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Brauer Diagrams, Updown Tableaux and Nilpotent Matrices

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Abstract. We interpret geometrically a variant of the Robinson-Schensted correspondence which links Brauer diagrams with updown tableaux, in the spirit of Steinberg's result [32] on the original Robinson-Schensted correspondence. Our result uses the variety of all $(N, \omega, \boldsymbol{V})$ where $\boldsymbol{V}$ is a complete flag in $\mathbb{C}^{2n}$, ω is a nondegenerate alternating bilinear form on $\mathbb{C}^{2n}$, and N is a nilpotent element of the Lie algebra of the simultaneous stabilizer of both ω and $\boldsymbol{V}$, instead of Steinberg's variety of $(N, \boldsymbol{V}, \boldsymbol{V}')$ where $\boldsymbol{V}$ and $\boldsymbol{V}'$ are two complete flags in $\mathbb{C}^n$ and N is a nilpotent element of the Lie algebra of the simultaneous stabilizer of both $\boldsymbol{V}$ and $\boldsymbol{V}'$.

Keywords: Robinson-Schensted correspondence, Brauer algebra, Young diagram, nilpotent matrix, symplectic form

1. Introduction

We will interpret the "updown analogue" of the Robinson-Schensted correspondence (initially given by R. Stanley (see [33, Lemma 8.3 and the footnote on p. 60]), then more generally by S. Sundaram ([33, Lemma 8.7] and [34]), and also later modified by T. Roby ([22])) in the spirit of R. Steinberg's result [32] for the original Robinson-Schensted correspondence, namely by way of parametrizing the irreducible components of an algebraic variety in two ways. Although many variants of the Robinson-Schensted correspondence have been devised by now, the only other analysis in this direction seems to have been given by M. van Leeuwen [38] for his orthogonal and symplectic group versions. (However, see Note 1 at the end.) In this section, we will briefly summarize the history of the Robinson-Schensted correspondence and Steinberg's interpretation, then introduce the objects involved in the updown version, describing how the following sections are organized. Let us express our gratitude to J. Matsuzawa, B. Srinivasan, S. Fomin, T. Kobayashi, T. Oshima, K. Koike, Y. Tanaka, M. Yamaguchi, R. Stanley, D. Vogan, C. Krattenthaler, M. van Leeuwen, J. Stembridge, T. Roby, R. Proctor, J. Stroomer, G. Benkart, N. Nakayama, M. Saks, S. Sundaram, and G. Tesler for valuable comments and discussions which brought us inspirations, encouragements and information. Finally we thank the referee for suggesting many improvements on the preliminary manuscript.

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1.1. The original Robinson-Schensted correspondence

A *partition* λ is a nonincreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of positive integers, and l, also denoted by $l(\lambda)$, is called its *length*. Write $|\lambda|$ for $\sum_{i=1}^{l} \lambda_i$. If λ is a partition and if $|\lambda| = n$, then λ is called a partition of n, and we write $\lambda \vdash n$. The *Young diagram* of λ is the subset of $\mathbb{N} \times \mathbb{N}$ consisting of all (i, j) satisfying $j \leqq \lambda_i$ (called its *cells*), often denoted by λ itself and visualized as in figure 1. The set of all partitions form a lattice, called *Young's lattice*, by containment of their Young diagrams. We write $\lambda \subset \mu$ for this partial order. In any poset $(\Pi, \prec)$, we write $x \dot{\prec} y$ if $x \precneqq y$ and there is no $z \in \Pi$ such that $x \precneqq z \precneqq y$. In Young's lattice, $\lambda \dot{\subset} \mu$ is equivalent to $\lambda \subset \mu$ and $|\mu| = |\lambda| + 1$. A *standard tableau* of *shape* $\lambda \vdash n$ is a labeling of the cells of λ by integers from 1 through n in such a way that the labels increase along its rows from left to right, and along its columns from top to bottom. The label of the cell (i, j) is denoted by $T(i, j)$. The set of all standard tableaux of shape λ will be denoted by STab(λ). The standard tableaux T of shape λ are in 1-1 correspondence with the saturated chains of partitions $\varnothing = \lambda^{(0)} \dot{\subset} \lambda^{(1)} \dot{\subset} \lambda^{(2)} \dot{\subset} \cdots \dot{\subset} \lambda^{(n)} = \lambda$ from $\varnothing$ to λ, where $\lambda^{(i)}$ is determined from T as the set of cells having labels $\leqq i$ in T (see figure 2).

We denote the symmetric group of degree n by $\mathfrak{S}_n$. The Robinson-Schensted correspondence, which associates with each $w \in \mathfrak{S}_n$ a pair $(P(w), Q(w))$ of standard tableaux, both of some shape $\lambda \vdash n$ which depends on w, was introduced initially by G. de B. Robinson [21] in an attempt to give a proof of the Littlewood-Richardson rule in the representation theory

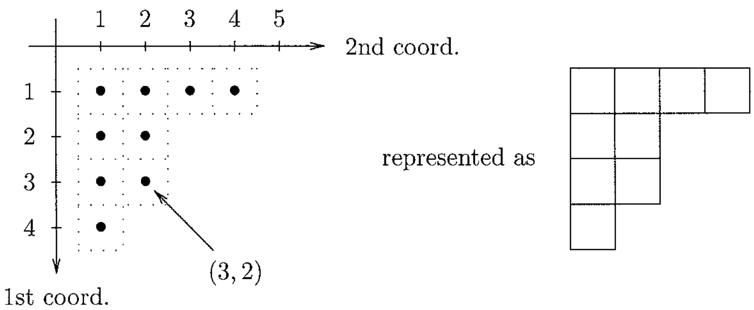


Figure 1. Young diagram of (4, 2, 2, 1).

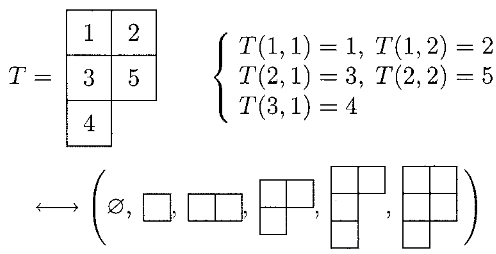


Figure 2. An example of a standard tableau of shape (2, 2, 1).

of the symmetric group, and later by C. Schensted [28] to analyze the longest increasing subsequences in permutations. Its interesting combinatorial structure has then been extensively studied by A. Lascoux and M.-P. Schützenberger, D. Knuth, C. Greene, S. Fomin, and many others. A connection with left cells and right cells of the symmetric group was given by D. Kazhdan and G. Lusztig, and a connection with the representations of a quantized general linear Lie algebra was given by E. Date, M. Jimbo, and T. Miwa, which was among what inspired a more general theory of crystal basis by M. Kashiwara.

1.2. Steinberg's interpretation

Let Z denote the algebraic variety consisting of all triples $(N, \boldsymbol{V}, \boldsymbol{V}')$, where N is a nilpotent n by n matrix and $\boldsymbol{V}$ and $\boldsymbol{V}'$ are complete flags in $\mathbb{C}^n$, such that the 1-parameter group $\{\exp tN \mid t \in \mathbb{C}\}$ fixes both $\boldsymbol{V}$ and $\boldsymbol{V}'$. Here a *complete flag* $\boldsymbol{V}$ in an n-dimensional vector space V is by definition a maximal chain in the lattice $\mathcal{L}(V)$ of all linear subspaces of V, ordered by containment, namely a sequence $(V_i)_{i=0}^n$ of subspaces of V such that $0 = V_0 \mathrel{\dot\subset} V_1 \mathrel{\dot\subset} V_2 \mathrel{\dot\subset} \cdots \mathrel{\dot\subset} V_n = V$, where $W \mathrel{\dot\subset} W'$ means that $W \subsetneqq W'$ and there is no subspace W'' satisfying $W \subsetneqq W'' \subsetneqq W'$ (which is equivalent to saying that $W \subset W'$ and $\dim W' = \dim W + 1$). Saying that $\{\exp tN \mid t \in \mathbb{C}\}$ fixes $\boldsymbol{V} = (V_i)$ is equivalent to saying that N lies in the Lie algebra of the stabilizer of $\boldsymbol{V}$ in G, or that N maps each component V_i of $\boldsymbol{V}$ into V_i itself. We also express this by calling $\boldsymbol{V}$ an N-*stable* flag.

Let $\operatorname{Irr} Z$ denote the set of irreducible components of Z. He gave two ways to parametrize $\operatorname{Irr} Z$: one by the permutations of n letters, and the other by the pairs of n-cell standard tableaux (T, T') such that the shapes of T and T' are the same. This establishes a 1-1 correspondence between $\mathfrak{S}_n$ and $\coprod_{\lambda \vdash n} \mathrm{STab}(\lambda) \times \mathrm{STab}(\lambda)$, and he showed that this coincides with the Robinson-Schensted correspondence. The way he parametrized $\operatorname{Irr} Z$ is by giving a partition of Z into irreducible locally closed subvarieties of the same dimension. Then the closures of these subvarieties constitute $\operatorname{Irr} Z$ by the following general argument.

Namely, in general, if Z is an algebraic variety over $\mathbb{C}$, and if $Z = \coprod_{\alpha \in A} Z_\alpha$ is a partition of Z into a finite number of irreducible locally closed subvarieties of the same dimension m, then $Z = \bigcup_{\alpha \in A} \overline{Z_\alpha}$ is the decomposition of Z into its irreducible components. One sees this by noting the following two facts. First, each $\overline{Z_\alpha}$ is irreducible since each Z_α is irreducible. Secondly, for any $\alpha \in A$, the union $\bigcup_{\substack{\beta \in A \\ \beta \neq \alpha}} \overline{Z_\beta}$ of the closures of the other pieces cannot contain the whole Z_α (and hence Z), since $\overline{Z_\beta} \cap Z_\alpha \subset \overline{Z_\beta} - Z_\beta$ has dimension strictly smaller than m for each β $(\neq \alpha)$, and an irreducible variety cannot be covered by a finite number of subvarieties of strictly lower dimensions.

Now let us return to Steinberg's variety Z. The first partition of Z is given by looking at the relative positions of the two complete flags. Namely, let $\pi_{X \times X}$ denote the projection $Z \ni (N, \boldsymbol{V}, \boldsymbol{V}') \mapsto (\boldsymbol{V}, \boldsymbol{V}') \in X \times X$, where X denotes the set of all complete flags in $\mathbb{C}^n$. The group $G = GL(n, \mathbb{C})$ has a natural (transitive) action on X, and hence acts diagonally on $X \times X$. The Bruhat decomposition of G shows that $X \times X$ is partitioned into the G-orbits $\mathcal{O}_w$, $w \in \mathfrak{S}_n$ (since $\mathfrak{S}_n$ is the Weyl group of G), which are irreducible locally closed subvarieties. Then the $Z_w = \pi_{X \times X}^{-1}(\mathcal{O}_w)$, $w \in \mathfrak{S}_n$, are locally closed subvarieties into which Z is partitioned, and each piece Z_w is actually irreducible because it is a vector bundle over $\mathcal{O}_w$. Their dimensions turn out to be all equal because the differences in dimension of the $\mathcal{O}_w$ are exactly complemented by the dimensions of the fibers.

The second partition of Z is given by looking at the Jordan types of N restricted to the subspaces constituting $\boldsymbol{V}$ and $\boldsymbol{V}'$. Earlier, N. Spaltenstein [29] had studied the variety X_N of N-stable complete flags. An N-stable flag $\boldsymbol{V} = (V_i)_{i=0}^n$ determines a sequence of partitions $(\lambda^{(i)})_{i=0}^n$, where $\lambda^{(i)}$ is the *Jordan type* of $N|_{V_i}$, namely the partition comprising the sizes of the blocks of its Jordan canonical form. It is a saturated chain from $\varnothing$ to λ in Young's lattice, to which one can associate a standard tableau T of shape λ by the rule described in Section 1.1. Let us call T the (N-) *type* (*tableau*) of $\boldsymbol{V}$, and let $X_{N,T}$ denote the collection of all N-stable flags of type T. Spaltenstein showed that, for fixed N of Jordan type λ, the $X_{N,T}$, $T \in \mathrm{STab}(\lambda)$, are irreducible locally closed subvarieties of the same dimension into which X_N is partitioned, so that their closures give the irreducible components of X_N. Now, for each pair (T, T') of standard tableaux of the same shape, let $Z_{T,T'}$ be the collection of $(N, \boldsymbol{V}, \boldsymbol{V}') \in Z$ such that $\boldsymbol{V}$ and $\boldsymbol{V}'$ have N-types T and T' respectively. It is a locally closed subvariety of Z, and is irreducible since there is a surjective map $G \times X_{N_0,T} \times X_{N_0,T'} \to Z_{T,T'}$, $(g, \boldsymbol{V}, \boldsymbol{V}') \mapsto (\mathrm{Ad}(g)N_0, g \cdot \boldsymbol{V}, g \cdot \boldsymbol{V}')$, where N_0 is a fixed nilpotent element of Jordan type λ. Moreover, its dimension turns out to be independent of T and T'. So the partition of Z into the $Z_{T,T'}$, $(T, T') \in \coprod_{\lambda \vdash n} \mathrm{STab}(\lambda) \times \mathrm{STab}(\lambda)$, has the desired property.

Steinberg gave a down-to-earth argument to show that the bijection determined by these parametrizations coincides with the Robinson-Schensted correspondence. Here we follow a result by M. Saks ([26, Theorems 3.1 and 3.2]) for posets, or a result obtained independently by E. Gansner ([9, Theorem 2.1]) and Saks ([27, Theorem 5.16]) for acyclic digraphs. We use the latter formulation, but we only state it for posets (which amount to "transitive" acyclic digraphs). If $(\Pi, \prec)$ is a finite poset, we follow the terminology in [9] and call a matrix $A = (a_{pq})$ with entries in $\mathbb{C}$ and with rows and columns indexed by Π a *generic matrix of* Π if (1) $a_{pq} = 0$ unless $p \precneqq q$, and (2) the a_{pq}, $p \precneqq q$, are algebraically independent over $\mathbb{Q}$. (Saks uses different terminology; see [26] and [27].) Their result says that the Jordan type $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of such a matrix A coincides with the Greene-Kleitman invariant of Π, namely λ_1 equals the maximum number of elements of Π contained in a chain, $\lambda_1 + \lambda_2$ equals the maximum number of elements of Π contained in a union of two chains, and so on. Now let $w \in \mathfrak{S}_n$, and let $\Pi(w) = (\{1, 2, \ldots, n\}, \prec_w)$ be the poset in which $p \prec_w q$ means $p \leqq q$ and $w^{-1}(p) \leqq w^{-1}(q)$. For each i, let $\Pi_i(w)$ (resp. $\Pi^i(w)$) denote the subset $\{1, 2, \ldots, i\}$ (resp. $\{w(1), w(2), \ldots, w(i)\}$) with the induced poset structure. (Note that these are order ideals in $\Pi(w)$.) Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)$ denote the standard basis of $\mathbb{C}^n$, and for each $w \in \mathfrak{S}_n$ let $\boldsymbol{V}^w = (V_i^w) \in X$ be defined by $V_i^w = \sum_{i'=1}^{i} \mathbb{C}\boldsymbol{e}_{w(i')}$. Then $(\boldsymbol{V}^e, \boldsymbol{V}^w)$ is a representative of $\mathcal{O}_w$, where e denotes the identity element of $\mathfrak{S}_n$. The nilpotent matrices stabilizing both $\boldsymbol{V}^e$ and $\boldsymbol{V}^w$ form a vector space with basis $\{E_{pq} \mid p \precneqq_w q\}$, where E_{pq} denotes the (p, q)th matrix unit, namely the matrix with 1 in the (p, q)th position and 0 in all other positions. The generic matrices of $\Pi(w)$ constitute a Zariski dense subset of this vector space (it is easy to see that such a subset of an affine space is Zariski dense by induction on the dimension). By the above result of Gansner and Saks, the Jordan type of A coincides with the Greene-Kleitman invariant of $\Pi(w)$. This in turn coincides with the common shape of $P(w)$ and $Q(w)$ by a result of Greene [11]. Moreover, $A|_{V_i^e}$ (resp. $A|_{V_i^w}$) is represented by the submatrix of A with row and column indices in $\Pi_i(w)$ (resp. $\Pi^i(w)$), which is again a generic matrix of $\Pi_i(w)$ (resp. $\Pi^i(w)$). Applying Greene's result for the subword of w consisting of the letters $\leqq i$ (resp. the initial i letters of the word w), one sees that its Jordan type coincides with the ith term of the chain of partitions corresponding to $P(w)$ (resp. $Q(w)$). This means that a dense subset of Z_w lies in $Z_{P(w),Q(w)}$, hence $\overline{Z_w} = \overline{Z_{P(w),Q(w)}}$ by the arguments above.

