Journal of Algebraic Combinatorics **14** (2001), 5–16

Graphs with Least Eigenvalue −2: The Star Complement Technique

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Received March 2, 1999; Revised March 14, 2001

Abstract. Let G be a connected graph with least eigenvalue -2, of multiplicity k. A *star complement* for -2 in G is an induced subgraph $H = G - X$ such that $|X| = k$ and -2 is not an eigenvalue of H. In the case that G is a generalized line graph, a characterization of such subgraphs is used to decribe the eigenspace of -2. In some instances, G itself can be characterized by a star complement. If G is not a generalized line graph, G is an *exceptional* graph, and in this case it is shown how a star complement can be used to construct G without recourse to root systems.

Keywords: graph, eigenvalue, eigenspace

1. Introduction

We take G to be an undirected graph without loops or multiple edges, with vertex set $V(G) = \{1, \dots, n\}$, and with (0, 1)-adjacency matrix A. Let P denote the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\mu)$ of A, and let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$. Since $\mathcal{E}(\mu)$ is spanned by the vectors $P\mathbf{e}_j$ $(j = 1, \dots, n)$ there exists $X \subseteq V(G)$ such that the vectors $P\mathbf{e}_j$ $(j \in X)$ form a basis for $\mathcal{E}(\mu)$. Such a subset X of $V(G)$ is called a *star set* for μ in G. (The terminology reflects the fact that the vectors $P\mathbf{e}_1, \dots, P\mathbf{e}_n$ form a eutactic star in the sense of Seidel [23]. In the context of star partitions [14, Section 7.1], star sets are called *star cells*.) An equivalent definition which is useful in a computational context is the following: if μ has multiplicity k then a star set for μ in G is a set X of k vertices of G such that μ is not an eigenvalue of $G - X$ [14, Theorem 7.2.9]. Here $G - X$ is the subgraph of G induced by $\overline{X}$, the complement of X in $V(G)$. Accordingly, the graph $G - X$ is called the *star complement* for μ corresponding to X. Such graphs are

This research was supported by EPSRC grant number GR/L94901.

called *μ-basic* subgraphs in [18]. (The dual terminology arises because some early results in this area were proved independently in [13, 18] and [20]; moreover the present authors were unaware of [18] when preparing the monograph [14].)

In this paper we discuss star complements for -2 in a connected graph having -2 as its least eigenvalue. It is proved in [5] that such a graph is either a generalized line graph (in particular a line graph) or a graph representable in the root system E_8, in the sense of [3, Section 3.11] (see also [6, Chapter 3]). The connected graphs which have least eigenvalue greater than or equal to -2, and which are not generalized line graphs, are called *exceptional*. It is shown in [5] that an exceptional graph has at most 36 vertices, and that this bound is best possible. Some further details of such graphs are given by Bussemaker and Neumaier [4].

In Section 2 we introduce the notion of a *foundation* for the root multigraph of a generalized line graph: it is used to characterize star complements for -2 and to describe the eigenspace of -2 in generalized line graphs. In Section 3, we show that a graph is exceptional if and only if it has an exceptional star complement for -2. By the Interlacing Theorem [8, Theorem 0.19] such a star complement has least eigenvalue greater than -2 and hence is one of 573 known graphs. We describe how the exceptional graphs can then be constructed as extensions of these graphs—this is the 'star complement technique' of the title. In fact, all the maximal extensions have now been found by these means using a computer [11]. We also point out that the exceptional graphs can be constructed in this way without recourse to root systems, although a mixture of old and new methods is more efficient in practice. Finally, in Section 4 we show how certain graphs with least eigenvalue -2 can be characterized by star complements for -2. The remainder of the present section is devoted to the preliminary results that we shall need.

