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Each Invertible Sharply d-Transitive Finite Permutation Set with $d \geq 4$ is a Group

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Abstract. All known finite sharply 4-transitive permutation sets containing the identity are groups, namely S_4, S_5, A_6 and the Mathieu group of degree 11. We prove that a sharply 4-transitive permutation set on 11 elements containing the identity must necessarily be the Mathieu group of degree 11. The proof uses direct counting arguments. It is based on a combinatorial property of the involutions in the Mathieu group of degree 11 (which is established here) and on the uniqueness of the Minkowski planes of order 9 (which had been established before): the validity of both facts relies on computer calculations. A permutation set is said to be invertible if it contains the identity and if whenever it contains a permutation it also contains its inverse. In the geometric structure arising from an invertible permutation set at least one block-symmetry is an automorphism. The above result has the following consequences. i) A sharply 5-transitive permutation set on 12 elements containing the identity is necessarily the Mathieu group of degree 12. ii) There exists no sharply 6-transitive permutation set on 13 elements. For $d \geq 6$ there exists no invertible sharply d-transitive permutation set on a finite set with at least $d + 3$ elements. iii) A finite invertible sharply d-transitive permutation set with $d \geq 4$ is necessarily a group, that is either a symmetric group, an alternating group, the Mathieu group of degree 11 or the Mathieu group of degree 12.

Keywords: sharply d-transitive permutation set, Mathieu groups of degrees 11 and 12, (B)-geometry arising from a permutation set, block-symmetry

1. Introduction

A permutation set H on X is a *subset* of the symmetric group $\mathrm{Sym}(X)$. If X is finite with $|X| = t$ we shall sometimes say that H is a permutation set *on t elements* or that t is the *degree* of H. For arbitrary elements $x_1, x_2, \ldots, x_r \in X$ we denote by $H_{x_1 \cdots x_r}$ the subset of H consisting of all permutations fixing each one of the given elements. If x, y are distinct elements of X we denote by $H(y \mapsto x)$ the subset of H consisting of all permutations mapping y to x. If $g \in \mathrm{Sym}(X)$ then $\mathrm{Fix}(g)$ is the set of all fixed points of g. The functional notation will be used for permutations, hence the permutation $g \in \mathrm{Sym}(X)$ maps each element x of X to $g(x)$; if $f, g \in \mathrm{Sym}(X)$ then fg is the permutation mapping

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each $x \in X$ to $f(g(x))$. If H is a permutation set on X and $f \in \mathrm{Sym}(X)$ then we write $Hf = \{hf : h \in H\}$ and $fH = \{fh : h \in H\}$.

The geometric structure arising from a permutation set H on X is called a (B)-*geometry* in the terminology of [2, 3]. Its points are the elements of the cartesian product $X \times X$; two points (a, b), (c, d) are said to be parallel if $a = c$ or $b = d$; the blocks of the (B)-geometry are simply the permutations in H; point-block incidence is given by set-theoretic inclusion if we view each permutation $f \in \mathrm{Sym}(X)$ as being a special subset of $X \times X$, namely the subset of all pairs $(x, f(x))$ as x varies in X. The points which are incident with any given permutation are pairwise non-parallel.

If H is a non-empty permutation set on X and h is any fixed permutation in H, then the bijective mapping $X \times X \to X \times X$, $(x, y) \mapsto (x, h^{-1}(y))$ maps H to $h^{-1}H$ bijectively, in particular it maps h to the identical permutation, and yields thus an isomorphism of the corresponding (B)-geometries. Hence it can always be assumed up to ismorphisms that the permutation set defining any given (B)-geometry contains the identical permutation.

We are interested here in the finite (B)-geometries arising from sharply $(m+2)$-transitive finite permutation sets. The sharp $(m+2)$-transitivity of the permutation set amounts to the request that any $(m+2)$ pairwise non-parallel points are simultaneously incident with a unique block of the (B)-geometry. We shall call these (B)-geometries *Minkowski* m*-structures* as in [7]. Other terminologies are ‘Minkowski $(m+2)$-planes’ in [1] and “$(m+2)$-reti in senso stretto” in [18]. A characterization of these geometries in terms of Buekenhout diagrams is also possible [13].

