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A Statistic on Involutions

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Abstract. We define a statistic, called *weight*, on involutions and consider two applications in which this statistic arises. Let $I(n)$ denote the set of all involutions on $[n](=\{1, 2, \ldots, n\})$ and let $F(2n)$ denote the set of all fixed point free involutions on $[2n]$. For an involution δ, let $|\delta|$ denote the number of 2-cycles in δ. Let $[n]_q = 1+q+\cdots+q^{n-1}$ and let $\binom{n}{k}_q$ denote the q-binomial coefficient. There is a statistic wt on $I(n)$ such that the following results are true.

(i) We have the expansion

$$\binom{n}{k}_q = \sum_{\delta \in I(n)} (q-1)^{|\delta|} q^{\mathrm{wt}(\delta)} \binom{n-2|\delta|}{k-|\delta|}.$$

(ii) An analog of the (strong) Bruhat order on permutations is defined on $F(2n)$ and it is shown that this gives a rank-$2\binom{n}{2}$ graded EL-shellable poset whose order complex triangulates a ball. The rank of $\delta \in F(2n)$ is given by $\mathrm{wt}(\delta)$ and the rank generating function is $[1]_q[3]_q \cdots [2n-1]_q$.

Keywords: permutation statistics, q-binomial coefficient, Bruhat order, involutions, fixed point free involutions

1. Introduction and statement of results

In this paper we define a statistic on involutions and consider two applications in which this statistic arises.

An *arc* or a *2-cycle* is a set consisting of two distinct positive integers. We write an arc as $[i, j]$, with $i < j$. For an arc $[i, j]$, we call i the *initial point* and j the *terminal point* of the arc. The *span* of an arc $[i, j]$ is defined as $\mathrm{span}\,[i, j] = j - i - 1$. A pair $\{[i, j], [k, l]\}$ of disjoint arcs is said to be a *crossing* if $i < k < j < l$ or $k < i < l < j$ (see figure 1).

An *involution* is a finite set of pairwise disjoint arcs. For nonnegative integers n, k, let $I(n)$ denote the set of all involutions whose arcs are contained in $[n](=\{1, 2, \ldots n\})$ and let $I(n, k)$ denote the set of involutions in $I(n)$ with k arcs. We will always write involutions in their *standard representation* which is in increasing order of initial points.

Let δ be an involution. The number of arcs in δ is denoted by $|\delta|$. The *crossing number* of δ, denoted $c(\delta)$, is the number of pairs of arcs of δ that are crossings. Define the *weight* of δ, denoted by $\mathrm{wt}(\delta)$, as follows:

$$\mathrm{wt}(\delta) = \left(\sum_{[i,j]\in\delta} \mathrm{span}[i, j]\right) - c(\delta).$$

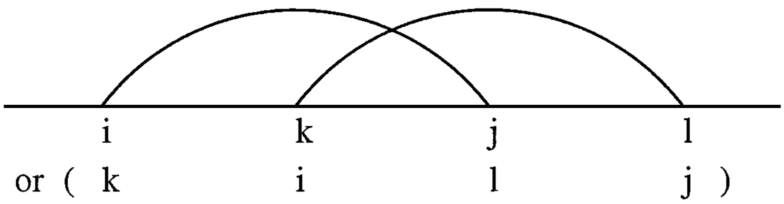

Figure 1. A crossing.

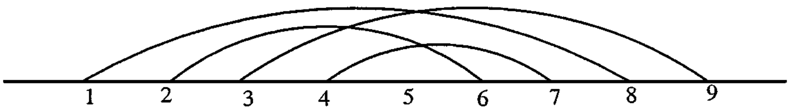

Figure 2. $\delta = \{[1, 8], [2, 6], [3, 9], [4, 7]\}$.

Example 1.1 Let $\delta = [1, 8][2, 6][3, 9][4, 7] \in I(9)$. Represent δ as shown in figure 2. Observe that there are 3 crossings. Thus $wt(\delta) = (8 - 1 - 1) + (6 - 2 - 1) + (9 - 3 - 1) + (7 - 4 - 1) - 3 = 13$.

In order to motivate our first application of the weight statistic, we define the notion of symmetric Boolean packings (see Björner [3, Exercise 7.36], and Ref. [13]). Let P be a finite graded rank-n poset with rank function $r : P \to \{0, 1, 2, \ldots, n\}$. For $0 \leq k \leq n$, let N_k denote the number of elements of P of rank k. We say that the elements $x_1, x_2, \ldots, x_h$ of P form a *symmetric chain* if x_{i+1} covers x_i for every $i < h$ and $r(x_1) + r(x_h) = n$. A *symmetric chain decomposition* (SCD) of P is a covering of P by pairwise disjoint symmetric chains. Let $B(n)$ denote the poset of all subsets of $[n]$, under inclusion. We say that a subset $Q \subseteq P$ is *symmetric Boolean* if

(i) Q, under the induced order, has a minimum element, say z, and a maximum element, say z'.
(ii) Q is order isomorphic to $B(r(z') - r(z))$.
(iii) $r(z') + r(z) = n$.