Let X be a star set for the eigenvalue μ in an arbitrary graph G. If a single vertex is deleted from G then by interlacing the multiplicity of an eigenvalue changes by 1 at most. Accordingly the deletion of any r vertices from X $(0 < r < k)$ results in a graph for which μ is an eigenvalue of multiplicity $k-r$. We can also make the following observations: (i) if K is an induced subgraph of G not having μ as an eigenvalue then G has a star complement for μ containing K [22, Proposition 1.1], (ii) a connected graph has a connected star complement for each eigenvalue. The second observation appears in [18, pp. 250–251], attributed to S. Penrice. We shall need the following hybrid of results (i) and (ii):

Proposition 1.1 *Let μ be an eigenvalue of the connected graph G, and let K be a connected induced subgraph of G not having μ as an eigenvalue. Then G has a connected star complement for μ containing K.*

Proof: Let $|V(K)| = r$. Since G is connected we may order its vertices so that each vertex after the first is adjacent to a predecessor. Since K is connected we may take $1, \ldots, r$ to be the vertices of K. Let A be the adjacency matrix of G and let $\{\mathbf{c}_j : j \in Y\}$ be the basis of the column space of $\mu I - A$ obtained by deleting each column which is a linear combination of its predecessors. Note that $\{1, \ldots, r\} \subseteq Y$ because μ is not an eigenvalue of K. By [22, Lemma 2.3] the principal submatrix of $\mu I - A$ determined by Y is invertible. Since $|Y| = \text{codim}\mathcal{E}(\mu)$, $\overline{Y}$ is a star set for μ and the subgraph H induced by Y is a star complement.

We prove that H is connected by showing that each vertex y of Y with $y > 1$ is adjacent to a previous vertex j of Y. We take j to be least such that j is adjacent to y in G. Then $j < y$ and the y-th entry of $\mathbf{c}_j$ is -1. On the other hand the y-th entry of all preceding columns is 0, and so $\mathbf{c}_j$ is not a linear combination of its predecessors. Accordingly $j \in Y$ as required. □

The following fundamental result combines the Reconstruction Theorem [14, Theorem 7.4.1] with its converse [14, Theorem 7.4.4]. The Reconstruction Theorem is well-known in the context of Schur complements (cf. [19, p. 47]); in the graph-theoretical context, it appears as [13, Theorem 4.6] and [18, Theorem 1.1].

Theorem 1.2 *Let X be a set of vertices in the graph G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^T(\mu I - C)^{-1}B. \tag{1}$$

Note that if X is a star set for μ then the corresponding star complement $H(= G - X)$ has adjacency matrix C, and Eq. (1) tells us that G is determined by μ, H and the H-neighbourhoods of vertices in X. In this situation, let $|V(H)| = t$ and define a bilinear form on $\mathbb{R}^t$ by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T(\mu I - C)^{-1}\mathbf{y}$. Denote the columns of B by $\mathbf{b}_u (u \in X)$. To discuss graphs with a prescribed star complement we shall use the following consequence of Theorem 1.2 (cf. [18, Corollary 2.1]).

Corollary 1.3 *Suppose that μ is not an eigenvalue of the graph H. There exists a graph G with a star set X for μ such that $G - X = H$ if and only if the characteristic vectors $\mathbf{b}_u (u \in X)$ satisfy*
(i) $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \mu$ *for all* $u \in X$, *and*
(ii) $\langle \mathbf{b}_u, \mathbf{b}_v \rangle \in \{-1, 0\}$ *for all pairs* u, v *of distinct vertices in* X.

