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Cyclic Characters of Symmetric Groups

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Dedicated to Dieter Blessenohl on the occasion of his sixtieth birthday

Abstract. We consider characters of finite symmetric groups induced from linear characters of cyclic subgroups. A new approach to Stembridge's result on their decomposition into irreducible components is presented. In the special case of a subgroup generated by a cycle of longest possible length, this amounts to a short proof of the Kraśkiewicz-Weyman theorem.

Keywords: symmetric group, Young tableau, multi major index, induced character, Lie idempotent

In a remarkable paper of 1987, Kraśkiewicz and Weyman described the decomposition of certain characters of the symmetric group S_n into irreducible components [6]. Let C be a subgroup generated by a cycle σ of order n. Denote by ψ_i the character of C mapping σ onto the i-th power of a primitive n-th root of unity. Then the multiplicity $({\psi_i}^{S_n}, \zeta^p)_{S_n}$ of the irreducible character ζ^p indexed by the partition p of n in ${\psi_i}^{S_n}$ equals the number of standard Young tableaux of shape p and major index congruent i modulo n. Another proof of this theorem has been given by Garsia [2], see also Chapter 8 in [8].

More generally, like Stembridge in [11] we consider characters ψ^{S_n} over the field $\mathbb{C}$ of complex numbers, where ψ is a linear character of an arbitrary cyclic subgroup Z. We call them *cyclic* characters of S_n. In order to give a combinatorial description of the occurring multiplicities $(\psi^{S_n}, \zeta^p)_{S_n}$ we use the notion of a *multi major index*, which is a tuple of major indices defined in segments. For the special case $Z = C$ we obtain exactly the result of Kraśkiewicz and Weyman, hence giving a new proof of it.

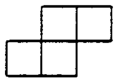
The method we use is different from that presented by Stembridge: Making use of a certain Lie idempotent introduced by Klyachko [5], our proof is based on the *noncommutative character theory of symmetric groups*, contained in the first author's thesis [4] that is shortly summarized in the first section. The second section contains the theorem and its proof.

1. The frame algebra

Let $\mathbb{N}$ ($\mathbb{N}_0$, resp.) be the set of all positive (nonnegative, resp.) integers and $\mathbb{N}^*$ a free monoid with alphabet $\mathbb{N}$. A word $q = q_1 \cdots q_k \in \mathbb{N}^*$ is called a composition of n iff $q_1 + \cdots + q_k = n$. We denote by C_q the conjugacy class containing all permutations $\pi \in S_n$

whose cycle partition is a rearrangement of q. Let ch_q be the class function of S_n such that $(\chi, \mathrm{ch}_q)_{S_n} = \chi(C_q)$ for all class functions χ of S_n, i.e., up to a scalar factor ch_q is the characteristic function of C_q in S_n. For the outer product $\bullet$ in the algebra $\mathcal{C} := \bigoplus_{n \in \mathbb{N}} \mathcal{C}\ell_{\mathbb{C}} S_n$ of all class functions we then have the multiplication rule $\mathrm{ch}_q \bullet \mathrm{ch}_r = \mathrm{ch}_{qr}$ for all $q, r \in \mathbb{N}^*$. Using this algebra $\mathcal{C}$, the character theory of symmetric groups can be elegantly described. For details, including a coproduct and hence a bialgebra structure on $\mathcal{C}$, see [3].

In the first author's thesis [4], a noncommutative analogue of this bialgebra $\mathcal{C}$ of class functions is presented. The main idea behind it is to consider algebraic structures consisting of Young tableaux: Let $\leq$ be the partial order on $\mathbb{Z} \times \mathbb{Z}$ ($\mathbb{Z}$ the set of all integers) defined by: $(u, v) \leq (x, y)$ iff $u \leq x$ and $v \leq y$. A finite subset R of $\mathbb{Z} \times \mathbb{Z}$ is called a *frame* if it is convex with respect to $\leq$. E.g., $S = \{(1, 2), (1, 3), (2, 1), (2, 2)\}$ is a frame and may be illustrated by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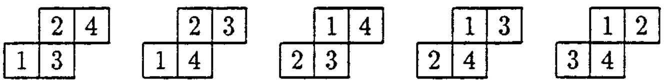
The following version of a well known concept is convenient for our purposes. Let R be a frame. A *standard Young tableau* of shape R is a permutation π with the following property: Filled into R row by row, starting from bottom left and ending at top right, π is increasing in rows (from left to right) and columns (downwards). The set of all these permutations is denoted by SYT^R. In the group ring $\mathbb{C}S_n$ of S_n (where $n = |R|$), we may then form the sum of all elements of SYT^R and set $\mathrm{Z}^R := \sum \mathrm{SYT}^R$. For the frame S mentioned above we have the following standard Young tableaux:



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Hence, $\mathrm{Z}^S = 1324 + 1423 + 2314 + 2413 + 3412 \in \mathbb{C}S_4$.