A *symmetric Boolean packing* (SBP) of P is a covering of P by pairwise disjoint symmetric Boolean subsets. De Bruijn, Tenbergen, and Kruyswijk [4] constructed a symmetric chain decomposition of $B(l)$, for $l \geq 0$. It follows that if P admits a SBP, then it has a SCD.

The existence of a SBP allows us to expand the rank numbers of P in terms of the binomial coefficients. Let $P = Q_1 \uplus Q_2 \uplus \cdots \uplus Q_t$ (disjoint union) be a SBP of P. Let z_i (respectively z_i') denote the minimum (respectively, maximum) element of Q_i, $i = 1, 2, \ldots, t$. Since Q_i is order isomorphic to $B(r(z_i') - r(z_i))$ and $r(z_i') + r(z_i) = n$ we have

$$N_k = \sum_{i=1}^{t} \binom{r(z_i') - r(z_i)}{k - r(z_i)} = \sum_{i=1}^{t} \binom{n - 2r(z_i)}{k - r(z_i)}. \tag{1.1}$$

Let q be a prime power and let $B_q(n)$ denote the poset of subspaces, under inclusion, of an n-dimensional vector space over $\mathbb{F}_q$ (the finite field with q elements). The number of elements of rank k in $B_q(n)$ is the q-binomial coefficient $\binom{n}{k}_q$. Griggs [8] proved that $B_q(n)$ has a SCD. An explicit construction of a SCD of $B_q(n)$ was a long-standing open problem. In their beautiful recent paper [16], Vogt and Voigt solve this problem. It is not difficult to see that their construction actually yields a SBP of $B_q(n)$. It follows that the q-binomial coefficients admit an expansion (in the form of identity 1.1 above) in terms of the binomial coefficients. Our first application of the weight statistic is the following explicit expansion $\binom{n}{k}$ and $\binom{n}{k}_q$ are taken to be zero if $n < 0$ or $k < 0$).

Theorem 1.2

$$\binom{n}{k}_q = \sum_{\delta \in I(n)} (q-1)^{|\delta|} q^{\mathrm{wt}(\delta)} \binom{n-2|\delta|}{k-|\delta|}.$$

For example, we have $\binom{5}{k}_q = \binom{5}{k} + (q-1)(4+3q+2q^2+q^3)\binom{3}{k-1} + (q-1)^2(3+4q+4q^2+3q^3+q^4)\binom{1}{k-2}$.

We originally had two proofs of Theorem 1.2: a simple manipulative proof based on permutation statistics and a bijective proof based on row reduced echelon forms. The bijective proof, however, does not yield a SBP of $B_q(n)$. We feel that a fuller understanding of the construction in [16] would yield a bijective proof of Theorem 1.2 (or even a generalization of Theorem 1.2) that actually implements a SBP of $B_q(n)$. Therefore, in Section 2, we are only presenting the manipulative proof of Theorem 1.2.

Let $F(2n)$ denote the set of all *fixed point free involutions* in $I(2n)$, i.e., involutions in $I(2n)$ having n arcs. The weight statistic defined on $I(2n)$ restricts to $F(2n)$. (We note that the crossing number statistic on fixed point free involutions was first defined by Stembridge [15] in his work on pfaffians.) We now define a partial order on $F(2n)$. This partial order can be seen as an analog of the (strong) Bruhat order on permutations (Edelman [6]).

Let $\delta = [a_1, b_1][a_2, b_2] \ldots [a_n, b_n] \in F(2n)$. We say that $\tau \in F(2n)$ is obtained from δ by an *interchange*, written $\delta \sim \tau$, if there exist $1 \le i < j \le n$ such that

(i) τ's standard representation is obtained from δ by exchanging b_i and a_j. or
(ii) τ's standard representation is obtained from δ by exchanging b_i and b_j.

Note that if $\delta \sim \tau$ then $\tau \sim \delta$. We say that τ is obtained from δ by a *weight increasing interchange* if $\delta \sim \tau$ and $\mathrm{wt}(\delta) < \mathrm{wt}(\tau)$.

Define a partial order $\le$ on $F(2n)$ as follows: Let $\delta, \tau \in F(2n)$. Then $\delta \le \tau$ if τ can be obtained from δ by a sequence of (zero or more) weight increasing interchanges. For example, the Hasse diagram of $F(6)$ is shown in figure 3.

For a finite graded poset P the rank generating function is defined by $R(P, q) = \sum_{k=0}^{n} N_k q^k$, where N_k is the number of elements of P of rank k and n is the rank of P (here q is an indeterminate). The following result is proved in Section 3 (below $[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$).