If G has H as a star complement for μ with corresponding star set X then each induced subgraph $G - Y (Y \subseteq X)$ also has H as a star complement for μ. Moreover any graph with H as a star complement for μ is an induced subgraph of such a graph G for which X is maximal, because H-neighbourhoods determine adjacencies among vertices in a star set. Accordingly, in determining all the graphs with H as a star complement for μ, it suffices to describe those for which a star set X is maximal. Such maximal graphs always exist when $\mu \neq -1, 0$ since then $|X|$ is bounded by 2^t: this is because distinct vertices of X have distinct H-neighbourhoods when $\mu \neq -1, 0$ [14, Corollary 7.3.6]. For $\Delta \subseteq V(H)$ let H_Δ denote the graph obtained from H by adding a vertex with Δ as its neighbourhood. For $\mu \in \mathbb{R}$ let $\mathcal{S}(H, \mu)$ be the set of those Δ for which μ is an eigenvalue of H_Δ but not of H. When $\mathcal{S}(H, \mu)$ is not empty it is convenient to define the *extendability graph* $\Gamma(H, \mu)$ on $\mathcal{S}(H, \mu)$ as follows: Δ_1 is adjacent to Δ_2 in $\Gamma(H, \mu)$ if and only if Δ_1, Δ_2 feature as H-neighbourhoods in a $(t + 2)$-vertex graph for which H is a star complement. If we identify a set Δ with its characteristic vector, this graph is the "compatibility graph" of [18, Algorithm 2.4]: the vertices of $\Gamma(H, \mu)$ are the $(0, 1)$-vectors $\mathbf{b}$ in $\mathbb{R}^t$ such that $\langle \mathbf{b}, \mathbf{b} \rangle = \mu$, and $\mathbf{b}_1 \sim \mathbf{b}_2$ if and only if $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle \in \{-1, 0\}$. The partial ordering by inclusion of the graphs G with a star set X for μ such that $G - X = H$ is determined by the partial ordering of cliques in $\Gamma(H, \mu)$. In particular, a graph G for which X is maximal corresponds to a maximal clique in $\Gamma(H, \mu)$; and if every maximal clique in $\Gamma(H, \mu)$ happens to determine a graph isomorphic to G then G is characterized by H. Certain generalized line graphs are characterized in this way in [15] by taking $\mu = -2$.

Finally, the following bound (see [21]) improves the exponential one noted above.

Theorem 1.4 *Let H be a star complement for μ in the graph G. If $\mu \neq -1, 0$ and H has t vertices ($t > 1$) then G has at most $t + \frac{1}{2}(t-1)(t+4)$ vertices.*

This bound of order $\frac{1}{2}t^2$ is asymptotically best possible as $t \to \infty$ because in $L(K_t)$ the eigenspace of -2 has codimension t.

2. Graph foundations

In order to introduce some convenient terminology, let μ be an eigenvalue of the line graph $L(G)$, and let Y be a set of edges of G. In accordance with the definition of line star partitions in [14, Section 7.8.3] we say that Y is a *line star set* for μ in G if it is a star set for μ in $L(G)$. In this situation $G \setminus Y$ (the spanning subgraph of G obtained by deleting the edges in Y) is the corresponding *line star complement* for μ in G. In particular, if $\mu = -2$ we call a line star complement a *foundation* for G. (A definition in the wider context of generalized line graphs is given in Definition 2.3.)

Example 1 The graph $L(K_5)$ has spectrum $6, 1^4, -2^5$ (where superscipts denote multiplicities), and a star complement for -2 has the form $L(F)$ where the foundation F is one of the graphs of figure 1. Here the graphs are shown in increasing order of the largest eigenvalue (or *index*).

In what follows the following definition will also be helpful.

Definition 2.1 An *orchid* is a unicyclic graph with an odd cycle, or a tree in which one pendant edge is doubled (i.e. replaced by a pendant double edge). An *orchid garden* is a graph whose components are orchids.

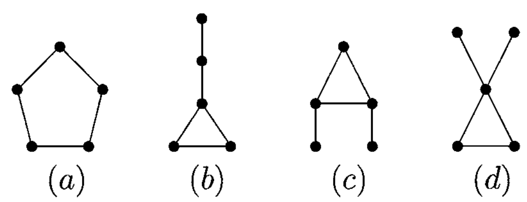

Figure 1. The foundations for K_5.

If G is a connected graph which is neither a tree nor an orchid, then the least eigenvalue of $L(G)$ is -2 [16]; moreover a foundation of G is a spanning tree of G if G is bipartite and a spanning orchid garden in G if G is non-bipartite [14, Theorem 7.8.13]. (Note that in the latter case all components in a foundation are unicyclic graphs with odd cycles.)