Consider the (B)-geometry defined by an arbitrary permutation set H on X. If $f \in H$ is any given block then the *symmetry* with respect to f is the mapping $X \times X \to X \times X$, $(x, y) \to (f^{-1}(y), f(x))$, see [2] or [3]. The image of a block $h \in H$ is the permutation $fh^{-1}f$. Hence although the symmetry with respect to f is always an involutory permutation of the point-set fixing the block f pointwise, it may well be the case that it is not an automorphism of the (B)-geometry, in the sense that the image of some other block is not necessarily a block. If there exists a block $f \in H$ such that its symmetry is actually an automorphism, then the (B)-geometry can be described by the permutation set $G = f^{-1}H$ and the block whose symmetry is an automorphism may be taken to be the identity. The mapping induced on the blocks is thus $G \to G$, $g \mapsto g^{-1}$.

Adopting the terminology of [15] we shall say that the permutation set G on X is *invertible* if G contains the identity and if whenever a permutation g lies in G then so does its inverse g^{-1}. A permutation *group* is clearly invertible and yields a (B)-geometry with the additional property that the symmetry with respect to *each* block is an automorphism.

A Minkowski 0-structure of order n is an affine plane of order n and arises from a sharply 2-transitive permutation set on n elements; a Minkowski 1-structure of order n is a Minkowski plane of order n and arises from a sharply 3-transitive permutation set on $n+1$ elements: there are infinitely many values of n for which such structures are known to exist.

A finite Minkowski 2-structure of order n arises from a sharply 4-transitive permutation set on X with $|X| = n + 2$. Very few examples are known and they all arise from groups, namely S_4, S_5, A_6 and the Mathieu group of degree 11: these are in fact the only sharply 4-transitive permutation groups (the finiteness assumption is not even required here, see [6, Thm. 5.8.1]).

Assume $|X| \geq 7$ and let G be a sharply 4-transitive permutation set on X containing the identity and having the property that each permutation $j \in G$ exchanging two elements and fixing three further elements of X is necessarily an involution. It was proved in [16] that under these assumptions we must have $|X| = 11$. It was also remarked that the above property is certainly satisfied if G, besides containing the identity, is such that $g \in G$ always implies $g^{-1} \in G$, that is G is an invertible permutation set.

Is G then necessarily a group, whence the Mathieu group of degree 11? Some partial answers have been given in [4, 17]: both papers focus attention on the involutions in G. In Section 2 we answer the question affirmatively under the mere assumption that G contains the identity, see Proposition 7: we also make essential use of some properties of the involutions in G.

This result is the starting point for the subsequent sections. A similar property is namely proved in Section 3: a sharply 5-transitive permutation set on 12 elements containing the identity is necessarily the Mathieu group of degree 12, see Proposition 9.

Higher degrees are handled in Section 4, in which we prove the non-existence of a sharply 6-transitive permutation set on 13 elements, see Proposition 10.

This result contradicts property (4) established by J.H. Conway for M_{13} in Section 2 of [5]: the fact alone that M_{13} is the union of 13 translates of M_{12}, one for each position of the hole, even with the correct cardinality $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$, does not imply 6-fold transitivity if M_{13} is only known to be a permutation *set*.

An inductive argument yields then the non-existence of an invertible sharply d-transitive permutation set of degree at least $d + 3$ for $d \geq 6$.

As a final consequence we have a classification of the finite invertible sharply d-transitive permutation sets for $d \geq 4$: each such permutation set turns out to be a group, hence either a symmetric group, an alternating group, the Mathieu group of degree 11 or the Mathieu group of degree 12, see Proposition 12.

We can reformulate this property as follows. For $m \geq 2$ a finite Minkowski m-structure in which some block-symmetry is an automorphism must necessarily arise from a sharply $(m+2)$-transitive permutation *group* (and so in turn *every* block-symmetry is an automorphism).