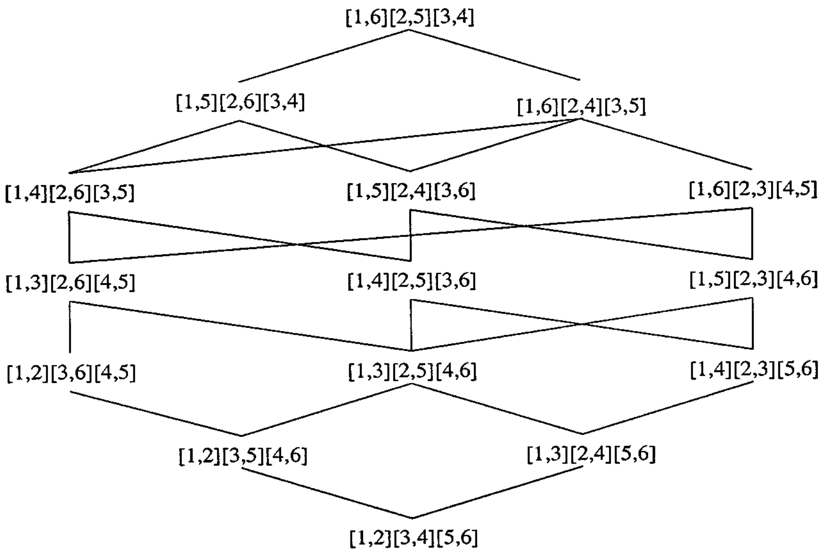

Figure 3. Hasse diagram of $F(6)$.

Theorem 1.3

(i) *For* $n \geq 1$, $(F(2n), \leq)$ *is a graded poset of rank* $2\binom{n}{2}$. *The rank of* $\delta \in F(2n)$ *is given by* $\mathrm{wt}(\delta)$ *and*

$$R(F(2n), q) = [1]_q[3]_q[5]_q \cdots [2n-1]_q.$$

(ii) *For* $n \geq 2$, $(F(2n), \leq)$ *is EL-shellable and its order complex triangulates the ball of dimension* $2\binom{n}{2} - 2$.

One of the referees has kindly informed us that there is a possible relationship between the poset structure on $F(2n)$ and the cell decomposition of the homogeneous space $GL(2n)/Sp(2n)$, considered in Howe and Kraft [9].

Representing set partitions by means of a suitable set of arcs and by slightly extending the definition of crossing of arcs, we can define a notion of weight for set partitions. This statistic turns out to be the same as that recently defined by Johnson [10, 11]. We shall treat this topic in a future paper.

2. *q*-Binomial coefficients

In this section we give a proof of Theorem 1.2 based on permutation statistics. For nonnegative integers n, k, define the generating function $i_q(n,k) = \sum_{\delta \in I(n,k)} q^{\mathrm{wt}(\delta)}$. We put $i_q(n,k) = 0$ if $n < 0$ or $k < 0$.

Proposition 2.1 *For nonnegative integers* n, k,

$$i_q(n+1,k) = i_q(n,k) + [n]_q i_q(n-1,k-1),$$

with $i_q(0,k) = \delta_{0,k}$.

Proof: We first recall the proof of the identity in the $q = 1$ case. Define a map

$$\Upsilon : I(n+1,k) \to I(n,k) \uplus ([n] \times I(n-1,k-1))$$

as follows: Given $\delta \in I(n+1,k)$, define $\Upsilon(\delta) = \delta$ if $n+1$ is not contained in any arc of δ. If $[i, n+1] \in \delta$ for some $i \in \{1, 2, \ldots, n\}$, delete arc $[i, n+1]$ from δ to get $\bar{\delta}$. Relabel the elements of $[n+1] - \{i, n+1\}$ as $\{1, 2, \ldots, n-1\}$, in increasing order. Perform the corresponding relabeling of $\bar{\delta}$ to get $\delta' \in I(n-1, k-1)$. Define $\Upsilon(\delta) = (i, \delta')$. The map Υ is easily seen to be a bijection. In the general case we check that if $\Upsilon(\delta) = (i, \delta') \in [n] \times I(n-1, k-1)$, then $\mathrm{wt}(\delta) = n - i + \mathrm{wt}(\delta')$. This will prove the identity.

Let $\Upsilon(\delta) = (i, \delta')$. Let p be the number of arcs in δ with terminal points belonging to $\{i+1, \ldots, n\}$. Then it is easily seen that $c(\delta) = c(\bar{\delta}) + p = c(\delta') + p$ and

$$\left(\sum_{[k,l] \in \bar{\delta}} \mathrm{span}[k,l] \right) = \left(\sum_{[k,l] \in \delta'} \mathrm{span}[k,l] \right) + p.$$

We have

$$\begin{aligned} \mathrm{wt}(\delta) &= \left(\sum_{[k,l] \in \bar{\delta}} \mathrm{span}[k,l] \right) + (n+1-i-1) - c(\delta) \\ &= \left(\sum_{[k,l] \in \delta'} \mathrm{span}[k,l] \right) - c(\delta') + n - i \\ &= \mathrm{wt}(\delta') + n - i. \end{aligned}$$

This completes the proof. □

Corollary 2.2 *For a nonnegative integer n,*

$$\sum_{\delta \in F(2n)} q^{\mathrm{wt}(\delta)} = i_q(2n, n) = [2n-1]_q [2n-3]_q \cdots [1]_q.$$

Proof: Since $i_q(2n-1, n) = 0$, we have by Proposition 2.1, $i_q(2n, n) = [2n-1]_q\, i_q(2n-2, n-1)$. The result now follows by induction. □

The following recurrence for the q-binomial coefficients was given by Goldman and Rota [7]. (See also [12]).