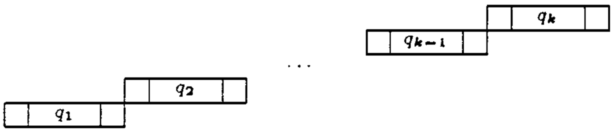
Corresponding to any partition $p = p_1 p_2 \cdots p_k \in \mathbb{N}^*$ ($p_1 \geq \cdots \geq p_k$) there is the frame $R(p) = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq i \leq k, 1 \leq j \leq p_i\}$. We write SYT^p, Z^p instead of $\mathrm{SYT}^{R(p)}$, $\mathrm{Z}^{R(p)}$ resp. .

In [4] the linear subspace $\mathcal{R}$ of $\mathbb{C}S := \bigoplus_{n \in \mathbb{N}} \mathbb{C}S_n$ is introduced as the $\mathbb{C}$-linear span of all elements Z^R (R frame). Furthermore, a product $\bullet$ on $\mathcal{R}$ and an algebra epimorphism $c : (\mathcal{R}, \bullet) \to (\mathcal{C}, \bullet)$ are defined such that $(\phi, \psi) = (c(\phi), c(\psi))_S$ for all $\phi, \psi \in \mathcal{R}$, where the bilinear mapping on the left hand side is given by

$$(\sigma, \tau) := \begin{cases} 1 & \text{if } \sigma = \tau^{-1} \\ 0 & \text{if } \sigma \neq \tau^{-1} \end{cases} \quad \text{for all permutations } \sigma, \tau$$

on $\mathbb{C}S$ and the one on the right hand side is the canonical orthogonal extension $(\cdot, \cdot)_S$ of the scalar products $(\cdot, \cdot)_{S_n}$.

If $q = q_1 q_2 \cdots q_k$ is a composition of $n \in \mathbb{N}$ and R is the frame illustrated by



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then the image of $\Xi^q := \mathrm{Z}^R$ under c is the permutation character $\xi^q = (1_Y)^{S_n}$ related to any Young subgroup Y of type q. Furthermore, $\Xi^q \bullet \Xi^r = \Xi^{qr}$ for all $q, r \in \mathbb{N}^*$. It should be mentioned that the so-called *frame algebra* $\mathcal{R}$ contains the direct sum $\mathcal{D}$ of all *descent algebras* $\mathcal{D}_n = \langle \Xi^q \mid q$ composition of $n \rangle_{\mathbb{C}}$ discovered by Solomon [9].

The crucial point is the fact that c is an extension of Solomon's epimorphism [9] and $c(\mathrm{Z}^p) = \zeta^p$ is the irreducible character of S_n corresponding to p for any partition p of n.

Now, let ω_n be the element of $\mathbb{C}S_n$ operating via Polya operation on any word $x_1 x_2 \cdots x_n$ of length n by $\omega_n\, x_1 x_2 \cdots x_n = [[\cdots[[x_1, x_2], x_3], \cdots], x_n]$, where $[x, y] = xy - yx$ denotes the Lie commutator of x and y. By the Dynkin-Specht-Wever theorem [1, 10, 12] ω_n is a Lie idempotent (up to the factor n), i.e., $\omega_n \omega_n = n\omega_n$. Furthermore, $\omega_n = \sum_{k=0}^{n-1} (-1)^k\, \mathrm{Z}^{(n-k)1^k} \in \mathcal{R}$, and $c(\omega_n) = \mathrm{ch}_n$.