1.3. The outline of our result

With preceding subsections as background, let us draw the outline of our case, namely the case regarding the correspondence between Brauer diagrams and updown tableaux. After providing some preliminaries, we quote the key results from later sections. They retain the same numbers (such as Theorem 6.2) as appear later in their proper places. Some of the statements are slightly rephrased, but the equivalence will be easily recognized.

Let n continue to denote a positive integer, and let $\mathcal{D}_{2n}$ denote the set of *Brauer diagrams* on $\{1, 2, \ldots, 2n\}$, by which we mean graphs with vertex set $\{1, 2, \ldots, 2n\}$ and degree sequence $(\overbrace{1, 1, \ldots, 1}^{2n})$. R. Brauer [4] used them in the representation theory of orthogonal groups, in the two-line notation as in figure 3(a), to represent the basis elements of what is now called the Brauer algebra. We continue to denote the symmetric group of degree n by $\mathfrak{S}_n$. It can be regarded as a subset of $\mathcal{D}_{2n}$ consisting of the permutation diagrams like figure 3(b). We write Brauer diagrams in one line as in figure 3(c).

The information carried by such a diagram is a set partition of $\{1, 2, \ldots, 2n\}$ into n blocks of size 2. It also represents a coset in $\mathfrak{S}_{2n}/W(B_n)$, where $W(B_n)$ is the Weyl group of type B_n (also called the hyperoctahedral group or the group of signed permutations) embedded into $\mathfrak{S}_{2n}$ as the centralizer of the element $w_0 = \left(\begin{smallmatrix} 1 & 2 & \cdots & 2n \\ 2n & 2n-1 & \cdots & 1 \end{smallmatrix}\right)$. One sees this by letting $\mathfrak{S}_{2n}$ act on $\mathcal{D}_{2n}$ from the left by permuting the vertices. This action is transitive, and $W(B_n)$ is the stabilizer of the element $d_0 \in \mathcal{D}_{2n}$ corresponding to the set partition $\{\{1, 2n\}, \{2, 2n-1\}, \ldots, \{n, n+1\}\}$.

For any i, $1 \leqq i \leqq 2n$, put $i' = w_0(i)$.

For each $d \in \mathcal{D}_{2n}$, we define an element w_d of $\mathfrak{S}_{2n}$, which will be a representative of a coset in $W(B_n)\backslash\mathfrak{S}_{2n}$ (rather than $\mathfrak{S}_{2n}/W(B_n)$ due to technical reasons) corresponding

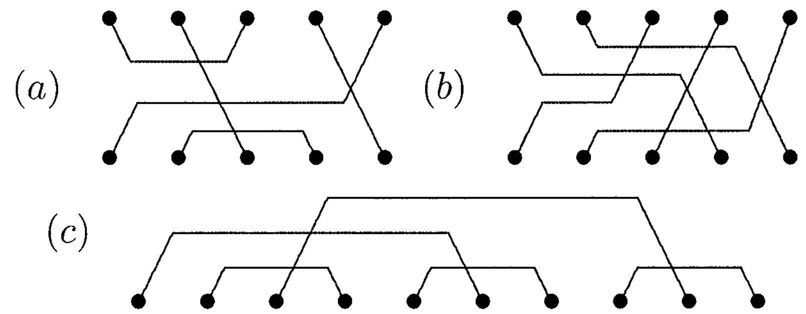
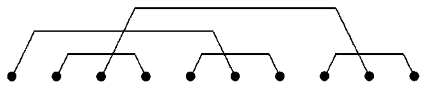


Figure 3. An example of a Brauer diagram on 10 points.

$d =$ 

position labeling $\to$	1	2	3	4	5	6	7	8	9	10
d-labeling $\to$	1	2	3	$2'$	4	$1'$	$4'$	5	$3'$	$5'$

$$w_d = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 9 & 4 & 10 & 7 & 5 & 8 & 6 \end{pmatrix}$$

$$i_d = (1\,6)(2\,4)(3\,9)(5\,7)(8\,10)$$

Figure 4. An example of w_d and i_d.

to d. See figure 4. Let $1, 2, \ldots, 2n$ be the original labeling of the vertices of d, which we call the "position labeling". We define another labeling of the vertices, which we call the "d-labeling", as follows: (1) label the n "left-end vertices" of the edges in d by $1, 2, \ldots, n$ from left to right; (2) for each i, $1 \leqq i \leqq n$, let i' label the "right-end vertex" linked with the "left-end vertex" having the d-label i; and (3) define w_d using the two-line notation, by putting the position labels in the upper row, and putting the d-labels in the lower row. Its inverse is easier to write down: if $a_1 < a_2 < \cdots < a_n$ are the position labels of the "left-end vertices" of d, and if b_i is the position label of the "right-end vertex" linked with the left-end vertex at position a_i, then $w_d^{-1} = \left(\begin{smallmatrix} 1 & 2 & \cdots & n & n+1 & \cdots & 2n-1 & 2n \\ a_1 & a_2 & \cdots & a_n & b_n & \cdots & b_2 & b_1 \end{smallmatrix}\right)$. The element w_d^{-1} sends d_0 to d by the action described above, so that w_d^{-1} lies in the coset in $\mathfrak{S}_{2n}/W(B_n)$ corresponding to d. Hence the set of w_d, obtained from all $d \in \mathcal{D}_{2n}$, constitutes a complete set of representatives of the cosets in $W(B_n)\backslash\mathfrak{S}_{2n}$. We denote this set by D_{2n}. It consists of all $w \in \mathfrak{S}_{2n}$ that satisfy $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(n)$ and $w^{-1}(i) < w^{-1}(i')$ for all $1 \leqq i \leqq n$.

People sometimes identify $d \in \mathcal{D}_{2n}$ with the fixed-point-free involution $(a_1\ b_1)(a_2\ b_2)\cdots(a_n\ b_n) \in \mathfrak{S}_{2n}$, where the a_i and the b_j are determined from d as above (also see figure 4). We denote this involution by i_d. Note that i_d is related to w_d by $i_d = w_d^{-1} w_0 w_d$, and that the collection of i_d constitutes the conjugacy class of w_0, namely the class of the products of n disjoint transpositions.

By an *updown tableau* of degree $2n$ we mean a sequence $M = (\mu^{(0)}, \mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(2n)})$ of partitions satisfying (1) $\mu^{(0)} = \mu^{(2n)} = \varnothing$, and (2) $\mu^{(i-1)} \dot\subset \mu^{(i)}$ or $\mu^{(i-1)} \dot\supset \mu^{(i)}$ for each $1 \leqq i \leqq 2n$. Let $\mathcal{M}_{2n}$ denote the set of all such sequences. The word updown tableau or oscillating tableau has been used by several authors, including in the original appearance in [2], with more generality. However, in this paper we only use the elements of $\mathcal{M}_{2n}$, and the term updown tableau will only refer to an element of $\mathcal{M}_{2n}$. Our updown tableaux generalize pairs of standard tableaux of the same shape, since if $\mu^{(i-1)} \dot\subset \mu^{(i)}$ holds for all $1 \leqq i \leqq n$ (and accordingly $\mu^{(i-1)} \dot\supset \mu^{(i)}$ for all $n+1 \leqq i \leqq 2n$), then the whole sequence can be regarded as an encoding of two standard tableaux, one corresponding to the saturated chain $\varnothing = \mu^{(0)} \dot\subset \mu^{(1)} \dot\subset \cdots \dot\subset \mu^{(n)}$, and the other corresponding to $\varnothing = \mu^{(2n)} \dot\subset \mu^{(2n-1)} \dot\subset \cdots \dot\subset \mu^{(n)}$, both of shape $\mu^{(n)}$.

The following variant of the Robinson-Schensted correspondence linking Brauer diagrams and updown tableaux was introduced by Stanley, Sundaram, and Roby (see the beginning of Section 1). Let d be a Brauer diagram in $\mathcal{D}_{2n}$, and let $w_d \in D_n \subset \mathfrak{S}_{2n}$ be the representative of the corresponding coset in $W(B_n)\backslash\mathfrak{S}_{2n}$ defined above. Starting with $T^{(2n)} = \varnothing$, apply the following instruction for $k = 2n, 2n-1, \ldots, 1$ successively in this order, to obtain a sequence of tableaux $T^{(2n-1)}, T^{(2n-1)}, \ldots, T^{(0)}$: If $w_d(k)$ is a primed number i', then let $T^{(k-1)}$ be the tableau obtained by row-inserting i to $T^{(k)}$ (see [26], for example, to see the meaning of row-insertion). If $w_d(k)$ is an unprimed number i, then let $T^{(k-1)}$ be the tableau obtained by removing i from $T^{(k)}$ (where, as it is easy to see, it occupies a corner). Then the output of the correspondence is the updown tableau of degree $2n$ obtained by listing the shapes of the tableaux $T^{(i)}$, $0 \leqq i \leqq 2n$. They showed that this defines a bijection from $\mathcal{D}_{2n}$ to $\mathcal{M}_{2n}$.

It generalizes the original Robinson-Schensted correspondence, in the sense that the permutation diagram representing w is mapped to the updown tableau encoding the pair $(P(w), Q(w))$ in the sense described at the end of the previous paragraph.

The above description is what one would see by viewing the whole process of Roby's modified version through a mirror (a special mirror that maps each tableau without change of orientation) put vertically either outside the right margin or the left margin of the entire Brauer diagram. Due to a reflection symmetry of this bijection, viewing through a mirror does not change the result (for example, see [22]). It is also essentially the same as writing Sundaram's version specialized to the empty ending shape (the inverse map of it, since she takes $\mathcal{M}_{2n} \to \mathcal{D}_{2n}$ as the forward direction) with all shapes transposed, namely using row insertion while she uses column insertion. This is essentially the same description as the one used by M.-P. Delest, S. Dulucq, and L. Favreau in [5] and [7], where they also show its reflection symmetry. An apparent difference is that, at a right-end vertex, they insert the position label of the corresponding left-end vertex. This incurs the same movement of letters and the same sequence of shapes as our description, since the insertion process is governed by the relative magnitudes of the letters only.

Our purpose is to find an interpretation of this bijection in the spirit of Steinberg's result, namely by way of two different parameterizations of the irreducible components of some algebraic variety. To do so, we first need to find some objects classified by $\mathcal{D}_{2n}$ instead of $\mathfrak{S}_n$. What we came upon was a list of combinatorial parametrizations of the orbits of certain Lie groups acting on flag manifolds by T. Matsuki and T. Oshima [17]. One case of their results amounts to the classification of what one could call the "relative positions" of non-degenerate alternating bilinear forms and complete flags in $\mathbb{C}^{2n}$.

What we mean by this is as follows. Let $V = \mathbb{C}^{2n}$ instead of $\mathbb{C}^n$, and let

$$X = \{\text{complete flags in } V \text{ (which is now } \mathbb{C}^{2n})\}, \quad \text{and}$$
$$Y = \{\text{nondegenerate alternating bilinear forms on } V\}.$$

Let G denote the group $GL(2n, \mathbb{C})$. Then G naturally acts on X and Y, and each of these actions is transitive. However, as we see below, the diagonal action of G on $Y \times X$ is not transitive unless $n = 1$, and we say that the pairs $(\omega, \boldsymbol{V})$ and $(\omega', \boldsymbol{V}') \in Y \times X$ have the same relative position if they lie in the same G-orbit, namely if there is an element $g \in GL(2n, \mathbb{C})$ such that $\omega' = g^*\omega$ and $\boldsymbol{V}' = g \cdot \boldsymbol{V}$.

To state the classification, let us introduce some more notation. Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{2n})$ be the standard basis of $V = \mathbb{C}^{2n}$. We fix a "standard" symplectic form $\omega_0 \in Y$, which is represented by the matrix $J = \left(\begin{smallmatrix} O & J_1 \\ -J_1 & O \end{smallmatrix}\right)$ with $J_1 = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. In other words, ω_0 is defined by $\omega_0(\boldsymbol{e}_i, \boldsymbol{e}_j) = \omega_0(\boldsymbol{e}_{i'}, \boldsymbol{e}_{j'}) = 0$ and $\omega_0(\boldsymbol{e}_i, \boldsymbol{e}_{j'}) = -\omega_0(\boldsymbol{e}_{j'}, \boldsymbol{e}_i) = \delta_{ij}$ for all $1 \leqq i, j \leqq n$. Also, for each $w \in \mathfrak{S}_{2n}$, let $\boldsymbol{V}^w \in X$ be defined as in Section 1.2, the only difference being the dimension of the whole space. We will say that a sequence of vectors $\boldsymbol{v} = (v_1, v_2, \ldots, v_{2n})$ is a *basis of the flag* $\boldsymbol{V} = (V_i)_{i=0}^{2n}$ or *spans the flag* $\boldsymbol{V}$ if $v_1, v_2, \ldots, v_i$ spans V_i for all i. We also write $\boldsymbol{V} = \mathrm{Fl}(\boldsymbol{v})$. In this terminology, $\boldsymbol{V}^w$ is the flag spanned by the sequence $(\boldsymbol{e}_{w(1)}, \boldsymbol{e}_{w(2)}, \ldots, \boldsymbol{e}_{w(2n)})$.

With this notation, the classification of the relative positions of the elements of $Y \times X$ can be stated as follows.

Proposition 2.2 *The G-orbits in $Y \times X$ are in* 1-1 *correspondence with $\mathcal{D}_{2n}$. More precisely, if $(\omega, \boldsymbol{V}) \in Y \times X$, then there exists a unique Brauer diagram $d \in \mathcal{D}_{2n}$ such that $(\omega, \boldsymbol{V})$ lies in the same G-orbit as $(\omega_0, \boldsymbol{V}^{w_d})$.*

Note that $(\omega, \boldsymbol{V})$ lies in the same G-orbit as $(\omega_0, \boldsymbol{V}^{w_d})$ if and only if there is a basis $(v_1, v_2, \ldots, v_{2n})$ of the flag $\boldsymbol{V}$ such that

$$\omega(v_i, v_j) = \begin{cases} 0 & (\text{if } i, j \text{ are not linked in } d), \\ 1 & (\text{if } i, j \text{ are linked in } d \text{ and } i < j), \\ -1 & (\text{if } i, j \text{ are linked in } d \text{ and } i > j). \end{cases}$$

Although this result is not our original, we will give an elementary proof of this fact in Section 2.

Motivated by this classification, we introduce the following algebraic variety, which will substitute Steinberg's variety in the updown case:

$$Z = \{\, (N, \omega, \boldsymbol{V}) \in \mathcal{N} \times Y \times X \mid \text{the group } \{\exp tN \mid t \in \mathbb{C}\} \text{ fixes } \omega \text{ and } \boldsymbol{V}\},$$

where

$$\mathcal{N} = \{N \in M_{2n}(\mathbb{C}) \mid N \text{ is nilpotent}\}.$$

The condition in the definition of Z is equivalent to saying that $\boldsymbol{V}$ is N-stable (as defined in Section 1.2) and that $\omega(Nv, v') + \omega(v, Nv') = 0$ for all $v, v' \in V$.

We parametrize the irreducible components of Z in two ways.