The corresponding property does not hold for finite affine planes ($m = 0$): non-nearfield planes admitting involutory perspectivities do exist. Whether it holds for finite Minkowski planes ($m = 1$) is still an open question as far as we know. If it is assumed that *every* block-symmetry of a finite affine plane or of a finite Minkowski plane is an automorphism, then the underlying sharply 2-transitive or sharply 3-transitive permutation set must be a group respectively: that follows essentially from the results in [10] and [14].

Observe that if X is an arbitrary infinite set and d is an arbitrary positive integer, then there always exists an invertible sharply d-transitive permutation set on X, which in general is not a group, see [8, 12].

We finally remark that, besides requiring the computer checks described in Sections 2, 3, 4, our proofs rely essentially on the uniqueness of the Minkowski planes of order nine [19], which in turn ultimately rests on the uniqueness of the projective planes of order nine [11], another computer result.

2. The Mathieu group of degree 11

Throughout this section we assume that G is a sharply 4-transitive permutation set on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ containing the identical permutation. We have $|\mathrm{Fix}(gf^{-1})| \leq 3$ for any two distinct permutations f, g in G. In particular each non-identical permutation in G has at most three fixed points.

We denote by M the Mathieu group of degree 11 in its sharply 4-transitive permutation representation. The Mathieu group is uniquely determined up to permutation isomorphism [6, §5.8], in other words the sharply 4-transitive subgroups of S_{11} form a single conjugacy class: we may take for M any specific version of the Mathieu group of degree 11. We denote by I the set of all involutions in M.

Let J be a subset of S_{11} with the following properties:

a1) $|J| = 165$;
a2) each permutation in J is an involution with three fixed points;
a3) $I_1 \subseteq J$;
a4) $|\mathrm{Fix}(ts)| \leq 3$ for any two distinct $t, s \in J$.

Proposition 1 *There are precisely two subsets of S_{11} satisfying the above properties, namely the set I of all involutions in M and a subset I^* such that $\langle I^* \rangle$ is conjugate to M in S_{11}.*

Proof: The fact that I satisfies properties a1), a2), a3), a4) follows easily from the property that M is a sharply 4-transitive permutation group of degree 11, [6, §5.8].

The rest of the assertion has been verified by computer through a MAGMA program. Full code is available by e-mail from the first author. We summarize here the relevant steps.

Let T denote the set of all involutions in S_{11} with precisely three fixed points. The centralizer in S_{11} of an involution in T is easily seen to have order $6 \cdot 24 \cdot 16$ and so the cardinality of T is 17325.

Set $I' = I \setminus I_1$ and define $I'' = \{h \in T \setminus I : |\mathrm{Fix}(hj)| \leq 3 \text{ for all } j \in I_1\}$. We have $|I''| = 120$ and the subgroup of S_{11} generated by $I^* = I'' \cup I_1$ is a conjugate M^* of M in S_{11}.

Form a graph Γ on the set of vertices $V(\Gamma) = I' \cup I''$: two distinct involutions $a, b \in I' \cup I''$ are declared to be adjacent if and only if $|\mathrm{Fix}(ab)| > 3$. That means a, b cannot sit together in a sharply 4-transitive permutation set.

Since I' is a subset of M and M is sharply 4-transitive, we see that no two vertices in I' are adjacent and so I' is an independent subset of size 120 in Γ. Similarly, I'' is another independent subset of size 120 in Γ. In particular Γ is a bipartite graph with bipartition $\{I', I''\}$. Now a candidate subset J with the required properties must be of the form $J = I_1 \cup J'$ where J' is an independent subset of size 120 in Γ.

The graph Γ is regular (of degree 16) and so the complement $J'' = V(\Gamma) \setminus J'$ is also an independent subset of size 120 in Γ. Clearly $\{J', J''\}$ is a bipartition of Γ.

As the graph Γ is connected, it admits precisely one bipartition, which means $\{J', J''\} = \{I', I''\}$ and the assertion follows. □

Proposition 2 *There exist two Minkowski planes of order* 9 *up to isomorphism. Precisely one of them, namely the non-miquelian one, is extendable to a Minkowski* 2*-structure of order* 9.