We now turn to generalized line graphs. As usual in this context, $CP(k)$ denotes a *cocktail party graph*, namely the regular graph of degree $2k-2$ on $2k$ vertices. Recall that if G is a graph with vertices $v_1, \ldots, v_n$ and if $a_1, \ldots, a_n$ are non-negative integers then the *generalized line graph* $L(G; a_1, \ldots, a_n)$ (or for short $L(G; a)$, where $a = (a_1, \ldots, a_n)$) consists of disjoint copies of $CP(a_1), \ldots, CP(a_n)$ and $L(G)$ together with edges from each vertex of $CP(a_i)$ to a vertex e of $L(G)$ whenever $v_i \in e$. If λ is the least eigenvalue of such a graph then $\lambda \geq -2$: this and other properties of generalized line graphs are described in [9]. In particular the multiplicity of λ is determined when $\lambda = -2$, and generalized line graphs are characterized by a family $\mathcal{F}$ of 31 forbidden subgraphs. (A graph is a generalized line graph if and only if it contains no member of $\mathcal{F}$.) The following result of Doob and Cvetković [17] is of interest in relation to graph foundations. (It appears as Theorem 1.3 of [7] with a misprint in part (v).)

Theorem 2.2 *If G is a connected graph with least eigenvalue greater than -2 then one of the following holds*:

(i) $G = L(T; 1, 0, \ldots, 0)$ *where T is a tree*;
(ii) $G = L(H)$ *where H is a tree or an odd unicyclic graph*;
(iii) *G is one of* 20 *graphs on* 6 *vertices represented by the root system E_6*;
(iv) *G is one of* 110 *graphs on* 7 *vertices represented by the root system E_7*;
(v) *G is one of* 443 *graphs on* 8 *vertices represented by the root system E_8*.

If G is a connected graph with least eigenvalue -2 then a connected star complement for -2 is necessarily a graph of one of the five types described above. We have already noted the role of graphs of type (ii) in line graphs; below we explain the role of graphs of type (i) and (ii) in generalized line graphs. In Section 3 we explain how the graphs of type (iii), (iv) and (v) can be constructed as extensions of certain generalized line graphs, without recourse to root systems.

Consider a generalized line graph $L(G; a)$, where G is connected and $\sum_{i=1}^{n} a_i > 0$. The *root graph* of $L(G; a)$ is defined in [9] as the multigraph H obtained from G by adding a_i pendant double edges at vertex v_i for each $i = 1, \ldots, n$. Then $L(G; a) = L(H)$ if we understand that in $L(H)$ two vertices are adjacent if and only if the corresponding edges in H have exactly one vertex in common. In the case that $L(H)$ has least eigenvalue -2 (i.e. $L(H)$ is not of type (i)) we say that the subgraph F of H is a *foundation* for H if $L(F)$ is a star complement for -2 in $L(H)$.

Definition 2.3 Let G be a multigraph whose generalized line graph has least eigenvalue -2. A *foundation* for G is a line star complement for -2 in G.

Example 2 Let H be the root multigraph of the generalized line graph $L(K_3; 1, 1, 0)$. Thus H consists of a triangle with a pair of double edges added to two vertices of a triangle. All non-isomorphic foundations of H are shown in figure 2.

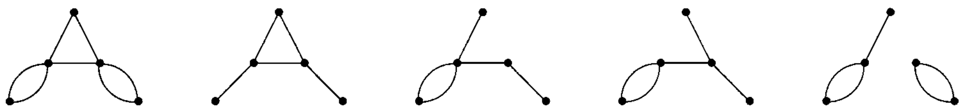

Figure 2. A multigraph and its foundations.