For each Brauer diagram $d \in \mathcal{D}_{2n}$, put

$$Z_d = \{(N, \omega, \boldsymbol{V}) \in Z \mid (\omega, \boldsymbol{V}) \in \mathcal{O}_d\} \quad \text{where} \quad \mathcal{O}_d = G \cdot (\omega_0, \boldsymbol{V}^{w_d}) \subset Y \times X.$$

Then by Proposition 2.2, Z is partitioned into the subsets Z_d, $d \in \mathcal{D}_{2n}$. We show that

Proposition 2.5 *For each $d \in \mathcal{D}_{2n}$, Z_d is an irreducible (Zariski) locally closed subvariety of Z of dimension $4n^2 - n$ regardless of d. $\{\overline{Z_d} \mid d \in \mathcal{D}_{2n}\}$ is the set of all irreducible components of Z.*

The second statement follows from the first by a general argument reviewed in the third paragraph of Section 1.2. Proposition 2.5 gives a parametrization of the irreducible components of Z by the Brauer diagrams.

On the other hand, we associate an updown tableau to each element of Z. Let $(N, \omega, \boldsymbol{V}) \in Z$, and put $\boldsymbol{V} = (V_i)_{i=0}^{2n}$. Restricted to V_i for each i, the form ω determines an alternating form $\omega|_{V_i}$ which may be degenerate. The radical of this form on V_i is also N-stable. Let $\mu^{(i)}$ denote the Jordan type of N restricted to this radical. Then we have

Proposition 3.1 *The sequence $(\mu^{(i)})_{i=0}^{2n}$ produced from $(N, \omega, \boldsymbol{V}) \in Z$ as above is always an updown tableau of degree $2n$.*

Thus the variety Z is also partitioned into the subsets Z_M, $M = (\mu^{(i)})_{i=0}^{2n} \in \mathcal{M}_{2n}$, where

$$Z_M = \left\{(N, \omega, \boldsymbol{V}) \in Z \mid N \text{ acts on the radical of } \omega\,|_{V_i} \text{ with Jordan type } \mu^{(i)} (\forall i)\right\}.$$

Moreover we have

Proposition 3.1 + Proposition 5.1 + Corollary 5.10 *For each updown tableau M of degree $2n$, Z_M is an irreducible, nonsingular (Zariski) locally closed subvariety of Z of dimension $4n^2 - n$. The subvarieties $\overline{Z_M}$, $M \in \mathcal{M}_{2n}$, give all irreducible components of Z.*

The proof of the irreducibility and smoothness of Z_M requires some detailed analysis, which will be carried out throughout Sections 4 and 5. The second statement follows from the first as in Proposition 2.5.

Thus the set of irreducible components of Z has two parametrizations, one by the set $\mathcal{D}_{2n}$ of Brauer diagrams on $2n$ points, and the other by the set $\mathcal{M}_{2n}$ of updown tableaux of degree $2n$. Hence the relation $\overline{Z_d} = \overline{Z_M}$ defines a bijection between the two parametrizing sets $\mathcal{D}_{2n} \xrightarrow{\sim} \mathcal{M}_{2n}$. Moreover, we have

Theorem 6.2 *The bijection from $\mathcal{D}_{2n}$ to $\mathcal{M}_{2n}$ defined by the relation $\overline{Z_d} = \overline{Z_M}$ coincides with the combinatorial bijection by Stanley, Sundaram and Roby.*

Namely our geometric construction gives an interpretation of the "updown analogue" of the Robinson-Schensted correspondence. We show this by constructing a series of posets from d, and then applying the result of Gansner and Saks reviewed in Section 1.2 to these posets (see Proposition 6.1).

2. Relative positions of symplectic forms and complete flags

In this section, we discuss the G-orbit decomposition of $Y \times X$, and thereby obtain a parametrization of the irreducible components of Z by the Brauer diagrams.

First we introduce some more notation. In general, if a group G acts on a set X and if x is an element of X, then G_x will denote the stabilizer of x in G, namely the subgroup $\{g \in G \mid g \cdot x = x\}$. Also, if G is a complex Lie group, then Lie G will denote the Lie algebra of G. The Lie algebra of $GL(2n, \mathbb{C})$ is $\mathfrak{gl}(2n, \mathbb{C})$ consisting of all $(2n) \times (2n)$ matrices with entries in $\mathbb{C}$ and equipped with the usual bracket operation of matrices. If G is a complex Lie subgroup of $GL(2n, \mathbb{C})$, then Lie G is the Lie subalgebra of $\mathfrak{gl}(2n, \mathbb{C})$ consisting of matrices A such that the one-parameter subgroup $\{\exp tA \mid t \in \mathbb{C}\}$ of $GL(2n, \mathbb{C})$ is contained in G. The complex Lie groups appearing in this article are also linear algebraic groups over $\mathbb{C}$, and the notion of Lie algebras of linear algebraic groups leads to the same Lie algebras.

Let us return to the situation where G denotes $GL(2n, \mathbb{C})$ and X denotes the set of all complete flags in $\mathbb{C}^{2n}$. For each $w \in \mathfrak{S}_{2n}$, let $\boldsymbol{V}^w$ be the flag defined in Section 1.3 before the quotation of Proposition 2.2. The stabilizer of the "standard" flag $\boldsymbol{V}^e$ in G, where e is the identity element of $\mathfrak{S}_{2n}$, is the subgroup B consisting of the upper triangular matrices in G. Since G acts on X transitively, X can be identified with G/B as a G-space. For each $w \in \mathfrak{S}_{2n}$, let $\dot{w}$ denote the permutation matrix representing w, namely $\sum_{j=1}^{2n} E_{w(j),j}$, where E_{ij} is the (i, j)th matrix unit as in Section 1.2. Then we have $\boldsymbol{V}^w = \dot{w} \cdot \boldsymbol{V}^e$ and $G_{\boldsymbol{V}^w} = \dot{w} B \dot{w}^{-1}$. If $g \in G$ has column vectors $v_1, v_2, \ldots, v_{2n}$ (we write $g = (v_1 \mid v_2 \mid \cdots \mid v_{2n})$), then we have $\mathrm{Fl}(\boldsymbol{v}) = g \cdot \boldsymbol{V}^e$ (see Section 1.3, before the quotation of Proposition 2.2) where $\boldsymbol{v}$ denotes the basis $(v_1, v_2, \ldots, v_{2n})$ of V. Namely if we regard Fl as a map from G to X, then it coincides with the natural projection $G \to G/B$ under the above identification. Next let H be the stabilizer of the "standard" symplectic form ω_0, introduced in Section 1.3 before the quotation of Proposition 2.2, in G. Then H is the symplectic group $Sp(2n, \mathbb{C})$ (or, according to an alternate convention, H is conjugate to $Sp(2n, \mathbb{C})$ in G). We have $Y \cong G/H$ as a G-space. Both X and Y are complex manifolds (resp. algebraic varieties over $\mathbb{C}$), and the actions of G on X and Y are holomorphic (resp. algebraic). Finally, the G-orbits on $Y \times X$ naturally correspond with the H-orbits on X, the B-orbits on Y, and the double cosets in $H \backslash G/B$.

We begin our argument by recalling the following characterization of the relative positions of two complete flags. For $w \in \mathfrak{S}_{2n}$ and $0 \leqq i, j \leqq 2n$, put $d_{ij}(w) = \#(\Pi_i(w) \cap \Pi^j(w))$ (see Section 1.2). Note that these numbers determine w. Now let $\boldsymbol{V} = (V_i)$ and $\boldsymbol{V}' = (V'_j)$ be two complete flags in $\mathbb{C}^{2n}$. Then $(\boldsymbol{V}, \boldsymbol{V}') \in \mathcal{O}_w$ (in the sense of Section 1.2) if and only if $\dim(V_i \cap V'_j) = d_{ij}(w)$ for all $0 \leqq i, j \leqq 2n$. From this one can also show that each $\mathcal{O}_w$ is Zariski locally closed in $X \times X$.

Now if $(\omega, \boldsymbol{V}) \in Y \times X$ and $\boldsymbol{V} = (V_i)_{i=0}^{2n}$, then write $\mathrm{Rad}(\omega\,|_{V_i})$ for the radical of $\omega\,|_{V_i}$, namely $\mathrm{Rad}(\omega\,|_{V_i}) = V_i \cap V_i^{\perp}$ (where $^{\perp}$ is taken with respect to ω). Let us call the sequence $(\mathrm{Rad}(\omega\,|_{V_i}))_{i=0}^{2n}$ the *ω-radical sequence* of $\boldsymbol{V}$, and denote it by $\boldsymbol{R}(\omega, \boldsymbol{V})$. If $\boldsymbol{W} = (W_i)_{i=0}^{2n}$ is a sequence of subspaces of V, then let us call $\boldsymbol{W}$ an *updown flag* of V if $W_0 = W_{2n} = 0$, and if either $W_{i-1} \dot{\subset} W_i$ or $W_{i-1} \dot{\supset} W_i$ holds for each i.

The following lemma is fundamental in the analysis that follows.

Lemma 2.1 *Let $(\omega, \boldsymbol{V}) \in Y \times X$, $\boldsymbol{V} = (V_i)_{i=0}^{2n}$, and put $W_i = V_i \cap V_i^{\perp} = \mathrm{Rad}(\omega\,|_{V_i})$ for each i (where $^{\perp}$ is taken with respect to ω). Then for each i, either $W_{i-1} \dot{\subset} W_i$ or $W_{i-1} \dot{\supset} W_i$ holds. In other words, the ω-radical sequence $\boldsymbol{R}(\omega, \boldsymbol{V})$ of $\boldsymbol{V}$ is an updown flag of V.*

Proof: We have $W_0 = 0$ since $V_0 = 0$, and W_{2n} is also 0 since ω is nondegenerate.

Now fix i. First suppose that V_i contains more vectors orthogonal to V_{i-1} than V_{i-1} does, and let v be any such vector. Then we have $\omega(v, V_i) = \omega(v, V_{i-1} \oplus \mathbb{C}v) = 0$. Hence a vector $u + cv$ of V_i $(u \in V_{i-1},\ c \in \mathbb{C})$ lies in $V_i^{\perp}$ if and only if $u \in V_i^{\perp} = (V_{i-1} \oplus \mathbb{C}v)^{\perp}$, which is equivalent to $u \in V_{i-1}^{\perp}$, since any $u \in V_{i-1}$ is orthogonal to v. Hence we have $W_i = W_{i-1} \oplus \mathbb{C}v \dot{\supset} W_{i-1}$.

Otherwise $V_i \cap V_{i-1}^{\perp}$ coincides with $V_{i-1} \cap V_{i-1}^{\perp} = W_{i-1}$. Now $W_i \subset V_i \cap V_{i-1}^{\perp} = W_{i-1}$, and the codimension is either 0 or 1. On the other hand, $(i-1) - \dim W_{i-1}$ and $i - \dim W_i$ are both even, so that $\dim W_{i-1} - \dim W_i$ must be odd. Hence we have $W_i \dot{\subset} W_{i-1}$. □

Now we turn to the classification of the relative positions of nondegenerate alternating bilinear forms on V and complete flags in V. Although this result is not our original (see Remark after the statement), we include an elementary proof for convenience.

Proposition 2.2 *The G-orbits in $Y \times X$ are in* 1-1 *correspondence with $\mathcal{D}_{2n}$. A complete set of representatives is given by* $\{(\omega_0, \boldsymbol{V}^{w_d}) \mid d \in \mathcal{D}_{2n}\}$.

In other words, we have a double coset decomposition

$$G = \coprod_{d \in \mathcal{D}_{2n}} H \dot{w}_d B.$$

Remark If we put $G' = SL(2n, \mathbb{C})$ and $B' = G' \cap B$, then G' is a complex simple Lie group containing H, and we have $X \cong G/B = G'B/B \cong G'/B'$ as G'-spaces. Hence the H-orbits on G/B are the same as the H-orbits on G'/B'. Moreover, B' is a Borel subgroup of G', and H is the group of the fixed points of the involutive automorphism $\sigma :\ g \mapsto J^{-1}({}^t g^{-1})J$ of G' (or G). Matsuki [16, Theorem 1, Corollary 1, and Theorems 2 and 3] gave a general solution to this kind of problem in the context of real Lie groups, namely the problem of parametrizing the H-orbits on G'/P where G' is a real semisimple Lie group, P is a minimal parabolic subgroup of G', and H is a subgroup of G' satisfying $(G^{\sigma})^o \subset H \subset G^{\sigma}$ for some involutive automorphism σ of G (where $(G^{\sigma})^o$ is the identity component of G^{σ}). (There is also a work by W. Rossmann [23], but Matsuki [16] gave a more complete result.) Since a complex simple Lie group is also a real simple Lie group, and a minimal parabolic subgroup of such a Lie group is a Borel subgroup, our problem is a special case of this general problem. Matsuki and Oshima [17, Theorem 4.1] gave the result of applying Matsuki's general solution to the cases where G' is a classical complex simple Lie group and σ is holomorphic (in order to apply such results to their problem in representation theory). Our case is their type AII. This kind of orbit decomposition was also studied in the context of algebraic groups in general by T. Springer and R. Richardson, starting with [30]. See [20] for more references.

The proof we include below is an elementary application of linear algebra. This proof also verifies that the classification for this case is valid over any field of characteristic different from 2, whether it is algebraically closed or not.

Put $\mathcal{O}_d = G \cdot (\omega_0, \boldsymbol{V}^{w_d}) \subset Y \times X$ for every $d \in \mathcal{D}_{2n}$. If $(\omega, \boldsymbol{V}) \in Y \times X$, then define $\boldsymbol{V}^{\perp} = (V_{2n-j}^{\perp})_{j=0}^{2n} \in X$, where $^{\perp}$ is taken with respect to ω. Our elementary proof depends on showing the following fact:

Lemma 2.3 *Let* $(\omega, \boldsymbol{V}) \in Y \times X$ *and* $d \in \mathcal{D}_{2n}$. *Then* $(\omega, \boldsymbol{V}) \in \mathcal{O}_d$ $(\subset Y \times X)$ *if and only if* $(\boldsymbol{V}, \boldsymbol{V}^{\perp}) \in \mathcal{O}_{i_d w_0}$ $(\subset X \times X)$ *in the sense of Section* 1.2, *where* $^{\perp}$ *is taken with respect to* ω.

Proof: For the implication of the latter condition by the former, it is enough to show this for $(\omega_0, \boldsymbol{V}^{w_d})$. Write $\boldsymbol{V} = (V_i)_{i=0}^{2n}$ for $\boldsymbol{V}^{w_d}$. Then $V_{2n-j}^{\perp}$, where $^{\perp}$ is taken with respect to ω_0, is spanned by the $\boldsymbol{e}_{w_d(k)}$ such that $i_d(k) > 2n - j$. Therefore $\dim(V_i \cap V_{2n-j}^{\perp}) = \#\{k \in \{1, 2, \ldots, 2n\} \mid k \leqq i \text{ and } i_d(k)(= i_d^{-1}(k)) > 2n - j\}$. Since this equals $d_{ij}(i_d w_0)$, we have $(\boldsymbol{V}, \boldsymbol{V}^{\perp}) \in \mathcal{O}_{i_d w_0}$.