Proof: The first assertion is Theorem B in [19]. The second assertion is Proposition 3 in [4]. □

The stabilizer G_x defines a Minkowski plane of order 9 which can be extended to a Minkowski 2-structure. Since the non-miquelian Minkowski plane of order 9 can be obtained from a group it satisfies the rectangle axiom, see [2, Thm. 4] or [3, III §4.3]. The Minkowski plane defined by G_x satisfies thus the rectangle axiom and since G_x contains the identical permutation we have in turn that G_x is a sharply 3-transitive group, [2, Thm. 4] or [3, III §4.3].

There are only two types of sharply 3-transitive groups of degree 10 [9, XI §2.6], they are namely $PGL(2, 9)$ and the group denoted by $M(3^2)$ in [9]. Proposition 2 shows that G_x cannot be $PGL(2, 9)$ otherwise the corresponding Minkowski plane would be miquelian. Hence G_x is isomorphic to $M(3^2)$ and admits thus a transitive extension which is precisely the Mathieu group. In other words we have that for each $x \in X$ the stabilizer G_x is a conjugate of M_x in S_{11}. After possibly replacing G by a suitable conjugate hGh^{-1} in S_{11}, we may assume $G_1 = M_1$.

Let J denote the set of involutions in G. Since an involution on eleven elements must necessarily have some fixed point, we have that J is the union of the sets J_x as x varies in X. In particular J is non-empty and since each involution in G_x has precisely two fixed points on $X\setminus\{x\}$ we see that every involution in J has precisely three fixed points on X.

The stabilizer of two points in G_x is a quaternion group of order 8. To any given three elements $x, y, z \in X$ there exists thus a unique involution in J fixing x, y and z. Distinct choices of x, y, z yield distinct involutions in J, as each non-identical permutation in G has at most three fixed points, whence

$$|J| = \binom{11}{3} = 165.$$

It is now clear that J satisfies properties a1), a2), a3) and a4) above. Proposition 1 shows that we only have two choices for J, namely $J = I$ or $J = I^*$: again, after possibly replacing G and M by suitable conjugates fGf^{-1}, fMf^{-1} with $f \in S_{11}$, we may limit our discussion to the former case.

So far the group M and the set G share the stabilizer of the element 1 and the involutions.

Proposition 3 *We have* $G_x = M_x$ *for each* $x \in X$.

Proof: The assertion is true if $x = 1$. Assume $x \neq 1$. As a 3-transitive permutation group on $X\setminus\{x\}$, the group M_x acts primitively on $X\setminus\{x\}$; in particular the stabilizer M_{1x} is a maximal subgroup of M_x and so, since I_x contains at least one involution not fixing 1, we have $\langle M_{1x}, I_x\rangle = M_x$.

Since both M_1 and I are in G, we have that G_x contains M_{1x} and I_x. We already remarked that G_x is a group, whence $M_x = \langle M_{1x}, I_x \rangle \le G_x$; the equality $|G_x| = |M_x|$ yields now $G_x = M_x$. □

Let F denote the subset of G consisting of all permutations in G with at least one fixed point. We have $F = \cup_{x \in X} G_x = \cup_{x \in X} M_x$.

Proposition 4 *We have* $|F \cap G(y \mapsto x)| = 444$ *for any two distinct elements* $x, y \in X$.

Proof: We have $F \cap G(y \mapsto x) = \bigcup_{\substack{z \in X \\ z \notin \{x,y\}}} G(y \mapsto x)_z$. The cardinality of the right-hand-side can be computed using the principle of inclusion-exclusion as

$$\sum_{\substack{z \in X \\ z \notin \{x,y\}}} |G(y \mapsto x)_z| - \sum_{\substack{z,u \in X \\ z,u \notin \{x,y\} \\ z \neq u}} |G(y \mapsto x)_{zu}| + \sum_{\substack{z,u,w \in X \\ z,u,w \notin \{x,y\} \\ |\{z,u,w\}|=3}} |G(y \mapsto x)_{zuw}|.$$

The sharp 4-transitivity of G on X yields $|G(y \mapsto x)_z| = 72$, $|G(y \mapsto x)_{zu}| = 8$, $|G(y \mapsto x)_{zuw}| = 1$, whence $|F \cap G(y \mapsto x)| = 9 \cdot 72 - \binom{9}{2} \cdot 8 + \binom{9}{3} \cdot 1 = 444$. □

Proposition 5 *We have* $G(y \mapsto x) = M(y \mapsto x)$ *for all pairs* x, y *of distinct elements in* X.

