the reasonings made to prove the last two results actually prove that

$$C_{3\,4\ldots n\,1\,2} = C_{(2,3)}C_{(3,4)}\cdots C_{(n-1,n)}C_{(1,2)}C_{(2,3)}\cdots C_{(n-2,n-1)}$$

in the Hecke algebra $\mathcal{H}$ of S_n, where C_σ denotes the Kazhdan-Lusztig element of $\mathcal{H}$ corresponding to $\sigma \in S_n$ (see [7], Chap. 7, for the definitions, and further information about, these objects). This, in theory, should allow one to compute explicitly the Kazhdan-Lusztig polynomials $P_{\sigma,3\,4\ldots n\,1\,2}$ for all $\sigma \in S_n$. However, we have ben unable to carry this out because of the complicated nature of multiplication in the Hecke algebra $\mathcal{H}$.

There are other pairs of permutations whose Kazhdan-Lusztig polynomials can be computed in a similar way.

Theorem 3.3 *Let $n \geq 5$. Then*

$$P_{e,3\,4\ldots n-2\,n\,n-1\,1\,2}(q) = F_{n-3}(q).$$

Proof: By Corollary 2.6 with $i = n-2$, we have

$$\begin{aligned} &P_{e,3\,4\ldots n-2\,n\,n-1\,1\,2}(q) \\ &\quad = q\,P_{(n-2,n-1),3\,4\ldots n-2\,n\,1\,n-1\,2}(q) + P_{e,3\,4\ldots n-2\,n\,1\,n-1\,2}(q) \\ &\qquad - \sum_{z\in E} q^{\frac{1}{2}(l(v)-l(z))}\mu(z, 3\,4\ldots n-2\;n\;1\;n-1\;2)P_{e,z}(q), \end{aligned} \tag{7}$$

where $E = E(3\,4\ldots n-2\;n\;1\;n-1\;2,\;n-2)$. We claim that E consists of only one permutation. To verify this claim, suppose $z \in E$. Thus, $z < 3\,4\ldots n-2\;n\;1\;n-1\;2$.

If $l(z) < l(3\ 4 \dots n-2\ n\ 1\ n-1\ 2) - 1$ then the condition on descents requires $z(n-3) > z(n-2) > z(n-1) > z(n)$. But together with Theorem 2.1, which implies $z^{-1}(n) \geq n-3$, this forces $z^{-1}(n) = n-3$. Now, from Theorem 2.1 we obtain further that $z^{-1}(n-1) \geq n-1$ which contradicts the condition on descents for z. Therefore if $z \in E$, then $z \lhd 3\ 4 \dots n-2\ n\ 1\ n-1\ 2$. But then z is obtained from $3\ 4 \dots n-2\ n\ 1\ \ n-1\ 2$ by transposing two elements and, since we need $z(n-2) > z(n-1)$, we conclude that $z = 3\ 4 \dots n-2\ 1\ n\ n-1\ 2$. Thus E is a singleton set as claimed and Eq. (7) can be rewritten explicitly as

$$P_{e,3\,4\dots n-2\,n\,n-1\,1\,2}(q) = q\,P_{(n-2,n-1),3\,4\dots n-2\,n\,1\,n-1\,2}(q) + P_{e,3\,4\dots n-2\,n\,1\,n-1\,2}(q) - q\,P_{e,3\,4\dots n-2\,1\,n\,n-1\,2}(q). \tag{8}$$

We now examine each of the terms on the right-hand-side of (8). Consider the Kazhdan-Lusztig polynomial from the first term, $P_{(n-2,n-1),3\,4\dots n-2\,n\,1\,n-1\,2}(q)$, and apply to it Corollary 2.6 with $i = n-3$. For simplicity of notation, let now $E' := E(3\ 4 \dots n-2\ 1\ n\ n-1\ 2,\ n-3)$. We show that $E' = \emptyset$. Suppose to the contrary that $z \in E'$. If $l(z) < l(3\ 4 \dots n-2\ 1\ n\ n-1\ 2) - 1$ then $z(n-4) > \cdots > z(n-1) > z(n)$ which implies that $z^{-1}(n) \leq n-4$ and this, by Theorem 2.1, contradicts the fact that $z \leq 3\ 4 \dots n-2\ 1\ n\ n-1\ 2$. If $z \lhd 3\ 4 \dots n-2\ 1\ n\ n-1\ 2$ then $z(n-3) > z(n-2)$ and this is again a contradiction. Therefore $E' = \emptyset$ and Corollary 2.6 with $i = n-3$ yields

$$P_{(n-2,n-1),3\,4\dots n-2\,n\,1\,n-1\,2}(q) = q\,P_{1\dots n-4\,n-1\,n-3\,n-2\,n,3\,4\dots n-2\,1\,n\,n-1\,2}(q) + P_{(n-2,n-1),3\,4\dots n-2\,1\,n\,n-1\,2}(q). \tag{9}$$

Now note that by Theorem 2.2, the first term on the right-hand-side of (9) is null, since (using, e.g., Theorem 2.1) $1 \dots n-4\ n-1\ n-3\ n-2\ n \not\leq 3\ 4 \dots n-2\ 1\ n\ n-1\ 2$. In turn, the second term on the right-hand-side of (9) can be evaluated explicitly:

$$\begin{aligned} P_{(n-2,n-1),3\,4\dots n-2\,1\,n\,n-1\,2}(q) &= P_{1\dots n-3\,n\,n-1\,n-2,3\,4\dots n-2\,1\,n\,n-1\,2}(q) \\ &= P_{1\dots n-3\,n-2,3\,4\dots n-2\,1\,2}(q) \\ &= F_{n-4}(q). \end{aligned} \tag{10}$$