Proposition 2.3 *For nonnegative integers n, k,*

$$\binom{n+1}{k}_q = \binom{n}{k}_q + \binom{n}{k-1}_q + (q^n - 1)\binom{n-1}{k-1}_q,$$

with $\binom{0}{k}_q = \delta_{0,k}$.

Proof of Theorem 1.2: We use the notation set up in the proof of Proposition 2.1, where a bijection $\Upsilon : I(n+1) \to I(n) \uplus ([n] \times I(n-1))$ was defined.

We will show that the right hand side satisfies the recurrence given in Proposition 2.3. We have

$$\begin{aligned}
&\sum_{\delta \in I(n+1)} (q-1)^{|\delta|} q^{\mathrm{wt}(\delta)} \binom{n+1-2|\delta|}{k-|\delta|} \\
&= \sum_{\delta \in I(n)} (q-1)^{|\delta|} q^{\mathrm{wt}(\delta)} \binom{n+1-2|\delta|}{k-|\delta|} \\
&\quad + \sum_{i=1}^{n} \left\{ \sum_{\substack{\delta \in I(n+1) \\ [i,n+1] \in \delta}} (q-1)^{1+|\delta'|} q^{(n-i)+\mathrm{wt}(\delta')} \binom{n+1-2(|\delta'|+1)}{k-(|\delta'|+1)} \right\} \\
&= \sum_{\delta \in I(n)} (q-1)^{|\delta|} q^{\mathrm{wt}(\delta)} \left\{ \binom{n-2|\delta|}{k-|\delta|} + \binom{n-2|\delta|}{k-1-|\delta|} \right\} \\
&\quad + \sum_{i=1}^{n} \left\{ \sum_{\delta \in I(n-1)} (q-1) q^{n-i} (q-1)^{|\delta|} q^{\mathrm{wt}(\delta)} \binom{n-1-2|\delta|}{k-1-|\delta|} \right\} \\
&= \left\{ \sum_{\delta \in I(n)} (q-1)^{|\delta|} q^{\mathrm{wt}(\delta)} \binom{n-2|\delta|}{k-|\delta|} \right\} + \left\{ \sum_{\delta \in I(n)} (q-1)^{|\delta|} q^{\mathrm{wt}(\delta)} \binom{n-2|\delta|}{k-1-|\delta|} \right\} \\
&\quad + \sum_{\delta \in I(n-1)} \left\{ \sum_{i=1}^{n} (q-1) q^{n-i} \right\} (q-1)^{|\delta|} q^{\mathrm{wt}(\delta)} \binom{n-1-2|\delta|}{k-1-|\delta|} \\
&= \left\{ \sum_{\delta \in I(n)} (q-1)^{|\delta|} q^{\mathrm{wt}(\delta)} \binom{n-2|\delta|}{k-|\delta|} \right\} + \left\{ \sum_{\delta \in I(n)} (q-1)^{|\delta|} q^{\mathrm{wt}(\delta)} \binom{n-2|\delta|}{k-1-|\delta|} \right\} \\
&\quad + (q^n - 1) \left\{ \sum_{\delta \in I(n-1)} (q-1)^{|\delta|} q^{\mathrm{wt}(\delta)} \binom{n-1-2|\delta|}{k-1-|\delta|} \right\}.
\end{aligned}$$

This completes the proof. □

The following identity (Theorem 3.4, [1]) follows as a corollary to Theorem 1.2.

Corollary 2.4 *For a nonnegative integer n,*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}_q = \begin{cases} (1-q)(1-q^3)\cdots(1-q^{2m-1}) & \text{if } n = 2m \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Proof:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q &= \sum_{k=0}^{n}(-1)^k\left(\left(\sum_{\delta\in I(n)}(q-1)^{|\delta|}q^{\mathrm{wt}(\delta)}\right)\binom{n-2|\delta|}{k-|\delta|}\right)\\ &= \sum_{\delta\in I(n)}(q-1)^{|\delta|}q^{\mathrm{wt}(\delta)}\left(\sum_{k=0}^{n}(-1)^k\binom{n-2|\delta|}{k-|\delta|}\right).\end{aligned}$$

The right hand side in this expression is zero except when $n-2|\delta|=0$ and $k-|\delta|=0$. In that case n is even, say $2m$, and hence $k=m$. Then in the summation only the terms corresponding to fixed point free involutions will survive and we have, by Corollary 2.2,

$$\begin{aligned}\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}_q &= (-1)^m(q-1)^m\sum_{\delta\in I(2m,m)}q^{\mathrm{wt}(\delta)}\\ &= (1-q)^m[1]_q[3]_q\cdots[2m-1]_q\\ &= (1-q)(1-q^3)\cdots(1-q^{2m-1}).\end{aligned}$$