2. Cyclic characters of symmetric groups

First of all, we present a construction of inverse images of the elements $\mathrm{ch}_q \in \mathcal{C}$ ($q \in \mathbb{N}^*$) under c based on Lie idempotents. Recall that $e \in \mathbb{C}S_n$ is a Lie idempotent up to the factor n iff $\omega_n e = ne$ and $e\omega_n = n\omega_n$.

Proposition 1 *For all $n \in \mathbb{N}$, let $e_n \in \mathcal{D}_n$ such that $\frac{1}{n}e_n$ is a Lie idempotent. Then, we have $c(e_{q_1} \bullet \cdots \bullet e_{q_k}) = \mathrm{ch}_q$ for all $q = q_1 \cdots q_k \in \mathbb{N}^*$.*

Proof: Let $n \in \mathbb{N}$. Then,

$$c(e_n) = \frac{1}{n} c(\omega_n e_n) = \frac{1}{n} c(\omega_n) c(e_n) = \frac{1}{n} c(e_n) c(\omega_n) = \frac{1}{n} c(e_n \omega_n) = c(\omega_n) = \mathrm{ch}_n$$

as c is an homomorphism with respect to the inner multiplication of $\mathcal{D}_n$ and $\mathcal{C}\ell_{\mathbb{C}} S_n$ by Solomon [9]. For any $q = q_1 \cdots q_k \in \mathbb{N}^*$, it follows that

$$c(e_{q_1} \bullet \cdots \bullet e_{q_k}) = c(e_{q_1}) \bullet \cdots \bullet c(e_{q_k}) = \mathrm{ch}_{q_1} \bullet \cdots \bullet \mathrm{ch}_{q_k} = \mathrm{ch}_q. \qquad \square$$

Let $n \in \mathbb{N}$. For all $\pi \in S_n$, we call $D(\pi) := \{ i \mid 1 \le i \le n-1 \text{ and } i\pi > (i+1)\pi \}$ the *descent set* of π. If $q = q_1 \cdots q_k \in \mathbb{N}^*$ is a composition of n, the *multi major index* of π with respect to q is defined to be the word of length n the j-th letter of which is

$$(\mathrm{maj}_q \pi)_j := \sum_{\substack{s_{j-1} < i < s_j \\ i \in D(\pi)}} (i - s_{j-1}) \qquad \text{for all } j \in \{1, \ldots, k\},$$

where $s_j := q_1 + \cdots + q_j$ for all $j \in \{0, \ldots, k\}$. In the special case of $q = n$, $\mathrm{maj}\,\pi = \mathrm{maj}_n\pi$ is the well known major index of π. For example, $\mathrm{maj}_{322}\, 5\,6\,2\,1\,3\,7\,4 = 2\,0\,1$ and $\mathrm{maj}_{43}\, 5\,6\,2\,1\,3\,7\,4 = 5\,2$. Let

$$\kappa_n(x) := \sum_{\pi \in S_n} x^{\mathrm{maj}\,\pi}\pi \qquad \text{(where } x \text{ is a variable).}$$

Then, for any primitive n-th root of unity ε, $\kappa_n(\varepsilon)$ is a Lie idempotent (up to the factor n) [5]. Let $q = q_1 \cdots q_k$ be a composition of n and

$$\kappa_q(x_1, \ldots, x_k) := \kappa_{q_1}(x_1) \bullet \cdots \bullet \kappa_{q_k}(x_k) \qquad \text{(where each } x_i \text{ is a variable).}$$

For any choice of primitive q_i-th roots of unity ε_i, we have $c(\kappa_q(\varepsilon_1, \ldots, \varepsilon_k)) = \mathrm{ch}_q$ by Proposition 1. We finally define, for all $j \in \mathbb{N}$,

$$q^{(j)} := \underbrace{\frac{q_1}{\gcd(q_1, j)} \cdots \frac{q_1}{\gcd(q_1, j)}}_{\gcd(q_1, j)\ \text{times}} \quad \cdots \quad \underbrace{\frac{q_k}{\gcd(q_k, j)} \cdots \frac{q_k}{\gcd(q_k, j)}}_{\gcd(q_k, j)\ \text{times}} \in \mathbb{N}^* \quad .$$

Then, if $\sigma \in S_n$ has cycle type q, $C_{q^{(j)}}$ is the conjugacy class of σ^j.