In (10), the first equality follows from Proposition 2.4 (applied with $i = n-1$, and $n-2$), the second one follows from Theorem 2.8 (suppressing n and $n-1$), and the third from Corollary 3.2. Consequently, we have obtained that the first term on the right-hand-side of Eq. (8) is

$$q\,P_{(n-2,n-1),3\,4\dots n-2\,n\,1\,n-1\,2}(q) = q F_{n-4}(q). \tag{11}$$

Similarly, since $E' = \emptyset$, the second term on the right-hand-side of (8) is

$$P_{e,3\,4\dots n-2\,n\,1\,n-1\,2}(q) = q\,P_{(n-3,n-2),3\,4\dots n-2\,1\,n\,n-1\,2}(q) + P_{e,3\,4\dots n-2\,1\,n\,n-1\,2}(q). \tag{12}$$

In (12), the Kazhdan-Lusztig polynomial in the first term of the right-hand-side is

$$\begin{aligned} P_{(n-3,n-2),3\,4\ldots n-2\;1\;n\;n-1\;2}(q) &= P_{1\ldots n-4\;n-2\;n\;n-1\;n-3,3\,4\ldots n-2\;1\;n\;n-1\;2}(q) \\ &= P_{1\ldots n-4\;n-2\;n-3,3\,4\ldots n-2\;1\;2}(q), \end{aligned} \tag{13}$$

by applying Proposition 2.4 and Theorem 2.8 (similarly to the situation in (10)). By Theorem 3.1,

$$\begin{aligned} P_{1\ldots n-4\;n-2\;n-3,3\,4\ldots n-2\;1\;2}(q) = q\; &P_{1\ldots n-5\;n-2\;n-4\;n-3,3\,4\ldots n-3\;1\;n-2\;2}(q) \\ &+ P_{1\ldots n-4\;n-2\;n-3,3\,4\ldots n-3\;1\;n-2\;2}(q). \end{aligned} \tag{14}$$

Since $1\ldots n-5\;n-2\;n-4\;n-3 \not\leq 3\;4\ldots n-3\;1\;n-2\;2$, the first term on the right-hand-side of (14) is null, and by Theorem 2.8, the second term is equal to $P_{1\ldots n-4\;n-3,3\,4\ldots n-3\;1\;2}(q)$. Also, this last polynomial is as in Corollary 3.2, so (14) becomes

$$P_{1\ldots n-4\;n-2\;n-3,3\,4\ldots n-2\;1\;2}(q) = F_{n-5}(q). \tag{15}$$

Combined with Eq. (13) this gives

$$P_{(n-3,n-2),3\,4\ldots n-2\;1\;n\;n-1\;2}(q) = F_{n-5}(q). \tag{16}$$

Finally, the second term on the right-hand-side of (12) can be computed similarly to the calculation in (10), and we obtain

$$P_{e,3\,4\ldots n-2\;1\;n\;n-1\;2}(q) = P_{1\ldots n-3\;n\;n-1\;n-2,3\,4\ldots n-2\;1\;n\;n-1\;2}(q) = F_{n-4}(q). \tag{17}$$

Substituting (16) and (17) into (12) we obtain

$$P_{e,3\,4\ldots n-2\;n\;1\;n-1\;2}(q) = q\;F_{n-5}(q) + F_{n-4}(q) = F_{n-3}(q). \tag{18}$$

Finally, the third term on the right-hand-side of (8) contains the same Kazhdan-Lusztig polynomial as in (17). By substituting (11), (18), and (17), into the initial Eq. (8) the proof is completed:

$$P_{e,3\,4\ldots n-2\;n\;n-1\;1\;2}(q) = q\;F_{n-4}(q) + F_{n-3}(q) - q\;F_{n-4}(q). \tag{19}$$

□

4. Conjectures and open problems

In this section we collect a variety of conjectures concerning Kazhdan-Lusztig polynomials which we have obtained empirically. Although many of the permutations appearing in these conjectures are quite similar to the ones considered in this note, the resulting Kazhdan-Lusztig polynomials are rather different, and we have been unable to prove them.

Conjecture 4.1 Let $i \in \mathbf{P}$. Then

$$P_{1\ n-1\ n-2 \dots n-i\ 2\ 3 \dots n-i-1\ n, n-1\ n\ n-2 \dots n-i\ 1\ 2 \dots n-i-1}(q) = 1 + (n - i - 2)q^i$$

for $n \geq i + 2$.

Conjecture 4.2 Let $n \geq 6$. Then

$$P_{e, n-2\ n-1\ n\ n-3 \dots 4\ 3\ 1\ 2}(q) = 1 + 2q^{n-4}.$$

Conjecture 4.3 Let $n \geq 6$. Then

$$P_{e, n-2\ n-1\ n\ n-3 \dots 4\ 2\ 1\ 3}(q) = 1 + 2q^{n-5} + q^{n-4}.$$

The above conjectures have been verified for $n \leq 8$. Note that Conjecture 4.1 generalizes both Conjectures 7.15 and 7.16 of [3].

Note added in proof: Conjecture 4.1 has been proved by P. Polo, *Construction of arbitrary Kazhdan-Lusztig polynomials in symmetric groups*, Representation Theory, **3** (1999), 90–104.

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