Theorem 2.4 *Let H be the root graph of the generalized line graph $L(G; a)$, where G is connected, $a \neq 0$ and $L(G; a)$ has least eigenvalue -2. A subgraph F of H is a foundation for H if and only if F is a spanning orchid garden of H.*

Proof: The multiplicity of -2 as an eigenvalue of $L(G; a)$ is $m - n + \sum_{i=1}^{n} a_i$, where m is the number of edges of G [9, Theorem 3.2], while the number of vertices of $L(G; a)$ is $m + 2\sum_{i=1}^{n} a_i$. Accordingly, a star complement for -2 has $n + \sum_{i=1}^{n} a_i$ vertices, the same number of vertices as H.

Now suppose that the subgraph F of H is a foundation for H. By Theorem 2.2 each component of $L(F)$ is of type (i) or (ii), since the graphs of types (iii), (iv), (v) are not generalized line graphs. It follows that each component of F is an orchid or a tree. In an orchid the number of vertices is the same as the number of edges. Accordingly, if a tree is present or if F is not a spanning subgraph then F has insufficient edges for $L(F)$ to be a star complement in $L(H)$. Hence F is a spanning orchid garden.

Conversely, suppose that F is a spanning orchid garden of H. Then $L(F)$ has least eigenvalue greater than -2 and the same number of vertices as H. Hence it is a star complement for -2 in $L(H)$, that is, F is a foundation for H. □

Remark In the situation of Theorem 2.4 we can construct a foundation F for the root graph of $L(G; a)$ from a foundation F' of G (if any) as follows. If G is not bipartite then F' is an orchid garden which spans G and we may take F to consist of F' together with a_i (single) pendant edges attached at vertex $v_i (i = 1, \ldots, n)$. If G is bipartite then F' is a tree which spans G: here we first modify F' by adding a_i pendant edges at vertex $v_i (i = 1, \ldots, n)$ and then obtain F by replacing one of these pendant edges by a double edge. Of course, not all foundations for the root graph of $L(G; a)$ can be constructed in this way.

Next we show how foundations of root graphs can be used to construct a basis for the eigenspace of -2 in generalized line graphs. This generalizes Doob's construction [16] of such a basis in the case of line graphs. We retain the notation of Theorem 2.4, and we use the term *supercycle* to mean either an odd cycle or a 2-cycle. There are $m - n + \sum_{i=1}^{n} a_i$ edges of H not in F, and since F is an orchid garden three possibilities arise when such an edge e is added to F: (1) the edge closes an even cycle, (2) the edge closes a supercycle (i.e. closes an odd cycle or doubles one pendant edge), (3) the edge joins a vertex of one orchid to a vertex of another orchid. We now ascribe weights to the edges of H as follows. In case (1) all weights are 0 except for 1 and -1 alternately on edges of the even cycle. In cases (2) and (3), $F + e$ contains a unique shortest path P between vertices of two different supercycles, and we first ascribe weights of 2 and -2 alternately to the edges of P.

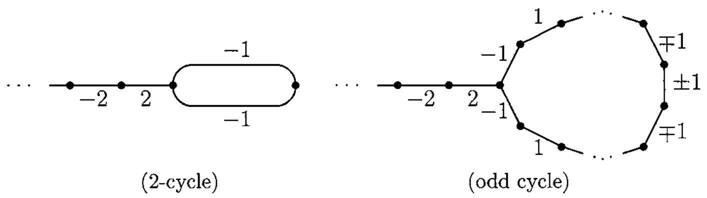

Figure 3. The construction of eigenvectors.

To within a unique choice of sign, weights are ascribed to the edges of the two supercycles as illustrated in figure 3, and all remaining weights are 0. (In all cases the construction may be seen as ascribing weights ± 1 alternately to the edges in a closed trail, with the assumption that double edges are assigned the same value; in edges traversed twice, the values are added.) In each case, the weights of edges in H are taken as co-ordinates of a vector whose entries are indexed by the corresponding vertices of $L(H)$. We call this vector the *characteristic vector* of the subgraph $F + e$, and we say that $F + e$ is *constructed* from F. It is straightforward to check that each characteristic vector is an eigenvector of $L(H)$ (that is, an eigenvector of $L(G; a)$) corresponding to -2. The $m - n + \sum_{i=1}^{n} a_i$ characteristic vectors are linearly independent because each of the aforementioned closed trails contains an edge not present in any of the others. Accordingly we have proved the following result.