In order to show the other implication, let $(\omega, \boldsymbol{V}) \in Y \times X$ be such that $(\boldsymbol{V}, \boldsymbol{V}^{\perp}) \in \mathcal{O}_{i_d w_0}$. We show that $(\omega, \boldsymbol{V})$ and $(\omega_0, \boldsymbol{V}^{w_d})$ lie in the same G-orbit. Since G acts transitively on Y, we may assume that $\omega = \omega_0$. Write $\boldsymbol{V} = g \cdot \boldsymbol{V}^e$, $g \in G$. Our goal is to show that $\dot{w}_d \in HgB$. We inductively claim that HgB contains an element $g^{(i)}$ whose first i columns coincide with those of $\dot{w}_d$. This claim trivially holds for $i = 0$ with the choice $g^{(0)} = g$. Now suppose $i > 0$. Write $\{w_d(1), w_d(2), \ldots, w_d(i-1)\} = I \cup J'$ where $I,\ J \subset \{1, 2, \ldots, n\}$ (J' is short for $\{j' \mid j \in J\}$). Recall the characterization of the elements of D_{2n}, which implies $I = \{1, 2, \ldots, r\}$ for some r and $J \subset I$. Put $i^* = i_d(i)$. By $(\boldsymbol{V}, \boldsymbol{V}^{\perp}) \in \mathcal{O}_{i_d w_0}$, we have

$$\dim\left(V_i \cap V_p^{\perp}\right) = \begin{cases} \dim\left(V_{i-1} \cap V_p^{\perp}\right) + 1 & \text{if } p < i^*, \quad \text{and} \\ \dim\left(V_{i-1} \cap V_p^{\perp}\right) & \text{if } p \geqq i^*. \end{cases} \tag{1}$$

Case 1 (i is a left-end vertex in d, or equivalently $i < i^*$) Using (1) for $p = i - 1$, we know that there is a vector $\boldsymbol{v}_1 \in V_i \backslash V_{i-1}$ which is orthogonal to V_{i-1}. This means that we can move from $g^{(i-1)}$ to

$$g_1 = \left(\boldsymbol{e}_{w_d(1)} \mid \boldsymbol{e}_{w_d(2)} \mid \cdots \mid \boldsymbol{e}_{w_d(i-1)} \mid \boldsymbol{v}_1 \mid \cdots\right)$$

by a right multiplication by B. The orthogonality of $\boldsymbol{v}_1$ with V_{i-1} means that $\boldsymbol{v}_1$ has no coefficients in the $\boldsymbol{e}_k$, $k \in I' \cup J$. Since $I \backslash J \subset \{w_d(1), w_d(2), \ldots, w_d(i-1)\}$, we can eliminate the coefficients in the $\boldsymbol{e}_k$, $k \in I \setminus J$, by a further right multiplication by B, yielding

$$g_2 = \left(\boldsymbol{e}_{w_d(1)} \mid \boldsymbol{e}_{w_d(2)} \mid \cdots \mid \boldsymbol{e}_{w_d(i-1)} \mid \boldsymbol{v}_2 \mid \cdots\right)$$

where $\boldsymbol{v}_2$ has coefficients in the central $2(n-r)$ positions only. A left multiplication by a matrix of the form $1_r \oplus h \oplus 1_r$, $h \in Sp(2n-2r)$, can leave the first $i-1$ columns unchanged and bring $\boldsymbol{v}_2$ to $\boldsymbol{e}_{r+1}$. Our claim for i is attained by choosing this result as $g^{(i)}$.

Case 2 (i is a right-end vertex in d, or equivalently $i^* < i$) Write $w_d(i^*) = j$, then we have $w_d(i) = j'$ and $j \in I \setminus J$. (1) applied for $p = i^* - 1$ shows the existence of a vector $\boldsymbol{v}_1 \in V_i \setminus V_{i-1}$ which is orthogonal to V_{i^*-1}. We can move from $g^{(i-1)}$ to

$$g_1 = \left(\boldsymbol{e}_{w_d(1)} \mid \boldsymbol{e}_{w_d(2)} \mid \cdots \mid \boldsymbol{e}_{w_d(i-1)} \mid \boldsymbol{v}_1 \mid \cdots\right)$$

by a right multiplication by B. Since $\boldsymbol{v}_1 \in V_{i^*-1}^{\perp}$, in particular $\boldsymbol{v}_1$ has no coefficients in $\boldsymbol{e}_{1'}, \boldsymbol{e}_{2'}, \ldots, \boldsymbol{e}_{(j-1)'}$. On the other hand, (1) applied for $p = i^*$ implies $\boldsymbol{v}_1 \notin V_{i^*}^{\perp}$, so $\boldsymbol{v}_1$ has a nontrivial coefficient in $\boldsymbol{e}_{j'}$. We may adjust $\boldsymbol{v}_1$ by a scalar multiplication (which is also a right multiplication by B) so that its coefficient in $\boldsymbol{e}_{j'}$ equals 1.

Next we can produce $g_2 = (\boldsymbol{e}_{w_d(1)} \mid \boldsymbol{e}_{w_d(2)} \mid \cdots \mid \boldsymbol{e}_{w_d(i-1)} \mid \boldsymbol{v}_2 \mid \cdots)$ in the same double coset, where $\boldsymbol{v}_2$ has coefficients only in $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ and $\boldsymbol{e}_{j'}$, as follows. Let A be the $n \times n$ upper unitriangular matrix $(\boldsymbol{e}_1 \mid \cdots \mid \boldsymbol{v}_1^< \mid \cdots \mid \boldsymbol{e}_n)$ where $\boldsymbol{v}_1^<$ denotes the lower half of $\boldsymbol{v}_1$. Then ${}^{t'}\!A \oplus A^{-1} \in H$ where ${}^{t'}\!A = J_1\, {}^t\!A J_1$, and its left multiplication onto g_1 has the following effect on its first i columns. To the lower half, it adds row j' into rows $(j+1)', (j+2)', \ldots, n'$ (with some coefficients) in such a way that eliminates entries in these rows in column i. Since the only entry in row j' in the first i columns of g_1 is in column i, the other $i-1$ columns are not affected by this multiplication. To the upper half, it adds rows $n, n-1, \ldots, j+1$ into row j with some coefficients. Since the upper half of the first $i-1$ columns of g_1 are $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_r$, interspersed with some zero vectors, the effect on this part can be undone by a further right multiplication by B. Thus we obtain g_2.

Next we can produce $g_3 = (\boldsymbol{e}_{w_d(1)} \mid \boldsymbol{e}_{w_d(2)} \mid \cdots \mid \boldsymbol{e}_{w_d(i-1)} \mid \boldsymbol{v}_3 \mid \cdots)$ in the same double coset, where $\boldsymbol{v}_3$ has coefficients only in $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_r$ and $\boldsymbol{e}_{j'}$ as follows. Let P be the $n \times n$ matrix $(\boldsymbol{0} \mid \cdots \mid -\boldsymbol{v}_2^{\ll} \mid \cdots \mid \boldsymbol{0})$ where $\boldsymbol{v}_2^{\ll}$ is the upper half of $\boldsymbol{v}_2$ whose topmost r components are replaced by 0, and $-\boldsymbol{v}_2^{\ll}$ is placed in column $n+1-j$ of P. Then $\left(\begin{smallmatrix} 1_n & P + {}^{t'}\!P \\ O & 1_n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1_n & {}^{t'}\!P \\ O & 1_n \end{smallmatrix}\right)\left(\begin{smallmatrix} 1_n & P \\ O & 1_n \end{smallmatrix}\right) \in H$ where ${}^{t'}\!P = J_1\, {}^t\!P J_1$, and its left multiplication onto g_2 does the following to its first i columns. It adds row j' into rows $n, n-1, \ldots, r+1$ with some coefficients in such a way that repels the entries in these rows in column i. As before, this only affects column i. It also adds rows $n', (n-1)', \ldots, (r+1)'$ into row j with some coefficients, but this has no effect in the first i columns, since these $n-r$ rows are all clear in these columns. Thus we obtain g_3.

Finally we can clear the rows 1 through r in column i by a right multiplication by B, thus attaining our claim for i. $\square$

Proof of Proposition 2.2: First note that the orbits $\mathcal{O}_d$ are different from one another. This is because $(\omega, \boldsymbol{V}) \in \mathcal{O}_d$ if and only if $(\boldsymbol{V}, \boldsymbol{V}^\perp) \in \mathcal{O}_{i_d w_o}$ (where $^\perp$ is taken with respect to ω) by Lemma 2.3, and the orbits $\mathcal{O}_{i_d w_o}$ are all different because the elements $i_d w_0 \in \mathfrak{S}_{2n}$ are all different.

In order to prove that $Y \times X = \cup_{d \in \mathcal{D}_{2n}} \mathcal{O}_d$, let $(\omega, \boldsymbol{V})$ be an arbitrary element of $Y \times X$, and let $w \in \mathfrak{S}_{2n}$ be such that $(\boldsymbol{V}, \boldsymbol{V}^\perp) \in \mathcal{O}_w \subset X \times X$. We show that $w = i_d w_0$ for some $d \in \mathcal{D}_{2n}$. We have $d_{ij}(ww_0) = i - d_{i,2n-j}(w) = i - \dim(V_i \cap V_j^\perp)$. By the usual dimension calculation, we find that $d_{ij}(ww_0) = d_{ji}(ww_0)$ for all i and j, namely that ww_0 is an involution. If ww_0 fixes j, then we must have $d_{j-1,2n-j+1}(w) = d_{j,2n-j}(w)$, in other words $\dim\, \mathrm{Rad}(\omega\,|_{V_{j-1}}) = \dim\, \mathrm{Rad}(\omega\,|_{V_j})$, which is impossible by Lemma 2.1. Therefore ww_0 must be of the form i_d, $d \in \mathcal{D}_{2n}$. By Lemma 2.3, this means that $(\omega, \boldsymbol{V}) \in \mathcal{O}_d$. Hence $Y \times X$ is covered by the $\mathcal{O}_d$, $d \in \mathcal{D}_{2n}$. $\square$

Note that the condition for N in the definition of Z (Section 1.3, after the quotation of Proposition 2.2) is equivalent to saying that N is a nilpotent element of the Lie algebra of $G_{(\omega,\boldsymbol{V})} = G_\omega \cap G_{\boldsymbol{V}}$, that is, the Lie algebra of the stabilizer in G of the point $(\omega, \boldsymbol{V})$ in $Y \times X$. If $(\omega, \boldsymbol{V}) = g \cdot (\omega_0, \boldsymbol{V}^{w_d})$ with $g \in G$, then this Lie algebra is conjugate to the Lie algebra of $G_{(\omega_0, \boldsymbol{V}^{w_d})}$ by $\mathrm{Ad}(g)$.

So let us determine the Lie algebra of $G_{(\omega_0, \boldsymbol{V}^{w_d})}$ for each $d \in \mathcal{D}_{2n}$. This Lie algebra turns out to be upper triangular as we see below, which makes it easy to set apart its nilpotent elements.

Put $\mathfrak{g} = \mathfrak{gl}(2n, \mathbb{C})$, and let $\mathfrak{b}$ be the Lie algebra of B. Then $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ consisting of all upper triangular matrices in $\mathfrak{g}$. Let $T \subset B$ be the maximal torus of G consisting of the diagonal matrices in G. Its Lie algebra $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices in $\mathfrak{g}$. Let $\varepsilon_i \in \mathfrak{t}^*$, $1 \leqq i \leqq 2n$, be defined by $\mathrm{diag}(h_1, h_2, \ldots, h_{2n}) \mapsto h_i$, and let Δ_+ denote the positive system of $\Delta(G, T)$ corresponding to B, namely $\Delta_+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqq i < j \leqq 2n\}$. Now let $\sigma : G \to G$ be the involution $g \mapsto J^{-1}({}^t g^{-1})J$ as above, which gives $G^\sigma = H$. It induces an involution on the Lie algebra $\mathfrak{g}$ by $A \mapsto J^{-1}(-{}^t A)J$, also denoted by σ. B and T are both σ-stable, and B^σ is a Borel subgroup of H containing a maximal torus T^σ of H. The Lie algebras of G^σ, B^σ, and T^σ coincide with the σ-fixed points $\mathfrak{g}^\sigma$, $\mathfrak{b}^\sigma$, and $\mathfrak{t}^\sigma$ in $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{t}$ respectively. $\mathfrak{g}^\sigma$ is (or is conjugate to) the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$, and $\mathfrak{t}^\sigma = \{\mathrm{diag}(h_1, h_2, \ldots, h_n, -h_n, \ldots, -h_2, -h_1) \mid h_1, h_2, \ldots, h_n \in \mathbb{C}\}$ is a Cartan subalgebra of $\mathfrak{g}^\sigma$. Let $\pi : \mathfrak{t}^* \to (\mathfrak{t}^\sigma)^*$ be the restriction map. Then $\pi(\Delta_+) = \{\pi(\varepsilon_i - \varepsilon_j) \mid 1 \leqq i < j \leqq n\} \cup \{\pi(2\varepsilon_i) \mid 1 \leqq i \leqq n\}$ is a positive system of $\Delta(H, T^\sigma)$ corresponding to B^σ.

For each $d \in \mathcal{D}_{2n}$, put $\bar{\Delta}(d) = \pi(\Delta_+ \cap w_d(\Delta_+) \cap \sigma(w_d(\Delta_+))) = \{\pi(\varepsilon_i - \varepsilon_j) \mid 1 \leqq i < j \leqq i',\ w_d^{-1}(i) < w_d^{-1}(j),\ w_d^{-1}(j') < w_d^{-1}(i')\}$, which is a closed subset of $\pi(\Delta_+)$. Actually the condition $w_d^{-1}(i) < w_d^{-1}(j)$ follows from $i < j \leqq i'$ by our choice of w_d. Also note that the $\pi(2\varepsilon_i)$, $1 \leqq i \leqq n$, are always contained in $\bar{\Delta}(d)$. Put $l_d = \#\bar{\Delta}(d)$.

Lemma 2.4 *Let $d \in \mathcal{D}_{2n}$, and put $B_d = G_{(\omega_0, \boldsymbol{V}^{w_d})} = H \cap \dot{w}_d B \dot{w}_d^{-1}$. Then its Lie algebra $\mathfrak{b}_d = \mathfrak{g}^\sigma \cap \mathrm{Ad}(\dot{w}_d)\mathfrak{b}$ admits a semidirect sum decomposition $\mathfrak{b}_d = \mathfrak{t}^\sigma \ltimes \mathfrak{u}_d$, where $\mathfrak{u}_d$ is the sum of the root spaces for $\bar{\Delta}(d)$, namely*

$$
\begin{aligned}
\mathfrak{b}_d &= \overbrace{\bigoplus_{i=1}^{n} \mathbb{C}(E_{ii} - E_{i'i'})}^{\mathfrak{t}^\sigma} \\
&\oplus \overbrace{\bigoplus_{\substack{1 \leq i < j \leq n \\ w_d^{-1}(j') < w_d^{-1}(i')}} \mathbb{C}(E_{ij} - E_{j'i'}) \oplus \bigoplus_{\substack{1 \leq i < j \leq n \\ w_d^{-1}(j) < w_d^{-1}(i')}} \mathbb{C}(E_{ij'} + E_{ji'}) \oplus \bigoplus_{i=1}^{n} \mathbb{C}E_{ii'}}^{\mathfrak{u}_d}.
\end{aligned}
$$

The subalgebra $\mathfrak{u}_d$ is exactly the set of nilpotent elements in $\mathfrak{b}_d$.

Proof: First we show that the group B_d is upper triangular. It suffices to show that, if $g \in G_{(\omega_0, \boldsymbol{V}^{w_d})}$, then g also fixes $\boldsymbol{V}^e$. Write $\boldsymbol{V}^e = (V_i^e)_{i=0}^{2n}$ and $\boldsymbol{V}^{w_d} = (V_j^{w_d})_{j=1}^{2n}$. It is sufficient to show that such g fixes V_i^e for $1 \leqq i \leqq n$ only, since $V_{2n-i}^e = (V_i^e)^\perp$ with respect to ω_0, and ω_0 is fixed by g. We proceed by induction on i. The claim is trivial if $i = 0$. Now suppose $i > 0$, and let j be such that $w_d(j) = i$, so that $\boldsymbol{e}_i \in V_j^{w_d}$. Since g fixes $\boldsymbol{V}^{w_d}$, we have $g\boldsymbol{e}_i \in V_j^{w_d}$. Since $w_d \in D_{2n}$, we have $\{w_d(1), w_d(2), \ldots, w_d(j)\} \subset \{1, 2, \ldots, i, 1', 2', \ldots, (i-1)'\}$, so that $g\boldsymbol{e}_i$ is a linear combination of $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_i$ and $\boldsymbol{e}_{1'}, \boldsymbol{e}_{2'}, \ldots, \boldsymbol{e}_{(i-1)'}$. By the induction hypothesis, we have $gV_{i-1}^e = V_{i-1}^e = \bigoplus_{k=1}^{i-1} \mathbb{C}\boldsymbol{e}_k$, which is contained in gV_i^e. Since V_i^e, hence gV_i^e, is isotropic with respect to $\omega_o = g^*\omega_0$, the vector $g\boldsymbol{e}_i$ cannot have nontrivial coefficients in $\boldsymbol{e}_{1'}$ through $\boldsymbol{e}_{(i-1)'}$. By linear independence $g\boldsymbol{e}_i$ does have a nontrivial coefficient in $\boldsymbol{e}_i$, therefore gV_i^e coincides with V_i^e.