□

3. Fixed point free involutions

In this section we give a proof of Theorem 1.3. In order to study the poset $(F(2n),\leq)$, we will realize it as an induced subposet of $S(2n)$ (the set of permutations of $[2n]$), with strong Bruhat order. Let P be a finite, graded poset with minimum element $\hat{0}$ and maximum element $\hat{1}$, $\hat{0}\neq\hat{1}$. Let $\bar{P}=P-\{\hat{0},\hat{1}\}$. By the *order complex* of P we mean the simplicial complex $\Delta(\bar{P})$ of chains in $\bar{P}$ (all our simplicial complexes contain $\emptyset$ and $\dim(\emptyset)=-1$). Let $\mathrm{cov}(P)=\{(x,y)\in P\times P \mid y \text{ covers } x\}$. An *edge labeling* of P is a map $\lambda:\mathrm{cov}(P)\to\Lambda$, where Λ is some poset. An unrefinable chain $c: x_0<x_1<\cdots<x_n$ in P gets the induced label $\lambda(c)=(\lambda(x_0,x_1),\lambda(x_1,x_2),\ldots,\lambda(x_{n-1},x_n))$. The chain c is said to be *rising* if $\lambda(x_0,x_1)\leq\lambda(x_1,x_2)\leq\cdots\leq\lambda(x_{n-1},x_n)$. We say that λ is an *EL-labeling* if the following two properties are satisfied:

1. For every $x,y\in P$, $x<y$, there is a unique rising, unrefinable chain $c_{x,y}$ from x to y.
2. If a is any other unrefinable chain from x to y, $a\neq c_{x,y}$, then $\lambda(c_{x,y})<_l\lambda(a)$ in the lexicographic order.

Björner [2] showed that if P admits an EL-labeling, then $\Delta(\bar{P})$ is shellable (for the definition of shellable complexes see [3]).

We shall need the following elementary result.

Proposition 3.1 *Let P be a finite graded poset with $\hat{0}$, $\hat{1}$, $\hat{0}\neq\hat{1}$. Let $\lambda:cov(P)\longrightarrow\Lambda$ be an EL-labeling of P. Let $Q\subseteq P$ contain $\hat{0}$ and also a maximum element z (in the induced order), with $\hat{0}\neq z$. Assume that Q satisfies the following property: For all $x,y\in Q$, $x<y$, the unique rising chain from x to y in P lies in Q. Then Q (with induced order) is a graded poset with the same rank function as P and λ, restricted to $cov(Q)$, is an EL-labeling of Q.*

Proof: We claim that a cover $x < y$ in Q is also a cover in P. If not, the unique rising x to y chain c in P has length ≥ 2 and $c \subseteq Q$, contradicting the cover in Q. It follows that Q is graded and has the same rank function as P. The fact that λ is an EL-labeling of Q is now clear. □

We now recall some results on the strong Bruhat order on permutations. Let $\pi = \pi_1\pi_2\ldots\pi_n$ be a permutation in $S(n),\ n \geq 2$. The number of inversions in π, i.e., the number of pairs (i, j) with $i < j$ and $\pi_i > \pi_j$ is denoted $i(\pi)$. For $\sigma \in S(n)$, we write $\pi \sim \sigma$ if σ can be obtained from π by interchanging two of the π_i's. We say σ is obtained from π by an *inversion increasing interchange* if $\pi \sim \sigma$ and $i(\pi) < i(\sigma)$. Define $\pi \leq \tau$ if τ can be obtained from π by a sequence of zero or more inversion increasing interchanges. It is well known that $(S(n), \leq)$ is a graded poset of rank $\binom{n}{2}$, with rank of a permutation π given by $i(\pi)$, and with rank generating function

$$R(S(n), q) = [1]_q[2]_q \cdots [n]_q.$$

Let Λ be the set of ordered pairs $(i, j) \in [n] \times [n]$ such that $i < j$. Linearly order Λ lexicographically. Let $\lambda : \mathrm{cov}(S(n)) \to \Lambda$ be the labeling

$$\lambda(\pi, \sigma) = (i, j), \qquad (*)$$

where i and j are interchanged in π to get σ and $i < j$.

We shall need the following facts about the strong Bruhat order on $S(n)$.