Theorem 2.5 *The eigenspace for the eigenvalue* -2 *of a generalized line graph is generated by the characteristic vectors of subgraphs constructed from any foundation of the corresponding root graph.*

Corollary 2.6 *A connected generalized line graph has least eigenvalue equal to* -2 *if and only if the corresponding root graph contains either an even cycle or two supercycles connected by a path* (*possibly of length* 0).

In the proof of Theorem 2.5, the characteristic vector of $F + e$ is an eigenvector of $L(H)$ with the property that the co-ordinate corresponding to e is non-zero and the co-ordinates corresponding to all other vertices outside $L(F)$ are zero. Deletion of these zero co-ordinates results in a vector which spans the eigenspace of -2 in $L(F + e)$. This illustrates a general property of star complements which we formulate below; in the general setting the nice features of generalized line graphs are lost.

Now let G be any graph, and H any induced subgraph of G. We define the G*-extension* of a vector $(x_i)_{i \in V(H)}$ to be the vector $(y_i)_{i \in V(G)}$ with $y_i = x_i$ when $i \in V(H)$ and $y_i = 0$ otherwise. Let X be a star set in G, with complement $\overline{X}$ in $V(G)$, and let $H = G - X$. If $x \in X$ then we call the subgraph of G induced by $\overline{X} \cup \{x\}$ a *one-vertex extension* of H. We can now reformulate the statement and proof of [14, Theorem 7.8.6] as follows.

Theorem 2.7 *Let* G *be a graph with* μ *as an eigenvalue*, *and let* H *be a star complement for* μ *in* G.

(i) *Each one-vertex extension of H has μ as a simple eigenvalue and a corresponding eigenvector whose co-ordinate for the vertex outside H is non-zero.*
(ii) *The eigenspace of μ in G is generated by G-extensions of μ-eigenvectors in all the one-vertex extensions of H.*

Proof: From the introductory remarks in Section 1, we know that a one-vertex extension of H has μ as a simple eigenvalue. Both assertions now follow from the observation [14, p. 164] that in Theorem 1.2, the nullspace of $\mu I - A$ consists of the vectors $\binom{\mathbf{x}}{(\mu I - C)^{-1} B\mathbf{x}}$, where $\mathbf{x} \in I\!R^k$ and k is the multiplicity of μ. (For (i), we apply this result in the case that $X = \{x\}$.) □

Recall that an eigenvalue is a *main* eigenvalue if the corresponding eigenspace contains an eigenvector which is not orthogonal to the all-1 vector $\mathbf{j}$. It is well known that -2 is a non-main eigenvalue in line graphs (see for example [14, Corollary 2.6.5]). We can use Theorem 2.7 to determine precisely when -2 is a main eigenvalue of a generalized line graph: a necessary and sufficient condition is that $\mathbf{j}$ is not orthogonal to a characteristic vector constructed from some foundation F of the root graph, equivalently the sum of corresponding weights on the edges of some $F + e$ is non-zero. We can now prove the following result by inspecting the possible forms of $F + e$.

Corollary 2.8 *Let $L(G; a)$ be a connected generalized line graph with $a \neq 0$ and least eigenvalue -2. Then -2 is a main eigenvalue if and only if the root graph of $L(G; a)$ has an odd cycle or two 2-cycles connected by a path of odd length.*

Proof: Since $\sum_{i=1}^{n} a_i > 0$, the root graph H of $L(G; a)$ has a 2-cycle C. If an odd cycle Z is also present then H has a foundation F and an edge e not in F such that both C and Z lie in a component of $F + e$. Now as above we can construct an eigenvector corresponding to -2 not orthogonal to $\mathbf{j}$. Similar remarks apply when H contains two 2-cycles connected by a path of odd length. Conversely, if H has no odd cycles and no pair of 2-cycles connected by a path of odd length then by Theorem 2.7(ii), the eigenspace of -2 is orthogonal to $\mathbf{j}$. □