It follows that $\mathfrak{b}_d$ is upper triangular, so that it is contained in

$$\begin{aligned}\mathfrak{b}^\sigma &= \bigoplus_{i=1}^{n}\mathbb{C}(E_{ii}-E_{i'i'})\oplus\bigoplus_{1\leqq i<j\leqq n}\mathbb{C}(E_{ij}-E_{j'i'})\\&\quad\oplus\bigoplus_{1\leqq i<j\leqq n}\mathbb{C}(E_{ij'}+E_{ji'})\oplus\bigoplus_{i=1}^{n}\mathbb{C}E_{ii'}.\end{aligned}$$

To obtain $\mathfrak{b}_d$, we take the intersection with $\mathrm{Ad}(\dot{w}_d)\mathfrak{b}=\bigoplus_{\substack{1\leqq i,j\leqq 2n\\ w_d^{-1}(i)\leqq w_d^{-1}(j)}}\mathbb{C}E_{ij}$. Since the matrix units involved in the summands of the above expression for $\mathfrak{b}^\sigma$ are all disjoint, we can check the condition term by term. Clearly the first sum survives, and by our choice of w_d the last sum also survives. In the second sum, the element $E_{ij}-E_{j'i'}$, $1\leqq i<j\leqq n$, lies in $\mathrm{Ad}(\dot{w}_d)\mathfrak{b}$ if and only if both $w_d^{-1}(i)<w_d^{-1}(j)$ and $w_d^{-1}(j')<w_d^{-1}(i')$ hold (we excluded the equalities since $i\neq j$). The first condition is automatic due to our choice of w_d, and the second condition persists. The third sum can be analyzed similarly, and we obtain the expression for $\mathfrak{b}_d$ in the statement. The one-dimensional summands in $\mathfrak{u}_d$ in this expression are exactly the root spaces for $\bar{\Delta}(d)$. Due to the upper triangularity, an element of $\mathfrak{b}_d$ is nilpotent if and only if its diaglnal entries are all zero, namely if it lies in $\mathfrak{u}_d$. □

Remark We can actually state a similar structure of the group B_d. Let U be the subgroup of B consisting of all upper unitriangular matrices. U is also σ-stable. We have $B^\sigma=T^\sigma\ltimes U^\sigma$ and $T^\sigma\subset B_d=H\cap\dot{w}_dB\dot{w}_d^{-1}$, so that we have $B_d=T^\sigma\ltimes U_d$ if we put $U_d=B_d\cap U^\sigma$. Now G^σ is a simple (hence reductive) algebraic group over $\mathbb{C}$, U^σ is the unipotent radical of a Borel subgroup B^σ of G^σ, and U_d is a Zariski closed subgroup of U^σ since it is the stabilizer of the point $\boldsymbol{V}^{w_d}$ under its algebraic action on X, and is stable under the conjugation action ty T^σ. Hence one can use [12, Proposition 28.1] or [3, Proposition 14.4] to conclude that U_d is connected, and is directly spanned by the root subgroups $(U^\sigma)_\beta$ whose corresponding root spaces are contained in $\mathfrak{u}_d$, namely $\beta\in\bar{\Delta}(d)$. This means that, if $\beta_1,\beta_2,\dots,\beta_{l_d}$ are the elements of $\bar{\Delta}(d)$ in any order and if E_{β_i} is a fixed root vector for the root β_i for each i (which, for example, can be taken to be the basis element appearing in the $\mathfrak{u}_d$ part of the expression in Lemma 2.4), then any element of U_d is expressed uniquely as $\exp(a_1E_{\beta_1})\exp(a_2E_{\beta_2})\cdots\exp(a_{l_d}E_{\beta_{l_d}})$, $a_1,a_2,\dots,a_{l_d}\in\mathbb{C}$. The map $\mathbb{C}^{l_d}\ni(a_1,a_2,\dots,a_{l_d})\mapsto\exp(a_1E_{\beta_1})\exp(a_2E_{\beta_2})\cdots\exp(a_{l_d}E_{\beta_{l_d}})\in U$ is moreover an isomorphism of varieties (see loc. cit.).

Proposition 2.5 *For each $d\in\mathcal{D}_{2n}$, put $Z_d=\{(N,\omega,\boldsymbol{V})\in Z\mid(\omega,\boldsymbol{V})\in\mathcal{O}_d\}$. Then Z_d is an irreducible (Zariski) locally closed subvariety of Z of dimension $4n^2-n$ regardless of d. Therefore $\{\overline{Z_d}\mid d\in\mathcal{D}_{2n}\}$ is the set of all irreducible components of Z.*

Proof: Let $d\in\mathcal{D}_{2n}$. The map $\phi:Y\times X\to X\times X$, $(\omega,\boldsymbol{V})\mapsto(\boldsymbol{V},\boldsymbol{V}^\perp)$, where $^\perp$ is taken with respect to ω, is a morphism of varieties, and by Lemma 2.3 we have $\mathcal{O}_d=\phi^{-1}(\mathcal{O}_{i_dw_0})$. Since $\mathcal{O}_{i_dw_0}$ is Zariski locally closed in $X\times X$, so is $\mathcal{O}_d$ in $Y\times X$. Let $\pi_{Y\times X}:Z\to Y\times X$ be the projection onto the second and third components. Then we have $Z_d=\pi_{Y\times X}^{-1}(\mathcal{O}_d)$. Therefore Z_d is a locally closed subvariety of Z.

Now let $q : G \times \mathfrak{u}_d \to Z_d$ be defined by $(g, N) \mapsto (\mathrm{Ad}(g)N, g^*\omega_0, g \cdot \boldsymbol{V}^{w_d})$. Since the Lie algebra $\mathfrak{b}_d$ of the stabilizer of $(\omega_0, \boldsymbol{V}^{w_d})$ consists of upper triangular matrices, the set of nilpotent elements in $\mathfrak{b}_d$ coincides with $\mathfrak{u}_d$, the set of strictly upper triangular matrices in $\mathfrak{b}_d$. It follows that the set of nilpotent elements in $\mathrm{Lie}(G_{g^*\omega_0} \cap G_{g \cdot \boldsymbol{V}^{w_d}})$, $g \in G$, equals $\mathrm{Ad}(g)\mathfrak{u}_d$, so that q is surjective. Since G and $\mathfrak{u}_d$ are irreducible, so is $\mathrm{im}\, q = Z_d$.

The fiber of q at an arbitrary point $(g^*\omega_0, g \cdot \boldsymbol{V}^{w_d})$ of $\mathcal{O}_d$, $g \in G$, is isomorphic to $\mathfrak{u}_d$, which is an l_d-dimensional vector space. On the other hand, we have $\dim \mathcal{O}_d = \dim G - \dim B_d = 4n^2 - (n + l_d)$. Therefore we have $\dim Z_d = \dim \mathcal{O}_d + \dim \mathfrak{u}_d = 4n^2 - n$, regardless of d.

The rest follows by the argument reviewed in the third paragraph of Section 1.2. $\square$

3. Types of radical sequences

Now let us see how we can produce an updown tableau (see Section 1.3) from an element of Z.

Let $\boldsymbol{W} = (W_i)_{i=0}^{2n}$ be an updown flag of V (see Section 2). Let us call the sequence $\varepsilon = (\varepsilon_i)_{i=1}^{2n}$ defined by $\varepsilon_i = \dim W_i - \dim W_{i-1} (\in \{\pm 1\})$ the *class* of $\boldsymbol{W}$. Note that $\sum_{p=1}^{i} \varepsilon_p \geqq 0$ holds for all $1 \leqq i \leqq 2n-1$, and also that $\sum_{p=1}^{2n} \varepsilon_p = 0$. Let $\mathcal{E}$ denote the set of all such ± 1 sequences of length $2n$. An element of $\mathcal{E}$ (or the corresponding $W_{\boldsymbol{\varepsilon}}$ defined in Section 4) is sometimes called a *Dyck path*. Also note that the elements of $\mathcal{E}$ are in 1-1 correspondence with the partitions ν whose Young diagrams are contained in that of the staircase partition $(n-1, n-2, \ldots, 1)$. To give an explicit correspondence, let $\varepsilon = (\varepsilon_i)_{i=1}^{2n} \in \mathcal{E}$, and let $a_1 < a_2 < \cdots < a_n$ (resp. $b_1 > b_2 > \cdots > b_n$) be the indices i with $\varepsilon_i = +1$ (resp. $\varepsilon_i = -1$). Then $\nu = (\nu_1, \nu_2, \ldots, \nu_l)$ (resp. its conjugate $\nu' = (\nu'_1, \nu'_2, \ldots, \nu'_{l'})$) is given by $\nu_i = a_{n+1-i} - (n+1-i)$ for all $1 \leqq i \leqq l$ (resp. $\nu'_j = n + j - b_{n+1-j}$ for all $1 \leqq j \leqq l'$), where l (resp. l') ($\geqq 0$) is the number of indices i (resp. j) such that $a_i > i$ (resp. $b_j < 2n + 1 - j$). The partition corresponding to ε in this manner will be denoted by ν_ε (see figure 5).

Note that, if $d \in \mathcal{D}_{2n}$ and $(\omega, \boldsymbol{V}) \in \mathcal{O}_d$, then the a_i (resp. b_j) corresponding to the class of $\boldsymbol{R}(\omega, \boldsymbol{V})$ in the fashion above are the same as the a_i that appeared in Section 2 in relation with w_d (resp. are obtained by rearranging the b_j in Section 2 in the decreasing order).

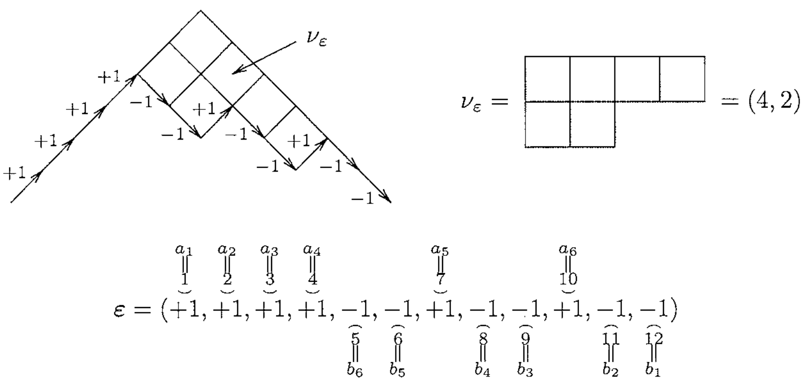


Figure 5. An example of ν_ε.

We write $\varepsilon \subset \varepsilon'$ if $\nu_\varepsilon \subset \nu_{\varepsilon'}$. This gives a poset isomorphism

$$\mathcal{E} \cong [\varnothing, (n-1, n-2, \ldots, 1)],$$

where the right-hand side is an interval in Young's lattice. The smallest element of $\mathcal{E}$ is $\varepsilon_0 = (\overbrace{1, 1, \ldots, 1}^{n}, \overbrace{-1, \ldots, -1, -1}^{n})$, and the largest element of $\mathcal{E}$ is $\varepsilon_{\max} = (1, -1, 1, -1, \ldots, 1, -1)$.

The term *class* will be used for an updown tableau in an obvious parallel manner.

If $(N, \omega, \boldsymbol{V}) \in Z$ and $\boldsymbol{V} = (V_i)_{i=0}^{2n}$, then $\mathrm{Rad}(\omega\,|_{V_i})$ is also N-stable for each i. Now let $N \in \mathcal{N}$, and let $\boldsymbol{W} = (W_i)_{i=0}^{2n}$ be an updown flag of V (see Section 2, before Lemma 2.1) consisting of N-stable subspaces. If $\mu^{(i)}$ denotes the Jordan type of $N|_{W_i}$, then let us call the sequence $(\mu^{(i)})_{i=0}^{2n}$ the (N-)*type* of $\boldsymbol{W}$, and denote it by $\mathrm{type}_N \boldsymbol{W}$.

Proposition 3.1 *If $(N, \omega, \boldsymbol{V}) \in Z$, then $\mathrm{type}_N \boldsymbol{R}(\omega, \boldsymbol{V})$ is an updown tableau of degree $2n$.*

For each $M \in \mathcal{M}_{2n}$ (see Section 1.3), the subset of Z defined by $Z_M = \{(N, \omega, \boldsymbol{V}) \in Z \mid \mathrm{type}_N \boldsymbol{R}(\omega, \boldsymbol{V}) = M\}$ forms a (Zariski) locally closed subvariety of Z.

Proof: Put $W_i = \mathrm{Rad}(\omega\,|_{V_i})$ for simplicity, and let $\mu^{(i)}$ be the Jordan type of $N\,|_{W_i}$ as above. By Lemma 2.1, $\boldsymbol{W} = (W_i)_{i=0}^{2n}$ is an updown flag of V. Clearly $\mu^{(0)} = \mu^{(2n)} = \varnothing$, and $W_{i-1} \mathrel{\dot\subset} W_i$ (resp. $W_{i-1} \mathrel{\dot\supset} W_i$) implies $\mu^{(i-1)} \mathrel{\dot\subset} \mu^{(i)}$ (resp. $\mu^{(i-1)} \mathrel{\dot\supset} \mu^{(i)}$) (see [32, Lemma 2.3] for example). Hence $\mathrm{type}_N(\boldsymbol{W})$ is an updown tableau.

Next note that, if k, k', k'' are fixed positive integers, then $\{(W, W') \in G_k(V) \times G_{k'}(V) \mid \dim(W \cap W') \geqq k''(\text{resp.} = k'')\}$ forms a Zariski closed (resp. locally closed) subvariety of $G_k(V) \times G_{k'}(V)$, where $G_k(V)$ and $G_{k'}(V)$ denote the Grassmannians of k and k'-dimensional subspaces of V respectively. Combining this with the map $(\omega, \boldsymbol{V}) \mapsto \boldsymbol{V}^\perp$, where $\boldsymbol{V}^\perp$ is taken with respect to ω, we see that the collection of $(\omega, \boldsymbol{V})$ (resp. $(N, \omega, \boldsymbol{V}) \in Z$) producing radical sequences of a fixed class $\varepsilon \in \mathcal{E}$ is Zariski locally closed in $Y \times X$ (resp. Z). Also, for fixed k and a fixed partition μ of k, the collection of $(N, W) \in \mathcal{N} \times G_k(V)$ such that W is N-stable and has the N-type μ forms a Zariski locally closed subvariety of $\mathcal{N} \times G_k(V)$ (note that the condition specifying the N-type can be written as equalities on the dimensions of $N^j W$ for various j). Again combining with this proves our second claim. $\square$

4. ε-transversal pairs and complete updown flags

It remains to show that each Z_M is irreducible, and that their dimensions are all the same. This will be completed in the next section, and this section provides preparatory results.

We first show that, for $(N, \omega, \boldsymbol{V}) \in Z$, the sum of the radicals $\sum_{k=0}^{2n} \mathrm{Rad}(\omega\,|_{V_k})$ is an N-stable maximal ω-isotropic subspace V (Lemma 4.1). With this in mind, we fix $\omega_0 \in Y$ and a maximal ω_0-isotropic subspace $\check{V}$, and define an algebraic variety $\check{Z}_M$ consisting of all pairs $(\check{N}, \boldsymbol{W})$, where $N \in \mathfrak{gl}(\check{V})$ is nilpotent, and $\boldsymbol{W}$ is an updown flag of $\check{V}$ made of $\check{N}$-stable subspaces summing up to $\check{V}$, and whose $\check{N}$-types consitute the updown tableau M. In this section, we show that $\check{Z}_M$ is nonsingular, irreducible, and of dimension $n^2 - n$

for every $M \in \mathcal{M}_{2n}$ (Proposition 4.5), and also show a lemma (Lemma 4.4) which will be used twice in the following section in deducing the irreducibility and the dimension of Z_M based on the irreducibility and the dimension of $\check{Z}_M$. For results like Lemma 4.4 and its application in the following section, a natural locale is the variety $\check{Z}_\varepsilon = \coprod \check{Z}_M$ with all M of a fixed class ε, which can be embedded into $\check{Z}$ (Steinberg's "Z" in the sense of Section 1.2 for the vector space $\check{V}$ instead of V) as an open subvariety, due to Lemma 4.2, Lemma 4.3, and the remarks after these lemmas. We also discuss this embedding, and a relation between some labelings of its irreducible components (Corollary 4.6).