Proof: Let g be an arbitrary permutation in $F \cap G(y \mapsto x)$. The permutation set $G(y \mapsto x)g^{-1}$ contains the identity, fixes x and acts sharply 3-transitively on $X \backslash \{x\}$. Since the Minkowski plane of order 9 arising from $G(y \mapsto x)g^{-1}$ can be obtained as a derived structure of a Minkowski 2-structure, namely the Minkowski 2-structure arising from the sharply 4-transitive permutation set Gg^{-1}, we see that $G(y \mapsto x)g^{-1}$ is a group. More precisely, since $G(y \mapsto x)g^{-1}$ fixes x, it is a conjugate in S_{11} of M_x fixing x, i.e. $G(y \mapsto x)g^{-1} = hM_x h^{-1}$ for some permutation $h \in S_{11}$ with $h(x) = x$. We have thus $G(y \mapsto x) = hM_x h^{-1} g$ and consequently $hM_x h^{-1} g = hM_x h^{-1} k$ for any two $g, k \in F \cap G(y \mapsto x)$. Since g and k also lie in the Mathieu group M we also have $M(y \mapsto x) = M_x g = M_x k$. We obtain $gk^{-1} \in M_x \cap hM_x h^{-1}$ and so the intersection $M_x \cap hM_x h^{-1}$ contains all 444 distinct permutations gk^{-1} obtained when g is fixed and k varies over the 444 permutations in $F \cap G(y \mapsto x)$. As both M_x and $hM_x h^{-1}$ are groups of order 720 we see that $M_x = hM_x h^{-1}$ is the unique possibility and the assertion follows. □

Proposition 6 *We have* $G = M$.

Proof: An immediate consequence of the above discussion and of the relations

$$G = G_x \cup \left(\bigcup_{\substack{y \in X \\ y \neq x}} G(y \mapsto x) \right), \qquad M = M_x \cup \left(\bigcup_{\substack{y \in X \\ y \neq x}} M(y \mapsto x) \right). \qquad \square$$

The above properties can be summarized in the following result.

Proposition 7 *Assume* $|X| = 11$ *and let* G *be a sharply* 4*-transitive permutation set on* X *containing the identity. Then* G *is a group, a copy of the Mathieu group of degree* 11.

3. The Mathieu group of degree 12

Throughout this section we assume that $\hat{G}$ is a sharply 5-transitive permutation set on $\hat{X} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ containing the identical permutation. We have $|\mathrm{Fix}(gf^{-1})| \leq 4$ for any two distinct permutations f, g in $\hat{G}$. In particular each non-identical permutation in $\hat{G}$ has at most four fixed points.

We denote by $\hat{M}$ the Mathieu group of degree 12 in its sharply 5-transitive permutation representation. The Mathieu group is uniquely determined up to permutation isomorphism [6, §5.8], in other words the sharply 5-transitive subgroups of S_{12} form a single conjugacy class: we may take for $\hat{M}$ any specific version of the Mathieu group of degree 12. We denote by $\hat{I}$ the set of all involutions in $\hat{M}$ with four fixed points.

Let $\hat{J}$ be a subset of S_{12} with the following properties:

b1) $|\hat{J}| = 495$;
b2) each permutation in $\hat{J}$ is an involution with four fixed points;
b3) $\hat{I}_1 \subseteq \hat{J}$;
b4) $|\mathrm{Fix}(ts)| \leq 4$ for any two distinct $t, s \in \hat{J}$.