3. Exceptional graphs

Our objective here is to show (a) that exceptional graphs can always be constructed from exceptional star complements, and (b) that consequently it is possible for the exceptional graphs to be constructed independently of root systems. The exceptional graphs with least eigenvalue greater than -2 are those appearing in parts (iii)–(v) of Theorem 2.2. Those of type (v) are one-vertex extensions of graphs of type (iv), which are in turn one-vertex extensions of graphs of type (iii). The 110 graphs of type (iv) are identified in [10] by means of the list of 7-vertex graphs in [7]. The twenty 6-vertex graphs of type (iii) are identified in [12], and are here denoted by $F_1, \ldots, F_{20}$. They belong to the family $\mathcal{F}$ of 31 minimal forbidden subgraphs which characterize generalized line graphs, the other eleven having least eigenvalue less than -2. Accordingly we can make the following assertion.

Proposition 3.1 *A graph is exceptional if and only if its least eigenvalue is greater than or equal to* -2 *and it contains as an induced subgraph one of the graphs* $F_1, \ldots, F_{20}$.

We can now prove the main result of this section.

Theorem 3.2 *Let G be a connected graph with least eigenvalue* -2*. Then G is exceptional if and only if it has an exceptional star complement for* -2.

Proof: If G has an exceptional star complement H, then H contains a forbidden subgraph as an induced subgraph. Since the same is now true of G, G is exceptional. Conversely, suppose that G is exceptional. By Proposition 3.1, G contains an induced subgraph F isomorphic to some F_j. By Proposition 1.1, G has a connected star complement H for -2 which contains F. Since H is exceptional, the Theorem follows. □

It is shown in [9], without using root systems, that if G is a minimal nongeneralized line graph then (i) G has at most 8 vertices, (ii) each vertex-deleted subgraph of G is of the form $L(H; a_1, \ldots, a_n)$ where either H is a tree and $\Sigma_i a_i \leq 2$ or H is unicylic and $\Sigma_i a_i = 1$. One can then find all the minimal nongeneralized line graphs by inspecting each connected one-vertex extension of each possible $L(H; a_1, \ldots, a_n)$. Those with least eigenvalue greater than -2 are the graphs $F_1, \ldots, F_{20}$. The connected one-vertex extensions of $F_1, \ldots, F_{20}$ with least eigenvalue greater than -2 are the graphs of type (iv); and the connected one-vertex extensions of these with least eigenvalue greater than -2 are the graphs of type (v). Finally we find the following.

Proposition 3.3 *Any connected one-vertex extension of any of the* 443 *graphs of type* (v) *has least eigenvalue smaller than or equal to* -2.

In this way we establish, without recourse to root systems, the following fact.

Theorem 3.4 *The set of exceptional graphs with least eigenvalue greater than* -2 *is finite.* (*There are* 573 *such graphs and they are of types* (iii)–(v) *from Theorem* 2.2.)

In view of Theorem 3.2, all exceptional graphs can be constructed as extensions of one of the 573 exceptional graphs with 6, 7 or 8 vertices which feature in Theorem 2.2. We know from Theorem 1.4 that the graphs which arise have at most 50 vertices; by the proof of [21, Theorem 2.1] that bound can be reduced to 44 vertices; but if we carry out the construction [11] then we find that the largest graph has 36 vertices. In any case we have the following theorem.

Theorem 3.5 *The set of exceptional graphs is finite.*

Computational results, described in [10], establish the following relevant facts. There are exactly 10 maximal graphs having one of $F_1, \ldots, F_{20}$ as a star complement for the eigenvalue -2. If we take the 110 exceptional graphs of type (iv) in the role of a star complement for -2 then we find exactly 39 mutually non-isomorphic maximal graphs.