For the moment, let us forget N until Proposition 4.5, concentrating on a relationship between a special kind of updown flags and pairs of complete flags. If $\check{V}$ is an n-dimensional subspace of V, let us abuse the terminology and say that $\boldsymbol{W} = (W_i)_{i=0}^{2n}$ is a *complete updown flag of* $\check{V}$ if it is an updown flag of $\check{V}$ and it satisfies $\sum_{i=0}^{2n} W_i = \check{V}$.

Lemma 4.1 *Let* $(\omega, \boldsymbol{V}) \in Y \times X$. *Then* $\boldsymbol{R}(\omega, \boldsymbol{V})$ *is a complete updown flag of a maximal isotropic subspace of* V.

Proof: Put $W_i = \mathrm{Rad}(\omega \mid V_i)$ for simplicity. First let us show that $\sum_{k=0}^{2n} W_k$ is an n-dimensional subspace of V. Note that $\sum_{k=0}^{2n} W_k = \sum_{i=1}^{n} W_{a_i}$. It is enough to show that $\sum_{i=1}^{p} W_{a_i}$ is strictly larger than $\sum_{i=1}^{p-1} W_{a_i}$ for any $1 \leqq p \leqq n$ (then it is larger by exactly one dimension). Fix p, and put $j = a_p$. Then we have $W_{j-1} \subsetneqq W_j$, and it was shown in the proof of Lemma 2.1 that, when this occurs, W_j is spanned by W_{j-1} and a vector v outside of V_{j-1}. Since the W_{a_i}, $i \leqq p-1$, are all subspaces of V_{j-1}, this means that v does not lie in $\sum_{i=1}^{p-1} W_{a_i}$, so that $\sum_{i=1}^{p} W_{a_i}$, which contains v, is strictly larger than $\sum_{i=1}^{p-1} W_{a_i}$.

Now let $\check{V}$ denote the sum of $\boldsymbol{W}$. In order to show that $\check{V}$ is isotropic for ω, it is enough to show that any $v \in W_j$ and $v' \in W_{j'}$, $1 \leqq j, j' \leqq n$, satisfy $\omega(v, v') = 0$. One can assume $j \leqq j'$, then we have $v \in W_j \subset V_j \subset V_{j'}$, and since v' is in the radical of $V_{j'}$ one has $\omega(v, v') = 0$. Since $\check{V}$ is n-dimensional, it is a maximal isotropic subspace of V, and $\boldsymbol{W}$ is a complete updown flag of $\check{V}$. □

If $\boldsymbol{K} = (K_i)_{i=0}^{n}$ and $\boldsymbol{K}' = (K'_i)_{i=0}^{n}$ are two complete flags in $\check{V}$, then put $\boldsymbol{K} \frown \boldsymbol{K}' = (K_0, K_1, K_2, \ldots, K_n = \check{V} = K'_n, \ldots, K'_2, K'_1, K'_0)$. If $\boldsymbol{K}$ and $\boldsymbol{K}'$ vary, this gives all complete updown flags in $\check{V}$ of class ε_0. Mapping (i, j) to $K_i \cap K'_j$, $0 \leqq i, j \leqq n$, defines a growth $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\} \to \mathcal{L}(\check{V})$, which we will call the *intersection growth* of $(\boldsymbol{K}, \boldsymbol{K}')$ and denote by $\boldsymbol{K} \cap \boldsymbol{K}'$. Here we follow [8] and [22] in adopting the term growth: if $(\Pi, \prec)$ and $(\Pi', \prec')$ are posets, then a map $g : \Pi \to \Pi'$ is called a *growth* if $x \mathrel{\dot{\prec}} y$ (see Section 1.1) implies $g(x) \mathrel{\dot{\prec}'} g(y)$ or $g(x) = g(y)$.

By a *cell* in Π (*crowned by* $D \in \Pi$) we mean a quadruple (A, B, C, D) of elements of Π such that $A \mathrel{\dot{\prec}} B$, $A \mathrel{\dot{\prec}} C$, $B \mathrel{\dot{\prec}} D$, $C \mathrel{\dot{\prec}} D$, and $B \neq C$. Note that this imples $A = B \wedge C$ and $D = B \vee C$. If (A, B, C, D) is a cell then so is (A, B, C, D), which we call its *conjugate*. A cell (A, B, C, D) will be called *rigid* (resp. an *atom*) under a growth $g : \Pi \to \Pi'$ if $(g(A), g(B), g(C), g(D))$ is again a cell in Π' (resp. $g(A) = g(B) = g(C) \mathrel{\dot{\prec}} g(D)$). The cells of $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$ are of the form $((i-1, j-1), (i-1, j), (i, j-1), (i, j))$ or its conjugate.

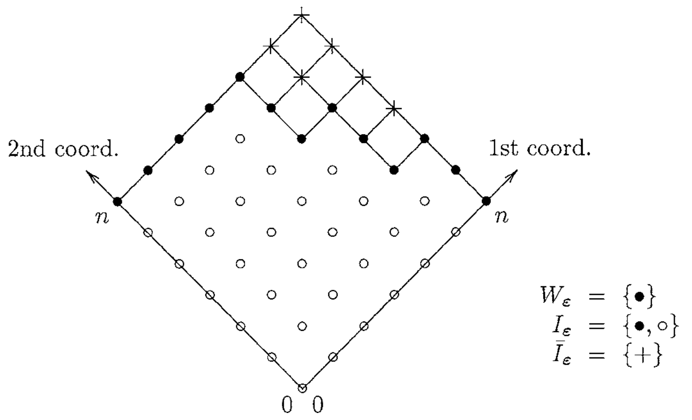


Figure 6. An example of W_ε, I_ε: $n = 6$, $\nu_\varepsilon = (4, 2)$.

For $\varepsilon \in \mathcal{E}$, the ε-*walk* $W_\varepsilon = \{(p_i, q_i)\}_{i=0}^{2n} \subset \{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$ is defined by

$$(p_i, q_i) = \begin{cases} (0, n) & \text{if } i = 0 \\ (p_{i-1} + 1, q_{i-1}) & \text{if } i > 0 \text{ and } \varepsilon_i = +1, \\ (p_{i-1}, q_{i-1} - 1) & \text{if } i > 0 \text{ and } \varepsilon_i = -1. \end{cases}$$

Note that we have $(p_{2n}, q_{2n}) = (n, 0)$. Let I_ε denote the order ideal of $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$ generated by W_ε, and $\overline{I_\varepsilon}$ the complement of I_ε. Observe that $\overline{I_\varepsilon}$, or the cells crowned by its elements, can be viewed as the Young diagram of ν_ε under suitable orientation (see figures 5 and 6), and that $\varepsilon \subset \varepsilon'$ (as defined in Section 3) is equivalent to $\overline{I_\varepsilon} \subset \overline{I_{\varepsilon'}}$ or $\overline{I_\varepsilon} \cup W_\varepsilon \subset \overline{I_{\varepsilon'}} \cup W_{\varepsilon'}$, not to $I_\varepsilon \subset I_{\varepsilon'}$. A pair $(\boldsymbol{K}, \boldsymbol{K}')$ of complete flags of $\check{V}$ will be said to be ε-*transversal* if all cells crowned by the elements of $\overline{I_\varepsilon}$ are rigid under the growth $\boldsymbol{K} \cap \boldsymbol{K}'$. (See remark after Lemma 4.2 for a concise set of conditions for ε-transversality.)

Lemma 4.2 *The ε-transversal pairs of complete flags of $\check{V}$ are in* 1-1 *correspondence with the complete updown flags of $\check{V}$ of class ε by taking the images of the ε-walk under the intersection growths of the pairs of complete flags.*

Proof: Let $\varepsilon \in \mathcal{E}$. Note that the cells in $\overline{I_\varepsilon} \cup W_\varepsilon$ are exactly the cells in $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$ crowned by the elements of $\overline{I_\varepsilon}$.

If $\boldsymbol{W}$ is an updown flag of class ε, one can define a growth $\bar{g} : \overline{I_\varepsilon} \cup W_\varepsilon \to \mathcal{L}(\check{V})$ by putting $\boldsymbol{W}$ on W_ε and proceeding cell by cell upwards inductively putting $\bar{g}(D) = \bar{g}(B) + \bar{g}(C)$ whenever (A, B, C, D) is a cell. The result is independent of the order of the procedure, since one always has $\bar{g}(i, j) = \sum_{k=b_j-1}^{a_i} W_k$ for all $(i, j) \in \overline{I_\varepsilon} \cup W_\varepsilon$ (where the a_i and the b_j correspond to ε as in Section 3). We claim that $\boldsymbol{W}$ is complete if and only if all cells in $\overline{I_\varepsilon} \cup W_\varepsilon$ are rigid under $\bar{g}$. Look at the dimensions of the $\bar{g}(i, j)$. For $(i, j) \in W_\varepsilon$, one always has $\dim \bar{g}(i, j) = i + j - n$ by the definition of an updown flag. Then the property of growth assures that for all points (i, j) one has $\dim \bar{g}(i, j) \leqq i + j - n$, and that $\dim \bar{g}(n, n) = n$ holds if and only if all cells in $\overline{I_\varepsilon} \cup W_\varepsilon$ are rigid. Therefore the claim holds.

Next suppose one has two growths $g, g' : \overline{I_\varepsilon} \cup W_\varepsilon \to \mathcal{L}(\check{V})$, and assume that all cells in $\overline{I_\varepsilon} \cup W_\varepsilon$ are rigid under g. We claim that, if $g|_{W_{\varepsilon_0}} = g'|_{W_{\varepsilon_0}}$ (resp. $g|_{W_\varepsilon} = g'|_{W_\varepsilon}$) holds, then one has $g = g'$. This is clear because one can reach W_ε from W_{ε_0} and vice versa by traversing cells, and the rigidity of the cell (A, B, C, D) determines $g(A)$ from $g(B)$, $g(C)$ by $g(A) = g(B) \cap g(C)$ in going downwards, and $g(D)$ from $g(B)$, $g(C)$ by $g(D) = g(B) + g(C)$ in going upwards.

Now if $(\boldsymbol{K}, \boldsymbol{K}')$ is an ε-transversal pair, let $\boldsymbol{W} = (W_i)_{i=0}^{2n}$ denote the sequence of subspaces attached to the points of W_ε under $\boldsymbol{K} \cap \boldsymbol{K}'$. The rigidity of the cells directly above W_ε assures the correct dimensions of the W_i to make $\boldsymbol{W}$ an updown flag of class ε. The growth $\bar{g}$ constructed from $\boldsymbol{W}$ coincides with $\boldsymbol{K} \cap \boldsymbol{K}'$ on W_ε, so all on $\overline{I_\varepsilon} \cup W_\varepsilon$, and hence is rigid. Therefore $\boldsymbol{W}$ is complete.

The correspondence $(\boldsymbol{K}, \boldsymbol{K}') \mapsto \boldsymbol{W}$ is injective since, because of rigidity, the value of $\boldsymbol{K} \cap \boldsymbol{K}'$ on W_ε determines $\boldsymbol{K} \cap \boldsymbol{K}'|_{\overline{I_\varepsilon} \cup W_\varepsilon}$, and hence $\boldsymbol{K}$ and $\boldsymbol{K}'$. It is also surjective since, if $\boldsymbol{W}$ is any complete updown flag of class ε, the $\bar{g}$ constructed from $\boldsymbol{W}$ determines a pair $(\boldsymbol{K}, \boldsymbol{K}')$ of complete flags on W_{ε_0} because of rigidity, and its intersection growth coincides with $\bar{g}$ on $\overline{I_\varepsilon} \cup W_\varepsilon$ so that $(\boldsymbol{K}, \boldsymbol{K}')$ is ε-transversal, and that $(\boldsymbol{K}, \boldsymbol{K}')$ corresponds to $\boldsymbol{W}$. □

Remark If $\varepsilon' \subset \varepsilon$ and $\boldsymbol{W}'$ is an updown flag of class ε', then one can define the "intersection growth of $\boldsymbol{W}'$", $g : I_{\varepsilon'} \to \mathcal{L}(\check{V})$, by putting $\boldsymbol{W}'$ on $W_{\varepsilon'}$ and proceeding cell by cell downwards inductively following the rule $g(A) = g(B) \cap g(C)$ whenever (A, B, C, D) is a cell (one always has $g(i, j) = \bigcap_{k=a_{i+1}-1}^{b_{j+1}} W_k$). $\boldsymbol{W}'$ can be called ε-transversal if all cells above W_ε and below $W_{\varepsilon'}$ are rigid under its intersection growth. The above correspondence can also be understood as a composition of bijections for pairs $(\varepsilon', \varepsilon'')$, $\varepsilon' \dot{\subset} \varepsilon'' \subset \varepsilon$, between the sets of ε-transversal complete updown flags of class ε' and those of class ε''. Here, if i_1 denotes the index such that ε' and ε'' only differ in the i_1th and $(i_1 + 1)$st positions, then $\boldsymbol{W}' = (W_i')_{i=0}^{2n}$ of class ε' corresponds to $\boldsymbol{W}''$ of class ε'' obtained from $\boldsymbol{W}'$ by replacing the i_1th component $W_{i_1}' = W_{i_1-1}' + W_{i_1+1}'$ by $W_{i_1-1}' \cap W_{i_1+1}'$.

Remark By an argument similar to the condition for completeness in the proof of Lemma 4.2, one can show that $(\boldsymbol{K}, \boldsymbol{K}')$ is ε-transversal if and only if $K_i + K_j' = \check{V}$ or equivalently $\dim(K_i \cap K_j') = i + j - n$ holds for all minimal points (i, j) of W_ε other than $(0, n)$ and $(n, 0)$. Namely, if we put $g = \boldsymbol{K} \cap \boldsymbol{K}'$, we have $\dim g(i, j) = i + j - n$ for $(i, j) \in W_{\varepsilon_0}$. Then the property of the growth assures that $\dim g(i, j) \geqq i + j - n$ for all (i, j) and that all cells in $\overline{I_\varepsilon} \cup W_\varepsilon$ are rigid if and only if $\dim g(i, j) = i + j - n$ holds for all minimal points (i, j) of $\overline{I_\varepsilon} \cup W_\varepsilon$. Since $g(0, n)$ and $g(n, 0)$ are always 0, one may exclude these points.

Lemma 4.3 *The pair $(\boldsymbol{K}, \boldsymbol{K}')$ of complete flags of $\check{V}$ is ε-transversal if and only if their relative position $w = w(\boldsymbol{K}, \boldsymbol{K}')$ satisfies $w(n + 1 - i) \leqq n - \nu_i'$, where ν stands for ν_ε, for all i in the range $1 \leqq i \leqq l(\nu_\varepsilon')$. For each fixed ε, the set of such w forms a coideal of $\mathfrak{S}_n$ with respect to the Bruhat order.*

Proof: The relative position $w(\boldsymbol{K}, \boldsymbol{K}')$ is the permutation $w \in \mathfrak{S}_n$ satisfying $d_{ij}(w) = \dim (K_i \cap K_j')$ for all i and j (see Section 2, before Lemma 2.1), so that the intersection growth $\boldsymbol{K} \cap \boldsymbol{K}'$ has atoms at the cells crowned by $(w(j), j)$, $1 \leqq j \leqq n$. Noting that rigid cells cannot be atoms, the above characterization of ε-transversality in terms of relative positions is clear. Now suppose w is such a permutation. Then it is known that w' is larger than w in the Bruhat order if and only if $d_{ij}(w') \leqq d_{ij}(w)$ for all i, j. Suppose w' satisfies this condition. We have $d_{ij}(w) = i + j - n$ for $(i, j) \in \overline{I_\varepsilon} \cup W_\varepsilon$ because of the rigidity of the cells above W_ε. However, as we saw in remark (the paragraph before Lemma 4.3), these are the lowest possible values of dimensions for these points. Therefore w' also satisfies $d_{ij}(w') = i + j - n$ for all $(i, j) \in \overline{I_\varepsilon} \cup W_\varepsilon$. This means that all cells above W_ε are rigid under the intersection growth of any pair of relative position w'. This proves our lemma. □

Remark This enables us to identify the collection of all complete updown flags of class ε with a Zariski open subvariety of the variety of all pairs of complete flags. (Note that the maps involved in the identification of these objects are morphism of algebraic varieties, since taking the sums and intersections of subspaces are morphisms for subspaces having the same dimension of intersections). Moreover, this identification is $GL(\check{V})$-equivariant, since the sum and intersection maps are $GL(\check{V})$-equivariant. Therefore the $GL(\check{V})$-orbits on the complete updown flags of $\check{V}$ of class ε are parametrized by the coideal of $\mathfrak{S}_n$ described in Lemma 4.3.