(1) λ is an EL-labeling of $(S(n), \leq)$ (see [6]).
(2) If $\pi = \pi_1\pi_2\ldots\pi_n$ then σ covers π if and only if σ is obtained from π by interchanging π_i and π_j where $i < j$ and $\pi_i < \pi_j$ and each element of the set $\{\pi_{i+1}, \ldots, \pi_{j-1}\}$ is either $< \pi_i$ or $> \pi_j$.
(3) The order complex of $(S(n), \leq)$ triangulates the sphere of dimension $\binom{n}{2} - 2$ and hence $(S(n), \leq)$ is Eulerian (see [6]).
(4) Let $\pi = a_1a_2\ldots a_n, \sigma = b_1b_2\ldots b_n, \pi < \sigma$. For $l \in [n]$, let $\pi^{-1}(l)$ and $\sigma^{-1}(l)$ denote the positions of l in π and σ, respectively. Let i be the smallest number such that $\pi^{-1}(i) < \sigma^{-1}(i)$. Then $\pi^{-1}(k) = \sigma^{-1}(k)$ for $k = 1, 2, \ldots, i-1$. Let j be the smallest number such that $j > i$ and $\pi^{-1}(i) < \pi^{-1}(j) \leq \sigma^{-1}(i)$. Then the second element in the unique rising chain from π to σ is obtained from π by interchanging i and j (see [6]).

Consider the linearly ordered set $[\bar{n}] = \{1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}\}$ and define $E(\bar{n})$ to be the set of all permutations of $[\bar{n}]$ satisfying:

1. for $i = 1, 2, \ldots, n$, i appears before $\bar{i}$.
2. $1, 2, \ldots, n$ appear in increasing order.

For instance, $E(\bar{2}) = \{1\bar{1}2\bar{2}, 12\bar{1}\,\bar{2}, 12\bar{2}\,\bar{1}\}$.

Proposition 3.2 *Consider the set of permutations of* $[\bar{n}]$, $(n \geq 2)$, *under strong Bruhat order and let* λ *be the edge labeling given by* $(*)$. *Then the subset* $E(\bar{n})$ *satisfies the assumption of Proposition* 3.1.

Proof: We have $\hat{0} = 1\bar{1}2\bar{2}\cdots n\bar{n} \in E(\bar{n})$. Let $z = 12\ldots n\bar{n}\overline{n-1}\ldots\bar{1} \in E(\bar{n})$. Consider $\pi = \pi_1\pi_2\ldots\pi_{2n} \in E(\bar{n})$. If $\pi_1\pi_2\ldots\pi_n \neq 12\ldots n$ then find the smallest $i \geq 2$ such that $\pi_1\pi_2\ldots\pi_{i-1} = 12\ldots(i-1)$ and $i = \pi_l$, $l > i$. Then $\{\pi_i, \pi_{i+1}, \ldots, \pi_{l-1}\} \subseteq \{\bar{1}, \bar{2}, \ldots \overline{\imath - 1}\}$. By a sequence of inversion increasing interchanges we can take π to $\pi_1\pi_2\ldots\pi_{i-1}\pi_l\pi_i\ldots\pi_{l-1}\pi_{l+1}\ldots\pi_{2n} \in E(\bar{n})$. Repeating this step we see that $\pi \leq \sigma \in E(\bar{n})$, where $\sigma = \sigma_1\sigma_2\ldots\sigma_{2n}$ satisfies $\sigma_1\sigma_2\ldots\sigma_n = 12\ldots n$. Now by another sequence of inversion increasing interchanges it follows that $\sigma \leq z$. Thus, z is the maximum element of $E(\bar{n})$.

Let $\delta, \tau \in E(\bar{n})$, with $\delta < \tau$. Let x be the least element of $[\bar{n}]$ such that $\delta^{-1}(x) < \tau^{-1}(x)$. We claim that $x \in \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$. Assume not and let $x \in \{1, 2, \ldots, n\}$. Let $p = \delta^{-1}(x)$. Since every $j \in [\bar{n}]$, $j < x$, appears in the same positions in δ and τ, we see that τ contains an entry $>x$ in the pth position. This entry thus appears before x in τ, a contradiction to the fact that $\tau \in E(\bar{n})$. Thus $x \in \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$. Let $x = \bar{\imath}$.

Now let y be the smallest element in $[\bar{n}]$ such that $y > \bar{\imath}$ and $\delta^{-1}(\bar{\imath}) < \delta^{-1}(y) \leq \tau^{-1}(\bar{\imath})$. So we can write $\delta = \alpha_1\bar{\imath}\alpha_2 y\,\alpha_3$, for some strings $\alpha_1, \alpha_2, \alpha_3$, and where every element of the string α_2 is either $<\bar{\imath}$ or $>y$. Two cases arise:

- (i) $y = \bar{t} \in \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$: Then t is not an element of the string α_2 since $\bar{\imath} < t < \bar{t}$. It follows that $\alpha_1 y\alpha_2\bar{\imath}\alpha_3 \in E(\bar{n})$.
- (ii) $y = t \in \{1, 2, \ldots, n\}$: In this case no element of the string α_2 is $>t$, as this contradicts the fact that $\delta \in E(\bar{n})$. Thus every element of the string α_2 is $<\bar{\imath}$ and is in fact a member of $\{\bar{1}, \bar{2}, \ldots, \overline{\imath - 1}\}$. It now follows that $\alpha_1 y\alpha_2\bar{\imath}\alpha_3 \in E(\bar{n})$.