These comprise one graph on 14 vertices (non-regular with spectrum $8, 1^6, -2^7$), 28 graphs on 17 vertices, five graphs on 18 vertices, one on 19 vertices, two on 20, one on 22 vertices and one on 27 vertices (the Schläfli graph with spectrum $16, 4^6, -2^{20}$). The graphs which arise in this way are not maximal exceptional graphs because they have representations in E_7.

It was an enormous task to generate all maximal graphs starting from the 443 star complements of type (v). (For a particularly difficult case a PC-586 computer took about 24 hours to produce all 1048580 maximal graphs which fall into 457 isomorphism classes.) The maximal graphs which arise here are the maximal exceptional graphs; there are 473 of them, and details appear in [11]. Their distribution by number of vertices is as follows.

Number of vertices	22	28	29	30	31	32	33	34	36
Number of graphs	1	1	432	25	7	3	1	2	1

4. Characterizations by star complements

We have noted that if $\mu \neq -1, 0$, there are only finitely graphs with a prescribed star complement. Accordingly we can try to characterize graphs by their star complements. For the most part, such characterizations depend on solving Eq. (1) for a given matrix C (cf. [10, 15]). In the case that $\mu = -2$ however we can exploit our knowledge of the graphs with least eigenvalue -2.

Theorem 4.1 *Let T be a tree on n vertices, and let G be a maximal graph with a star complement $L(T)$ for the eigenvalue -2. If $n \geq 37$ then $G = L(K_{p,q})$, where $p + q = n$ and $V(T)$ can be partitioned into colour classes of sizes p, q.*

We omit the proof since it is similar to the proof of the next theorem.

Theorem 4.2 *Let G be a maximal graph with a star complement C_t (t odd) for the eigenvalue -2. If $t \geq 37$ then $G = L(K_t)$.*

Proof: From the theory of star complements we know that G is connected [22, Proposition 2.1(i)] and has least eigenvalue -2 (by interlacing). Since $t \geq 37$, G is not exceptional. If G is a generalized line graph then C_t is a spanning orchid garden of the corresponding root graph only when G is the line graph of a t-vertex graph. Since G is maximal, $G = L(K_t)$. □

In the above proof we have relied on root system arguments. If we restrict ourselves to only the star complement technique then *we need to assume that* $t \geq 45$. However this restriction can be eliminated because Bell [1] has recently proved independently that the conclusion of Theorem 4.2 holds for all odd $t > 3$.

In a similar fashion we can prove the following more general result.

Theorem 4.3 *Let G be a connected maximal graph with a star complement $L(O)$ for the eigenvalue -2, where O is an orchid garden on $n(>3)$ vertices with p pendant edges and q 2-cycles. If the number of vertices of G is sufficiently large then $G = L(K_s; a_1, \dots, a_s)$, where $s + \sum_{i=1}^{s} a_i = n$ and $\sum_{i=1}^{s} a_i \leq p + q$.*

Here the inequality $\sum_{i=1}^{s} a_i \leq p + q$ follows from the fact that any pendant edge of O (not yet paired) may be one of a pair of double edges in the root graph of a generalized line graph. The edges of O not paired in this way belong to the complete graph K_s. This explains the essential difference between the roles of the graphs proposed in [2] as canonical star complements for -2 in $L(K_t)$. These are the star complements $L(O)$ for which the spectral radius (or *index*) of O is extremal. For odd t the graphs O in question are C_t and the graph R_t obtained from the star $K_{1,t-1}$ by adding an edge. For odd $t \geq 37$, $L(K_t)$ is the only maximal graph with $L(C_t)(= C_t)$ as a star complement for -2. On the other hand, if we regard R_t as a triangle on vertices 1, 2, 3 with $t-3$ pendant edges added at vertex 1, then its line graph is a star complement for -2 not only in $L(K_t)$ but also in each of the (maximal) graphs $L(K_{t-a}; a, 0, \dots, 0)$ $(a = 1, \dots, t-3)$.

Acknowledgment

The authors are grateful to M. Lepović for his help in the computer generation of maximal graphs.

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