The definitions given so far lead to the following surprising result for $\kappa_q(x_1, \ldots, x_k)$:

Proposition 2 *Let $j \in \mathbb{N}, q = q_1 \cdots q_k \in \mathbb{N}^*$ and ε_i be an arbitrary q_i-th root of unity for all $i \in \{1, \ldots, k\}$. Then,*

$$\kappa_{q^{(j)}}\big(\underbrace{\varepsilon_1^j, \ldots, \varepsilon_1^j}_{\gcd(q_1, j)\ \text{times}}, \ldots, \underbrace{\varepsilon_k^j, \ldots, \varepsilon_k^j}_{\gcd(q_k, j)\ \text{times}}\big) = \kappa_q\big(\varepsilon_1^j, \ldots, \varepsilon_k^j\big).$$

Proof: For $q = n$, $\kappa_{d^{n/d}}(\varepsilon_1^j, \ldots, \varepsilon_1^j) = \kappa_n(\varepsilon_1^j)$ is a special case of [7], Proposition 4.1, where $d = q_1 / \gcd(q_1, j)$ and ε_1^j is a d-th root of unity. For arbitrary q, let $d_i := q_i / \gcd(q_i, j)$ for all $i \in \{1, \ldots, k\}$. Then, using the result of the special case in each factor, we obtain

$$\begin{aligned}
&\kappa_{q^{(j)}}\big(\varepsilon_1^j, \ldots, \varepsilon_1^j, \ldots, \varepsilon_k^j, \ldots, \varepsilon_k^j\big) \\
&= \kappa_{d_1^{q_1/d_1}}\big(\varepsilon_1^j, \ldots, \varepsilon_1^j\big) \bullet \cdots \bullet \kappa_{d_k^{q_k/d_k}}\big(\varepsilon_k^j, \ldots, \varepsilon_k^j\big) \\
&= \kappa_{q_1}\big(\varepsilon_1^j\big) \bullet \cdots \bullet \kappa_{q_k}\big(\varepsilon_k^j\big) \\
&= \kappa_q\big(\varepsilon_1^j, \ldots, \varepsilon_k^j\big).
\end{aligned}$$

□

We are now in a position to state and prove the main result about cyclic characters of symmetric groups:

Theorem *Let $n \in \mathbb{N}, q = q_1 \cdots q_k$ be a composition of n, $\upsilon := \mathrm{lcm}(q_1, \ldots, q_k)$, η a primitive υ-th root of unity and $e_1, \ldots, e_k \in \mathbb{N}_0$ such that η^{e_j} is a primitive q_j-th root* *of unity for all* $j \in \{1, \ldots, k\}$. *Let* $\sigma \in C_q$, Z *be the subgroup of* S_n *generated by* σ, $i \in \{0, \ldots, v-1\}$ *and* $\psi_i : Z \longrightarrow K$, $\sigma^j \longmapsto \eta^{ij}$. *Then,*

$$\mathrm{M}^q_{(i)} := \sum \left\{ \pi \in S_n \;\middle|\; \sum_{j=1}^{k} e_j (\mathrm{maj}_q \pi)_j \equiv i \quad \text{modulo } v \right\}$$

is an element of $\mathcal{D}$, *and we have*

$$c\big(\mathrm{M}^q_{(i)}\big) = \psi_i^{S_n} \quad .$$

In particular, for any partition p *of* n,

$$\begin{aligned}\big(\psi_i^{S_n}, \zeta^p\big)_{S_n} &= \big(\mathrm{M}^q_{(i)}, \mathrm{Z}^p\big) \\ &= \left| \left\{ \pi \in \mathrm{SYT}^p \;\middle|\; \sum_{j=1}^{k} e_j \big(\mathrm{maj}_q \pi^{-1}\big)_j \equiv i \quad \text{modulo } v \right\} \right| \quad .\end{aligned}$$