From fact (4) listed previously we get that the second element of the unique rising chain from δ to τ (in $S(\bar{n})$) belongs to $E(\bar{n})$. By induction, the entire rising chain belongs to $E(\bar{n})$. □

We now define a map $\phi : F(2n) \to E(\bar{n})$ as follows: Let $\delta \in F(2n)$ with $\delta = [a_1, b_1][a_2, b_2]\ldots[a_n, b_n]$. In the permutation $123\ldots(2n)$, replace $a_1, a_2, \ldots, a_n$ by $1, 2, \ldots, n$ respectively and replace $b_1, b_2, \ldots, b_n$ by $\bar{1}, \bar{2}, \ldots, \bar{n}$ respectively to get $\phi(\delta)$. It is easily seen that $\phi(\delta) \in E(\bar{n})$. For instance, $[1, 6][2, 3][4, 7][5, 8] \in F(8)$ gets mapped to $12\bar{2}34\bar{1}\ \bar{3}\ \bar{4} \in E(\bar{4})$. It is also easily seen that ϕ is a bijection.

Proposition 3.3 *For all* $\delta \in F(2n)$, $\mathrm{wt}(\delta) = i(\phi(\delta))$.

Proof: Let $\delta \in F(2n)$, with $\delta = [a_1, b_1][a_2, b_2]\ldots[a_n, b_n]$. Write $\phi(\delta) = \pi = \pi_1\pi_2\ldots\pi_{2n} \in E(\bar{n})$. Then $\pi_1 = 1$ and $\pi_{b_1} = \bar{1}$.

Define $\pi' \in E(\overline{n-1})$ as follows: Consider $\pi_2 \dots \pi_{b_1-1}\pi_{b_1+1} \dots \pi_{2n}$, replace i by $i-1$ and $\bar{i}$ by $\overline{i-1}$ for $i = 2, 3, \dots, n$ to get π'. Clearly,

$$i(\pi) = i(\pi') + (b_1 - 2).$$

Define $\delta' \in F(2n-2)$ as follows: Consider $[a_2, b_2][a_3, b_3] \dots [a_n, b_n]$, subtract 1 from all numbers $<b_1$ and subtract 2 from all numbers $>b_1$ to get δ'. Then

$$\mathrm{wt}(\delta) = \mathrm{wt}(\delta') + (b_1 - 2)$$

and $\phi(\delta') = \pi'$. The result now follows by induction. □

Proposition 3.4 *The map ϕ is an order isomorphism.*

Proof: We first show that ϕ^{-1} is order preserving. Let $\pi, \sigma \in E(\bar{n})$ with $\pi < \sigma$ and $i(\sigma) = i(\pi) + 1$. Write $\pi = \pi_1\pi_2 \dots \pi_{2n} \in E(\bar{n})$ and $\phi^{-1}(\pi) = [a_1, b_1][a_2, b_2] \dots [a_n, b_n]$. Let σ be obtained from π by interchanging π_i and π_j, where $i < j$ and $\pi_i < \pi_j$. If $\pi_i, \pi_j \in \{1, 2, \dots, n\}$, we cannot interchange π_i and π_j and remain in $E(\bar{n})$. A similar situation holds when $\pi_i \in \{1, 2, \dots, n\}$ and $\pi_j \in \{\bar{1}, \bar{2}, \dots, \bar{n}\}$. Thus, $\pi_i \in \{\bar{1}, \bar{2}, \dots, \bar{n}\}$. Let $\pi_i = \bar{l}$. We consider two cases:

(a) $\pi_j \in \{\bar{1}, \bar{2}, \dots, \bar{n}\}$: let $\pi_j = \bar{t}$ for some $t > l$ (see figure 4).
In this case exchanging b_l and b_t in $\phi^{-1}(\pi)$ gives $\phi^{-1}(\sigma)$. Since ϕ^{-1} takes number of inversions to weight we have $\phi^{-1}(\pi) < \phi^{-1}(\sigma)$.

(b) $\pi_j \in \{1, 2, \dots, n\}$: let $\pi_j = t$ for some $t > l$. Then $b_l < a_t$ (see figure 5).

Since $i(\sigma) = i(\pi) + 1$, it follows that all elements of the set $\{\pi_{i+1}, \dots, \pi_{j-1}\}$ are either $<\bar{l}$ or $>t$. This implies that no number in the set $\{b_l + 1, \dots, a_t - 1\}$ is the initial point

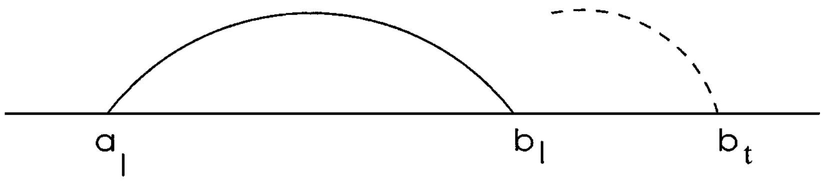

Figure 4.