Lemma 4.4 *Let $\varepsilon \in \mathcal{E}$, and let $\nu = \nu_\varepsilon$ be the corresponding partition defined in Section 3. Let $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ be two bases of $\check{V}$ such that the complete flags $\mathrm{Fl}(\boldsymbol{v})$ and $\mathrm{Fl}(\boldsymbol{w})$ (see the beginning of Section 2) are ε-transversal. Then the $|\nu|$ linear forms $v^*_{n+1-i} \otimes w^*_{n+1-j}$, $(i, j) \in \nu$, on $\check{V} \otimes \check{V}$ restrict to linearly independent linear forms on $S^2(\check{V})$, the space of symmetric tensors on $\check{V}$ of rank 2.*

Proof: For simplicity, let $\grave{\boldsymbol{v}} = (\grave{v}_i)_{i=1}^n$ and $\grave{\boldsymbol{w}} = (\grave{w}_j)_{j=1}^n$ denote the bases obtained by reverting the numbering of $\boldsymbol{v}$ and $\boldsymbol{w}$ respectively. Let $A = (a_{ji})$ be the transition matrix from the basis $\grave{\boldsymbol{v}}$ to $\grave{\boldsymbol{w}}$, namely

$$\grave{v}_i (= v_{n+1-i}) = \sum_{j=1}^n a_{ji} \grave{w}_j \left(= \sum_{j=1}^n a_{ji} w_{n+1-j} \right).$$

One can take the $s_{ii} = \grave{v}_i \otimes \grave{v}_i$, $1 \leqq i \leqq n$, and the $s_{ij} = \grave{v}_i \otimes \grave{v}_j + \grave{v}_j \otimes \grave{v}_i$, $1 \leqq i < j \leqq n$, as a basis of $S^2(\check{V})$. The forms in question take values on these basis vectors as in Table 1. (Table 1 only shows the first $\nu_1 + \nu_2 + \nu_3$ of its columns.) Note that ${}^tA$ is the transition matrix of the dual bases $\grave{\boldsymbol{w}}^*$ to $\grave{\boldsymbol{v}}^*$.

Claim For each i in the range $1 \leqq i \leqq l = l(\nu)$, the ν_i column vectors of dimension $n+1-i$ appearing in the rows labeled (i, i) through (i, n) and columns labeled $(i, 1)$ through (i, ν_i) are linearly independent.

It follows from the ε-transversality of the pair $(\boldsymbol{K}, \boldsymbol{K}')$ that $\dim(K_{n-i} \cap K'_{n-\nu_i}) = n - i - \nu_i$, in other words the linear independence of $\grave{v}^*_1, \ldots, \grave{v}^*_i, \grave{w}^*_1, \ldots, \grave{w}^*_{\nu_i}$. Representing these (dual) vectors in the basis $\grave{\boldsymbol{v}}^*$, one knows the linear independence of the following $i + \nu_i$ column

Table 1. The values of the s_{ij} at the $\check{v}_k^* \otimes \check{w}_l^*$.

		(k, l)										
		$(1, 1)$	$\cdots$	$(1, \nu_1)$	$(2, 1)$	$\cdots$	$(2, \nu_2)$	$(3, 1)$	$\cdots$	$(1, \nu_3)$	$\cdots$
	$(1, 1)$	a_{11}	$\cdots$	$a_{\nu_1 1}$							
	$(1, 2)$	a_{12}	$\cdots$	$a_{\nu_1 2}$	a_{11}	$\cdots$	$a_{\nu_2 1}$				
	$(1, 3)$	a_{13}	$\cdots$	$a_{\nu_1 3}$				a_{11}	$\cdots$	$a_{\nu_3 1}$	
	$\vdots$	$\vdots$	$\ddots$	$\vdots$							$\ddots$
	$(1, n)$	a_{1n}	$\cdots$	$a_{\nu_1 n}$							
	$(2, 2)$				a_{12}	$\cdots$	$a_{\nu_2 2}$				
	$(2, 3)$				a_{13}	$\cdots$	$a_{\nu_2 3}$	a_{12}	$\cdots$	$a_{\nu_3 2}$	
(i, j)					$\vdots$	$\ddots$	$\vdots$				$\ddots$
	$(2, n)$				a_{1n}	$\cdots$	$a_{\nu_2 n}$				
	$(3, 3)$							a_{13}	$\cdots$	$a_{\nu_3 3}$	
	$\vdots$							$\vdots$	$\ddots$	$\vdots$	$\ddots$
	$(3, n)$							a_{1n}	$\cdots$	$a_{\nu_3 n}$	
	$\vdots$										$\ddots$

vectors of dimension n:

$$\begin{array}{ccccccc}
1 & & 0 & a_{11} & \ldots & a_{\nu_i 1} \\
 & \ddots & & \vdots & \ddots & \vdots \\
0 & & 1 & a_{1i} & \ldots & a_{\nu_i i} \\
0 & \cdots & 0 & a_{1,i+1} & \ldots & a_{\nu_i, i+1} \\
\vdots & & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & a_{1n} & \ldots & a_{\nu_i n},
\end{array}$$

from which follows the linear independence of the ν_i column vectors of dimension $n - i$ appearing in the bottom-right block. This is enough to show the claim.

This claim is in turn enough to show the linear independence of the $|\nu|$ column vectors of dimension $\binom{n+1}{2}$ in the whole matrix. □

Let $\check{\mathcal{N}}$, $\check{X}$, $\check{\mathcal{O}}_w$, $\check{Z}$, $\check{Z}_w$ and $\check{Z}_{T,T'}$ denote the objects corresponding to $\mathcal{N}$, X, $\mathcal{O}_w$, Z, Z_w, $Z_{T,T'}$ in the sense of Section 1.2 respectively, for the vector space $\check{V}$ instead of V. If $\check{N} \in \check{\mathcal{N}}$, let us say that a sequence $\boldsymbol{W} = (W_k)$ of subspaces of $\check{V}$ is $\check{N}$-stable if all W_k are $\check{N}$-stable. For each $\varepsilon \in \mathcal{E}$ {let $\check{X}_\varepsilon$ denote the set of all complete updown flags of class ε, and put $\check{Z}_\varepsilon = \{(\check{N}, \boldsymbol{W}) \in \check{\mathcal{N}} \times \check{X}_\varepsilon \mid \boldsymbol{W}$ is $\check{N}$-stable$\}$ and $\mathcal{M}_\varepsilon = \{M \in \mathcal{M}_{2n} \mid M$ is of class $\varepsilon\}$, so that $\mathcal{M}_{2n} = \coprod_{\varepsilon \in \mathcal{E}_{2n}} \mathcal{M}_\varepsilon$. For each $M \in \mathcal{M}_\varepsilon$ put $\check{Z}_M = \{(\check{N}, \boldsymbol{W}) \in \check{Z}_\varepsilon \mid \mathrm{type}_{\check{N}} \boldsymbol{W} = M\}$.

Clearly $\check{Z}_\varepsilon = \coprod_{M \in \mathcal{M}_\varepsilon} \check{Z}_M$. Let us show that the $\check{Z}_M$ are irreducible and nonsingular, and all of dimension $n^2 - n$ (Proposition 4.5). This fact will be used in the next section. As a byproduct, this shows that the closures of the $\check{Z}_M$, $M \in \mathcal{M}_\varepsilon$, give all irreducible components of $\check{Z}_\varepsilon$ (see Corollary 4.6, where we also discuss relationship with other labelings of the irreducible components).

Proposition 4.5 *Fix $\varepsilon \in \mathcal{E}$ and $M \in \mathcal{M}_\varepsilon$. Then $\check{Z}_M$ is a nonsingular irreducible locally closed subvariety of $\check{Z}_\varepsilon$ of dimension $n^2 - n$.*

Proof: First of all, we note that $\check{Z}_M$ is a locally closed subvariety of $\check{Z}_\varepsilon$, since the condition $\text{type}_{\check{N}} \boldsymbol{W} = M$ can be given in terms of equalities on the dimensions of $\check{N}^j W_k$ for various j and k (where $\boldsymbol{W} = (W_k)_{k=0}^{2n}$).

Now let (p_k, q_k), $0 \leqq k \leqq 2n$, be as above. We have $p_k = \#\{k' \in \{1, 2, \dots, k\} \mid \varepsilon_{k'} = +1\}$ and $q_k = \#\{k' \in \{k+1, k+2, \dots, 2n\} \mid \varepsilon_{k'} = -1\}$. Also put $d_k = p_k + q_k - n = |\mu^{(k)}|$. For any sequence $\boldsymbol{W} = (W_k)$ of subspaces of $\check{V}$, let $\Sigma \boldsymbol{W}$ denote the sum of all its constituents: $\Sigma \boldsymbol{W} = \sum_k W_k$.

For each k, $0 \leqq k \leqq 2n$, put $M(k) = (\mu^{(k')})_{k'=0}^{k}$, and let $\check{X}_\varepsilon(k)$ be the set of sequences $(W_{k'})_{k'=0}^{k}$ of subspaces of $\check{V}$ such that $\varepsilon_{k'} = +1$ implies $W_{k'-1} \dot{\subset} W_{k'}$ and $\varepsilon_{k'} = -1$ implies $W_{k'-1} \dot{\supset} W_{k'}$ for all $1 \leqq k' \leqq k$, and such that $\dim \Sigma \boldsymbol{W} = p_k$, and finally let $\check{Z}_M(k)$ be the set of all $(\check{N}, \boldsymbol{W})$ such that $\boldsymbol{W} \in \check{X}_\varepsilon(k)$, $\check{N} \in \mathfrak{gl}(\Sigma \boldsymbol{W})$ is nilpotent, $\boldsymbol{W}$ is $\check{N}$-stable and $\text{type}_{\check{N}} W_{k'} = \mu^{(k')}$ holds for all $0 \leqq k' \leqq k$. Note that $M(2n) = M$, $\check{X}_\varepsilon(2n) = \check{X}_\varepsilon$, and $\check{Z}_M(2n) = \check{Z}_M$. For each k, $\check{X}_\varepsilon(k)$ and $\check{Z}_M(k)$ are algebraic varieties over $\mathbb{C}$.

We show that $\check{Z}_M(k)$ is irreducible and nonsingular by induction on k. This is equivalent to claiming that it is a connected complex manifold under the ordinary topology. Since $\check{Z}_M(0)$ is a single point, let us assume that $k > 0$. Note that, if $\boldsymbol{W} = (W_{k'})_{k'=0}^{k} \in \check{X}_\varepsilon(k)$, then we have $\boldsymbol{W}' = (W_{k'})_{k'=0}^{k-1} \in \check{X}_\varepsilon(k-1)$, since $\dim \Sigma \boldsymbol{W} = p_k$ implies $\dim \Sigma \boldsymbol{W}' = p_{k-1}$. Hence we have a natural projection $\check{Z}_M(k) \to \check{Z}_M(k-1)$.

We distinguish two cases.

Case 1 ($\varepsilon_k = -1$) Then $\check{Z}_M(k)$ can be written as a fibered product in the following manner. Let $\dot{Z}$ denote the collection of all $(\check{N}_{k-1}, W_{k-1})$ such that W_{k-1} is a d_{k-1}-dimensional subspace of V and $\check{N}_{k-1} \in \mathfrak{gl}(W_{k-1})$ is nilpotent of type $\mu^{(k-1)}$. Also let $\ddot{Z}$ denote the set of all $(\check{N}_{k-1}, W_{k-1}, W_k)$ such that $W_{k-1} \supset W_k$ are subspaces of $\check{V}$ of dimension d_{k-1} and d_k respectively, and $\check{N}_{k-1} \in \mathfrak{gl}(W_{k-1})$ is nilpotent such that $\text{type}_{\check{N}_{k-1}}(W_{k-1}) = \mu^{(k-1)}$ and $\text{type}_{\check{N}_{k-1}}(W_k) = \mu^{(k)}$. Then we have $\check{Z}_M(k) \cong \check{Z}_M(k-1) \underset{\dot{Z}}{\times} \ddot{Z}$.

Now let us show that $\ddot{Z} \to \dot{Z}$ is an analytic fiber bundle. The group $\check{G} = GL(\check{V})$ acts transitively on $\dot{Z}$, so that if S is the stabilizer of an arbitrarily chosen point $\dot{P}^0 = (\check{N}_{k-1}^0, W_{k-1}^0) \in \dot{Z}$, then we have $\dot{Z} \cong \check{G}/S$, and hence $\dot{Z}$ is a connected complex manifold. Then $\pi : \ddot{Z} \to \dot{Z}$ is an analytic fiber bundle, since we can find an analytic neighborhood U of 0 in a subspace $\mathfrak{c}$ of the tangent space of $\check{G}$ at the indentity which complements that of S, such that $\phi : X \mapsto (\exp X) \cdot \dot{P}^0$ is a bihomomorphic map from U onto a neighborhood $\dot{U}$ of $\dot{P}^0$ in $\dot{Z}$, and π is biholomorphically trivialized over $\dot{U}$ by $\pi^{-1}(\dot{U}) \ni ((\exp X) \cdot \check{N}^0, (\exp X) \cdot W_{k-1}^0, W_k) \mapsto ((\exp X) \cdot \check{N}^0, (\exp X) \cdot W_{k-1}^0), (\check{N}^0, W_{k-1}^0, (\exp X)^{-1} \cdot W_k)) \in \dot{U} \times \pi^{-1}(\dot{P}^0)$, and we can translate $\dot{U}$ by the action of $\check{G}$ to cover $\dot{Z}$.

If we write $F = \pi^{-1}(\dot{P}^0)$ for simplicity, then $\check{Z}_M(k)$ is also an analytic fiber bundle over $\check{Z}_M(k-1)$ with fiber F. By induction hypothesis, the base space is a connected complex manifold. The fiber F is also a connected complex manifold, since it is isomorphic to the set of hyperplanes of W_{k-1} having a prescribed $\check{N}$-type $\mu^{(k)}$, which is isomorphic to $\mathbb{P}^{r-1} - \mathbb{P}^{r'-1}$ where (r, c) is the position of the corner of the Young diagram $\mu^{(k-1)}$ which gets deleted in $\mu^{(k)}$ and (r', c') is the "next" corner of $\mu^{(k-1)}$ above (r, c). Hence the whole space $\check{Z}_M(k)$ is also a connected complex manifold in this case. We also have dim $\check{Z}_M(k) -$ dim $\check{Z}_M(k-1) =$ dim $F = r - 1$ in this case.