Proposition 8 *There is precisely one subset of* S_{12} *satisfying the above properties, namely the set* $\hat{I}$ *itself.*

Proof: The fact that $\hat{I}$ satisfies properties b1), b2), b3), b4) follows easily from the property that $\hat{M}$ is a sharply 5-transitive permutation group of degree 12, [6, §5.8].

Let $\hat{T}$ denote the set of all involutions in S_{12} with precisely four fixed points. The centralizer in S_{12} of an involution in $\hat{T}$ is easily seen to have order $24 \cdot 24 \cdot 16$ and so the cardinality of $\hat{T}$ is 51975, whence $|\hat{T} \backslash \hat{I}| = 51480$.

Define $\hat{I}'' = \{h \in \hat{T} \backslash \hat{I} : |\mathrm{Fix}(hj)| \leq 4 \text{ for all } j \in \hat{I}_1\}$. A second MAGMA program has verified that $\hat{I}''$ is empty: full code is available by e-mail from the first author. □

The stabilizer $\hat{G}_x$ of an element $x \in \hat{X}$ is a sharply 4-transitive permutation set on $\hat{X} \setminus \{x\}$ containing the identical permutation. By Proposition 7 the stabilizer $\hat{G}_x$ is a group, a copy of the Mathieu group of degree 11. In particular, for each $x \in \hat{X}$ the stabilizer $\hat{G}_x$ is a conjugate of $\hat{M}_x$ in S_{12}. After possibly replacing $\hat{G}$ by a suitable conjugate $h\hat{G}h^{-1}$ in S_{12}, we may assume $\hat{G}_1 = \hat{M}_1$.

Let $\hat{J}$ denote the subset of $\hat{G}$ consisting of the involutions with four fixed points. We have that $\hat{J}$ is the union of the sets $\hat{J}_x$ as x varies in $\hat{X}$. In particular $\hat{J}$ is non-empty.

The stabilizer of three points in $\hat{G}_x$ is a quaternion group of order 8. To any given four elements $x, y, z, u \in \hat{X}$ there exists thus a unique involution in $\hat{J}$ fixing x, y, z and u. Distinct choices of $\{x, y, z, u\}$ yield distinct involutions in $\hat{J}$, as each non-identical permutation in

$\hat{G}$ has at most four fixed points, whence

$$|\hat{J}| = \binom{12}{4} = 495.$$

It is now clear that $\hat{J}$ satisfies properties b1), b2), b3) and b4) above. Proposition 8 shows that we only have one choice for $\hat{J}$, namely $\hat{J} = \hat{I}$.

We can now replace X, G, M, F, I by $\hat{X}, \hat{G}, \hat{M}, \hat{F}, \hat{I}$ respectively and introduce the obvious necessary changes in the arguments of Section 2. In analogy with Proposition 3 we have $\hat{G}_x = \hat{M}_x$ for each $x \in \hat{X}$, while the analogue of Proposition 4 yields $|\hat{F} \cap \hat{G}(y \mapsto x)| = 4710$ for any two distinct elements $x, y \in \hat{X}$. If we argue as in Proposition 5 we obtain $\hat{G}(y \mapsto x) = \hat{M}(y \mapsto x)$ for all pairs x, y of distinct elements in $\hat{X}$, while the analogue of Proposition 6 finally shows that $\hat{G} = \hat{M}$ holds. These properties can be summarized in the following result.

Proposition 9 *Assume* $|\hat{X}| = 12$ *and let* $\hat{G}$ *be a sharply* 5*-transitive permutation set on* $\hat{X}$ *containing the identity. Then* $\hat{G}$ *is a group, a copy of the Mathieu group of degree* 12.

4. Higher degrees

Throughout this section we assume that H is a sharply 6-transitive permutation set on $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ containing the identical permutation. We have $|\mathrm{Fix}(gf^{-1})| \leq 5$ for any two distinct permutations f, g in H. In particular each non-identical permutation in H has at most five fixed points.