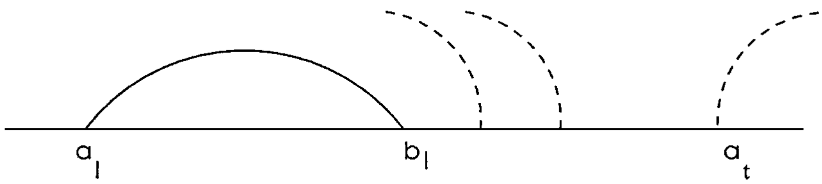

Figure 5.

of an arc in $\phi^{-1}(\pi)$. Thus, exchanging b_l and a_t in $\phi^{-1}(\pi)$ gives $\phi^{-1}(\sigma)$ in the standard representation. Since ϕ^{-1} takes number of inversions to weight we have $\phi^{-1}(\pi) < \phi^{-1}(\sigma)$. It follows that ϕ^{-1} is order preserving.

Now we show that ϕ is order preserving. Let $\delta \in F(2n)$ with $\delta = [a_1, b_1][a_2, b_2] \ldots [a_n, b_n]$. Let $\tau \sim \delta$. If τ is obtained from δ by exchanging b_i and a_j, $i < j$, then $\phi(\tau)$ is obtained from $\phi(\delta)$ by exchanging $\bar{\imath}$ and j. Thus, $\phi(\tau) \sim \phi(\delta)$. If, on the other hand, τ is obtained from δ by exchanging b_i and b_j, $i < j$, then $\phi(\tau)$ is obtained from $\phi(\delta)$ by exchanging $\bar{\imath}$ and $\bar{j}$. Thus, in this case also, $\phi(\tau) \sim \phi(\delta)$.

Since ϕ takes weight to the number of inversions, a weight increasing interchange corresponds to an inversion increasing interchange. $\square$

Proof of Theorem 1.3

(i) That $(F(2n), \leq)$ is a graded poset, with rank function given by weight, follows from Propositions 3.1, 3.2, 3.3 and 3.4. The rank of $(F(2n), \leq) = i(12 \ldots n\bar{n}\overline{(n-1)} \ldots \bar{1}) = 2\binom{n}{2}$. The rank generating function of $F(2n)$ follows from Corollary 2.2.

(ii) That $(F(2n), \leq)$ is EL-shellable follows from Propositions 3.1, 3.2 and 3.4. Now $\dim(\Delta(F(2n), \leq)) = 2\binom{n}{2} - 2$. To prove that the order complex of $F(2n)$ triangulates a ball we proceed as follows.

Consider the EL-labeling of $E(\bar{n})$ given by $(*)$. We claim that there is no unrefinable chain from $\hat{0} = 1\bar{1}2\bar{2} \ldots n\bar{n}$ to $\hat{1} = 12 \ldots n\bar{n}\,\overline{n-1} \ldots \bar{1}$ with a descent at every level. Suppose there were such a chain. Since $\bar{1}$ is the least element of $[\bar{n}]$ which changes its position from $\hat{0}$ to $\hat{1}$, the last few labels of this chain must be of the form $(\bar{1}, a)$ and all other labels (i, j) must satisfy $i \neq \bar{1}$. Thus this chain splits up into $\hat{0}$-π and π-$\hat{1}$ chains, where $\pi = 1\bar{1}23 \ldots n\bar{n}\ \overline{n-1} \ldots \bar{2}$. Then in the π-$\hat{1}$ chain the label $(\bar{1}, 2)$ occurs before $(\bar{1}, \bar{2})$. So somewhere in between there is no descent. Thus we arrive at a contradiction. Now for a poset P with an EL-labeling, $\mu_P(\hat{0}, \hat{1})$ is the number of unrefinable $\hat{0} - \hat{1}$ chains with descent at every level (see [14]). It follows that $\mu_{F(2n)}(\hat{0}, \hat{1}) = 0$.

Any $2\binom{n}{2} - 3$ dimensional face of $\Delta(E(\bar{n}))$ is a maximal chain in $\overline{E(\bar{n})}$ minus one element, say of rank i. Let $c : x_1 < x_2 < \cdots < x_{i-1} < x_{i+1} < \cdots < x_{2\binom{n}{2}-1}$ be such a face. Consider the rank 2 interval $[x_{i-1}, x_{i+1}]$. In the poset of all permutations of $[\bar{n}]$, $[x_{i-1}, x_{i+1}]$ has exactly 2 elements of rank i (since this poset is Eulerian). Thus, in $E(\bar{n})$ there are at most 2 elements of rank i in $[x_{i-1}, x_{i+1}]$. Thus, c is contained in at most two facets of $\Delta(F(2n))$. Now, a result of Danaraj and Klee [5] states that if Δ is a (pure) shellable complex, and if every face of dimension $\dim \Delta - 1$ is contained in at most 2 facets, then $|\Delta|$ is either a sphere or a ball of dimension $\dim \Delta$. Since $\mu_{F(2n)}(\hat{0}, \hat{1}) = 0$, it follows that $|\Delta(F(2n))|$ is a ball of dimension $2\binom{n}{2} - 2$. $\square$

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