Proof: Note first that $\sum a_\pi \pi \in \mathbb{C}S_n$ is an element of $\mathcal{D}_n$ iff $a_\pi = a_\sigma$ for all $\pi, \sigma \in S_n$ such that $D(\pi) = D(\sigma)$. This implies $\mathrm{M}^q_{(i)} \in \mathcal{D}_n$. Furthermore, for an arbitrary v-th root of unity φ it is easy to see that

$$\begin{aligned}\kappa_q(\varphi^{e_1}, \ldots, \varphi^{e_k}) &= \sum_{\pi_1 \in S_{q_1}} \cdots \sum_{\pi_k \in S_{q_k}} \varphi^{e_1 \mathrm{maj}\, \pi_1 + \cdots + e_k \mathrm{maj}\, \pi_k} \pi_1 \bullet \cdots \bullet \pi_k \\ &= \sum_{l=0}^{v-1} \varphi^l \mathrm{M}^q_{(l)}\end{aligned}$$

as

$$\sum_{\pi_1 \in S_{q_1}} \cdots \sum_{\pi_k \in S_{q_k}} \pi_1 \bullet \cdots \bullet \pi_k = \Xi^{1^{q_1}} \bullet \cdots \bullet \Xi^{1^{q_k}} = \Xi^{1^n} = \sum_{\pi \in S_n} \pi .$$

Hence, by Frobenius' reciprocity law, the two propositions and the preliminary remarks in Section 1, for any partition p of n,

$$\begin{aligned}\big(\psi_i^{S_n}, \zeta^p\big)_{S_n} &= \frac{1}{v} \sum_{j=0}^{v-1} \psi_i(\sigma^{-j})\, \zeta^p(\sigma^j) \\ &= \frac{1}{v} \sum_{j=0}^{v-1} \eta^{-ij}\, (\mathrm{ch}_{q^{(j)}}, \zeta^p)_{S_n} \\ &= \frac{1}{v} \sum_{j=0}^{v-1} \eta^{-ij} (\kappa_{q^{(j)}}((\eta^{e_1})^j, \ldots, (\eta^{e_1})^j, \ldots, (\eta^{e_k})^j, \ldots, (\eta^{e_k})^j), \mathrm{Z}^p) \\ &= \frac{1}{v} \sum_{j=0}^{v-1} \eta^{-ij} (\kappa_q((\eta^{e_1})^j, \ldots, (\eta^{e_k})^j), \mathrm{Z}^p)\end{aligned}$$

$$= \left(\frac{1}{v} \sum_{l=0}^{v-1} \sum_{j=0}^{v-1} \eta^{-ij} \eta^{jl} \mathrm{M}^q_{(l)},\ \mathrm{Z}^p \right)$$
$$= \left(\mathrm{M}^q_{(i)}, \mathrm{Z}^p\right)$$
$$= \left(c\left(\mathrm{M}^q_{(i)}\right), \zeta^p\right)_{S_n} \quad ,$$

and the theorem is proved. □

Corollary (Kraśkiewicz, Weyman [6]) *Let τ be a cycle of order n in S_n and ε be a primitive n-th root of unity. Let $i \in \{0, \ldots, n-1\}$ and write ψ_i for the character of the cyclic subgroup generated by τ such that $\psi_i(\tau) = \varepsilon^i$. Then the multiplicity of the irreducible character of S_n indexed by the partition p is given by*

$$\left(\psi_i^{S_n}, \zeta^p\right)_{S_n} = |\{\pi \in \mathrm{SYT}^p \mid \mathrm{maj}\, \pi^{-1} \equiv i \quad \text{modulo } n\}|.^1$$

Remark We consider the special case of the theorem where $e_i = v/q_i$ for all $i \in \{1, \ldots, k\}$. As the proof of the theorem shows, we then have, with the correct powers of η used for $\kappa_{q^{(j)}}$, for all $j \in \mathbb{N}$:

$$\zeta^p(\sigma^j) = (\kappa_{q^{(j)}}(\ldots), \mathrm{Z}^p) = \sum_{l=0}^{v-1} \eta^{jl} \left(\mathrm{M}^q_{(l)}, \mathrm{Z}^p\right) = \sum_{\pi \in SYT^p} (\eta^j)^{\sum \frac{v}{q_i} (\mathrm{maj}_q \pi^{-1})_i}.$$

Taking into account that $\mathrm{ind}_q \pi = \sum \frac{v}{q_i} (\mathrm{maj}_q \pi^{-1})_i$ for the q-index of the tableau π defined by Stembridge, we obtain a new proof of Theorem 3.3 in [11] by means of Proposition 1.1 in the same paper.

Note

1. Note that j is a descent of π^{-1} iff j stands strictly above of $j+1$ for $\pi \in \mathrm{SYT}^p$ filled into the frame $R(p)$. This is the link to the original version of the theorem.

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