The stabilizer H_y of an element $y \in Y$ is a sharply 5-transitive permutation set on $Y \setminus \{y\}$ containing the identical permutation. By Proposition 9 the stabilizer H_y is a group, a copy of the Mathieu group of degree 12. In particular, if we denote by $\hat{M}$ any specific version of the Mathieu group of degree 12 acting sharply 5-transitively on $\{2, 3, \dots, 13\}$ and fixing 1, for each $y \in Y$ the stabilizer H_y is a conjugate of $\hat{M}$ in S_{13}. After possibly replacing H by a suitable conjugate kHk^{-1} in S_{13}, we may assume $H_1 = \hat{M}$.

Let L denote the subset of H consisting of the involutions with five fixed points. We have that L is the union of the sets L_y as y varies in Y. In particular L is non-empty.

The stabilizer of four points in H_y is a quaternion group of order 8. To any given five elements $y, z, u, v, w \in Y$ there exists thus a unique involution in L fixing y, z, u, v and w. Distinct choices of $\{y, z, u, v, w\}$ yield distinct involutions in L, as each non-identical permutation in H has at most five fixed points, whence

$$|L| = \binom{13}{5} = 1287.$$

The subset L of S_{13} has thus the following properties.

c1) $|L| = 1287$;
c2) each permutation in L is an involution with five fixed points;

c3) $L \cap \hat{M} = L_1$ is precisely the set of all involutions in $\hat{M}$ fixing 1 and precisely four further elements in $\{2, 3, \ldots, 13\}$;
c4) $|\mathrm{Fix}(ts)| \leq 5$ for any two distinct $t, s \in L$.

A third MAGMA program has shown that such a set L cannot exist: full code is available by e-mail from the first author. We have thus the following result.

Proposition 10 *There exists no sharply* 6*-transitive permutation set on* 13 *elements.*

Proposition 11 *Let d be an integer, $d \geq 6$. There exists no invertible sharply d-transitive permutation set on a finite set of cardinality at least $d + 3$.*

Proof: Assume $d = 6$; let G be an invertible sharply 6-transitive permutation set on a finite set X with $|X| \geq 9$. The stabilizer G_{xy} of two elements $x, y \in X$ is an invertible sharply 4-transitive finite permutation set on $|X| - 2 \geq 7$ elements. By the result in [16] we have $|X| - 2 = 11$, whence $|X| = 13$, contradicting Proposition 10. The result now follows easily by induction on d since the one-point-stabilizer of a sharply $(d + 1)$-transitive invertible permutation set is an invertible permutation set which is sharply d-transitive on the remaining elements. □

Proposition 12 *Let G be an invertible sharply d-transitive permutation set on a finite set X. If $d \geq 6$ then G is either S_d, S_{d+1} or A_{d+2}. If $d = 5$ then G is either S_5, S_6, A_7 or the Mathieu group of degree* 12*. If $d = 4$ then G is either S_4, S_5, A_6 or the Mathieu group of degree* 11*.*

Proof: Assume $d \geq 6$. It follows from the previous Proposition that $|X| \leq d + 2$. If $|X| = d$ or $|X| = d + 1$ then $G = \mathrm{Sym}(X)$. If $|X| = d + 2$ then $G = \mathrm{Alt}(X)$ by Proposition 6 in [17].

Assume $d = 5$. If $|X| \geq 8$ then the stabilizer G_x of an element $x \in X$ is an invertible sharply 4-transitive permutation set on $|X| - 1 \geq 7$ elements. By the result in [16] we have $|X| - 1 = 11$, whence $|X| = 12$ and Proposition 13 yields that G is a copy of the Mathieu group of degree 12. If $|X| = 7$ then $G = \mathrm{Alt}(X)$ by Proposition 6 in [17]. If $|X| = 6$ or 5 then $G = \mathrm{Sym}(X)$.

Assume $d = 4$. If $|X| \geq 7$ then we have $|X| = 11$ by the result in [16] and Proposition 7 yields that G is a copy of the Mathieu group of degree 11. If $|X| = 6$ then $G = \mathrm{Alt}(X)$ by Proposition 6 in [17]. If $|X| = 5$ or 4 then $G = \mathrm{Sym}(X)$. □

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