Case 2 ($\varepsilon_k = +1$) Let $\dot{Z}$ be as in the previous case, and this time let $\ddot{Z}$ be the collection of all $(\check{N}_k, W_{k-1}, W_k)$ such that $W_{k-1} \subset W_k$ are subspaces of dimension d_{k-1} and d_k respectively, and $\check{N}_k \in \mathfrak{gl}(W_k)$ is nilpotent with type$_{\check{N}_k}(W_{k-1}) = \mu^{(k-1)}$ and type$_{\check{N}_k}(W_k) = \mu^{(k)}$. Then $\check{Z}_M(k)$ can be identified with a Zariski open subset of $\check{Z}_M(k-1) \underset{\dot{Z}}{\times} \ddot{Z}$ defined by $W_k \not\subset \Sigma \boldsymbol{W}'$ for $((\check{N}', \boldsymbol{W}'), (\check{N}_k, W_{k-1}, W_k)) \in \check{Z}_M(k-1) \underset{\dot{Z}}{\times} \ddot{Z}$ since, under this condition, $\check{N}$ is determined by its restriction on $\Sigma \boldsymbol{W}'$ and W_k. Thus, if we know that $\ddot{Z} \to \dot{Z}$ is an analytic fiber bundle whose fiber is a connected complex manifold, then, together with the induction hypothesis, $\check{Z}_M(k-1) \underset{\dot{Z}}{\times} \ddot{Z}$ is a connected complex manifold, and hence is irreducible and nonsingular. Since $\check{Z}_M(k)$ is Zariski open in $\check{Z}_M(k-1) \underset{\dot{Z}}{\times} \ddot{Z}$, it is also irreducible and nonsigular. (The above argument does not eliminate the possibility that $\check{Z}_M(k)$ might be empty. We will give a separate argument in the final three paragraphs of this proof to show that $\check{Z}_M$ is nonempty after all).

The proof of the fact that $\ddot{Z} \to \dot{Z}$ is an analytic fiber bundle goes in the same manner as in the previous case. What remains is to show that the fiber F of a point $\dot{P}^0 = (\check{N}^0_{k-1}, W^0_{k-1}) \in \dot{Z}$ is a connected complex manifold. Let F_1 be the collection of all d_k-dimensional subspaces containing W^0_{k-1}, which is isomorphic to $\mathbb{P}(\check{V}/W^0_{k-1})$. Let $\pi_1 : F \to F_1$ be the natural projection. Let $W_k \in F_1$. Then W_k has a Zariski open neighborhood U_1 in F_1 such that the tautological line bundle over F_1 has a nowhere vanishing section $\bar{s}$ on U_1. Let $s : U_1 \to \check{V}$ be a morphism which "lifts" $\bar{s}$, namely such that $W'_k = W^0_{k-1} \oplus \mathbb{C} \cdot s(W'_k)$ for every $W'_k \in U_1$. If (r, c) denotes the position of the corner of the Young diagram of $\mu^{(k)}$ added to $\mu^{(k-1)}$, then $A \in \mathfrak{gl}(W_k)$ with $A|_{W^0_{k-1}} = \check{N}^0_{k-1}$ is nilpotent of type $\mu^{(k)}$ if and only if $A(s(W_k)) \in F_2 = (\ker(\check{N}^0_{k-1})^{c-1} + \mathrm{im}\check{N}^0_{k-1}) - (\ker(\check{N}^0_{k-1})^{c-2} + \mathrm{im}\check{N}^0_{k-1}) \subset W^0_{k-1}$ (if $c = 1$, the subtrahend is understood to be empty). Therefore we have an algebraic isomorphism $\pi_1^{-1}(U_1) \cong U_1 \times F_2$, so that π_1 is an algebraic fiber bundle with fiber F_2. Since the base space F_1 ($\cong \mathbb{P}^{n-d_{k-1}-1}$) and the fiber F_2($\cong \mathbb{A}^{d_{k-1}-(r-1)} - \mathbb{A}^{d_{k-1}-r''}$, where (r'', c'') is the "uppermost" corner of $\mu^{(k-1)}$ in the rows r and below) are both connected complex manifolds, so is the total space F. We also see that dim $\check{Z}_M(k) -$ dim $\check{Z}_M(k-1) =$ dim $F =$ dim $F_1 +$ dim $F_2 = (n - d_{k-1} - 1) + (d_{k-1} - (r-1)) = n - r$ in this case.

Summing up the dimensions, we have dim $\check{Z}_M = \sum_{\varepsilon_k=+1}(n - r_k) + \sum_{\varepsilon_k=-1}(r_k - 1)$, where r_k denotes the row number of the cell either added or deleted in obtaining $\mu^{(k)}$ from $\mu^{(k-1)}$. Note that the r_k cancel in total, since all cells included eventually get removed. Since we have n summands in each sum, we have dim $\check{Z}_M = n^2 - n$.

As we remarked before, the above argument does not eliminate the possibility of $\check{Z}_M$ being empty. To see that this cannot happen, we first apply Fomin's theory (see [8]) to the convex subset I_ε of the poset $\square_n = \{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$. Note that W_ε is the upper boundary of I_ε. Let us call a growth into Young's lattice a *growth of partitions*, and a growth $g : (\Pi, \prec) \to (\Pi', \prec')$ *faithful* if $x \dot{\prec} y$ implies $g(x) \dot{\prec}' g(y)$. Then the updown tableaux of class ε are exactly the faithful growths of partitions on W_ε with the empty shape at both ends. Therefore [8, Theorem B] applied to I_ε provides a bijection between the $w \in C_\varepsilon$, where C_ε denotes the coideal of $\mathfrak{S}_n$ described in Lemma 4.3, and the $M \in \mathcal{M}_\varepsilon$. Moreover, [8, Theorem H] shows that one such bijection is given by defining the output M to be the restriction to W_ε of the growth of partitions g_w on $\square_n$ which maps each (p, q) to the Greene-Kleitman invariant of the poset $\Pi_p(w) \cap \Pi^q(w)$.

For the moment, let w be any element of $\mathfrak{S}_n$, and let $\check{N}$ be a generic matrix of the poset $\Pi(w)$ (see Section 1.2). For each $0 \leqq p \leqq n$ (resp. $0 \leqq q \leqq n$), let $\check{V}_p^e$ (resp. $\check{V}_q^w$) denote the span of $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_p$ (resp. $\boldsymbol{e}_{w(1)}, \boldsymbol{e}_{w(2)}, \ldots, \boldsymbol{e}_{w(q)}$). Then for each p and q, the subspace $\check{V}_p^e \cap \check{V}_q^w$ is $\check{N}$-stable, and the restriction $\check{N}|_{\check{V}_p^e \cap \check{V}_q^w} V$ is represented by the submatrix of $\check{N}$ consisting of the rows and columns indexed by $\Pi_p(w) \cap \Pi^q(w)$. This is a generic matrix of this poset. By the result of Gansner and Saks (see Section 1.2), the Jordan type of this matrix coincides with the Greene-Kleitman invariant of this poset, namely $g_w(p, q)$.

Now let $w \in C_\varepsilon$ be the permutation sent to M by the above bijection, and put $\boldsymbol{W}^M = (V_{p_k}^e \cap V_{q_k}^w)_{k=0}^{2n}$. Then $(\check{N}, \boldsymbol{W}^M) \in \check{Z}_M$ since g_w restricts to M on $W_\varepsilon = \{(p_k, q_k)\}_{k=0}^{2n}$. Hence $\check{Z}_M$ is not empty. $\square$

Note that the restriction of g_w (the growth of partitions defined in the final part of the above proof) on W_{ε_0} (the upper boundary of $\square_n$) is the updown tableau of class ε_0 obtained by "concatenating" the pair of tableaux (T, T') (see Section 1.3, after the definition of updown tableaux) produced from w by the Robinson-Schensted correspondence. Moreover, it is the essense of the results in [32] that the closures of $\check{Z}_w$ and $\check{Z}_{T,T'}$ give the same irreducible component of $\check{Z}$ if w corresponds to (T, T') in this manner. Hence it is natural to expect that, also in the case of $\check{Z}_\varepsilon$, the bijection $C_\varepsilon \to \mathcal{M}_\varepsilon$ (C_ε was defined in the final part of the above proof), $w \mapsto g_w|_{W_\varepsilon}$, not only gives the equality of numbers, but also represents the actual correspondence between the two parametrizations of the irreducible components of $\check{Z}_\varepsilon$.

Let us show that this is the case.

Note that the open embedding $\iota_\varepsilon : \check{X}_\varepsilon \to \check{X} \times \check{X}$ defined by Lemma 4.2 induces an open embedding $\tilde{\iota}_\varepsilon : \check{Z}_\varepsilon \to \check{Z}$ since, if $\boldsymbol{W} \in \check{X}_\varepsilon$ and $\iota_\varepsilon(\boldsymbol{W}) = (\boldsymbol{K}, \boldsymbol{K}')$, then $\boldsymbol{W}$ is $\check{N}$-stable if and only if $\boldsymbol{K}$ and $\boldsymbol{K}'$ are $\check{N}$-stable.

Corollary 4.6 *Let $\varepsilon \in \mathcal{E}$, and identify $\check{Z}_\varepsilon$ with its image in $\check{Z}$ under the open embedding $\tilde{\iota}_\varepsilon$ defined above. The relative closures of the $\check{Z}_w$ inside $\check{Z}_\varepsilon$, $w \in C_\varepsilon$, give all irreducible components of $\check{Z}_\varepsilon$, so do the closures of the $\check{Z}_M$ in $\check{Z}_\varepsilon$, $M \in \mathcal{M}_\varepsilon$. The closures of $\check{Z}_w$ and $\check{Z}_M$ in $\check{Z}_\varepsilon$ coincide if and only if w corresponds to M by $M = g_w|_{W_\varepsilon}$ (where g_w is the growth of partitions on $\square_n$ defined in the final part of the proof of Proposition* 4.5).

Proof: First note that $\check{Z}_\varepsilon = \coprod_{M \in \mathcal{M}_\varepsilon} \check{Z}_M$. Combined with the results in Proposition 4.5, the argument reviewed in Section 1.2 shows that the closures of the $\check{Z}_M$, $M \in \mathcal{M}_\varepsilon$, give all irreducible components of $\check{Z}_\varepsilon$. On the other hand, Lemma 4.3 implies that $\check{Z}_\varepsilon$, embedded into $\check{Z}$ by $\tilde{\iota}_\varepsilon$, coincides with $\coprod_{w\in C_\varepsilon} \check{Z}_w$. It is a part of the results in [32] that each $\check{Z}_w$ is an irreducible (nonempty, nonsingular) locally closed subvariety of $\check{Z}$ of dimension $n^2 - n$. Hence the (relative) closures of the $\check{Z}_w$ inside $\check{Z}_\varepsilon$, $w \in C_\varepsilon$, also give all irreducible components of $\check{Z}_\varepsilon$.

It was shown towards the end of the proof of Proposition 4.5 that, so long as $\check{N}$ is a generic matrix of the poset $\Pi(w)$, we have $(\check{N}, \boldsymbol{W}^M) \in \check{Z}_M$, where $M = g_w|_{W_\varepsilon}$. Note that $\boldsymbol{W}^M$ is identified with $(\check{\boldsymbol{V}}^e, \check{\boldsymbol{V}}^w)$ (the counterparts of "$\boldsymbol{V}^e$" and "$\boldsymbol{V}^w$" in the sense of Section 1.2 for the vector space $\check{V}$ rather than V) via $\tilde{\iota}_\varepsilon$. Also, the set of generic matrices of $\Pi(w)$ is Zariski dense in the vector space spanned by the matrix units E_{ij} with $i \precneqq_w j$, namely the space of all nilpotent matrices stabilizing both $\check{\boldsymbol{V}}^e$ and $\check{\boldsymbol{V}}^w$ (see Section 1.2). Hence the union of all $GL(\check{V})$-translates of $(\check{N}, \check{\boldsymbol{V}}^e, \check{\boldsymbol{V}}^w)$, where $\check{N}$ runs through all generic matrices of $\Pi(w)$, is Zariski dense in $\check{Z}_w$. Since this subset lies in $\check{Z}_M$, the closure of $\check{Z}_M$ contains $\check{Z}_w$, and hence contains the closure of $\check{Z}_w$ (where closures are all taken inside $\check{Z}_\varepsilon$). Since this is an inclusion relation between irreducible components of $\check{Z}_\varepsilon$, this is actually an equality. Since the closures of the $\check{Z}_w$, $w \in C_\varepsilon$ (resp. the $\check{Z}_M$, $M \in \mathcal{M}_\varepsilon$) are all different, we conclude that the closures of $\check{Z}_w$ and $\check{Z}_M$ coincide if and only if w corresponds to M in this manner. □

Remark The irreducible components of $\check{Z}_\varepsilon$ admit still another parametrization. If $w \in C_\varepsilon$, then the closure of $\check{Z}_w$ in $\check{Z}_\varepsilon$, in other words $\overline{\check{Z}_w} \cap \check{Z}_\varepsilon$ (where $^-$ denotes the closure in $\check{Z}$), can also be written as $\overline{\check{Z}_{T,T'}} \cap \check{Z}_\varepsilon$, or the closure of $\check{Z}_{T,T'} \cap \check{Z}_\varepsilon$ in $\check{Z}_\varepsilon$, where (T, T') comes from w by the Robinson-Schensted correspondence. Thus, if $\mathcal{T}_\varepsilon$ denotes the set of pairs of standard tableaux (T, T') that come from the elements of C_ε by the Robinson-Schensted correspondence, then the irreducible components of $\check{Z}_\varepsilon$ are also parametrized by $\mathcal{T}_\varepsilon$. The transfer from the parametrization by $\mathcal{M}_\varepsilon$ to that by $\mathcal{T}_\varepsilon$ can be attained by applying Fomin's theory to the region $\overline{I_\varepsilon} \cup W_\varepsilon$, without going back to w. One starts by putting M on its lower boundary W_ε, constructs a growth of partitions on $\overline{I_\varepsilon} \cup W_\varepsilon$ by Fomin's local rules, and obtains the pair (T, T') in a concatenated manner (see Section 1.3, the end of the paragraph containing the definition of updown tableaux) on the upper boundary W_{ε_0}. The growth thus constructed is actually the restriction of g_w for $w \in C_\varepsilon$ hidden behind. However, all cells above W_ε are rigid under such g_w, so that the construction of the part above W_ε only needs the rules that deal with rigid cells. Namely, suppose (A, B, C, D) is a cell in $\overline{I_\varepsilon} \cup W_\varepsilon$, and suppose we know $g(A) = \alpha$, $g(B) = \beta$, and $g(C) = \gamma$. Then the following subset of Fomin's rules determines $g(D) = \delta$:

$$\begin{cases} \text{If } \beta \neq \gamma, \text{ then } \delta = \beta \cup \gamma. \\ \text{If } \beta = \gamma, \text{ and } \beta - \alpha = \gamma - \alpha \text{ lies in the } r\text{th row,} \\ \quad \text{then } \delta - \beta = \delta - \gamma \text{ lies in the } (r+1)\text{st row.} \end{cases} \tag{2}$$

Although we do not use it in the sequel, the argument above can be used to show that any of $\check{Z}_w$, $\check{Z}_{T,T'}$, or $\check{Z}_M$ has a generic $\check{N}$-type of all intersections $K_p \cap K'_q$ for its elements $(\check{N}, \boldsymbol{K}, \boldsymbol{K}')$, where $\boldsymbol{K} = (K_p)_{p=0}^n$ and $\boldsymbol{K}' = (K'_q)_{q=0}^n$, in the following sense.

Corollary 4.7 *Let $w \in \mathfrak{S}_n$, and put $\check{Z}(g_w) = \{(\check{N}, \boldsymbol{K}, \boldsymbol{K}') \in \check{Z} \mid \mathrm{type}_{\check{N}}(K_p \cap K'_q) = g_w(p, q)$ $(0 \leqq \forall p, q \leqq n)\}$ where $\boldsymbol{K} = (K_p)_{p=0}^n$, $\boldsymbol{K}' = (K'_q)_{q=0}^n$ and g_w is the growth of partitions on*

$\square_n$ *defined in the final part of the proof of Proposition* 4.5. *Also let* (T, T') *be the pair of standard tableaux coming from* w *by the Robinson-Schensted correspondence*, *and* λ *the shape of* T *and* T'. *Then the following hold.*

(1) $\check{Z}(g_w)$ *is a Zariski open dense subset of* $\check{Z}_w$.

(2) *Let* $\varepsilon \in \mathcal{E}$ *be such that* $w \in C_\varepsilon$, *and put* $M = g_w|_{W_\varepsilon}$. *Then* $\check{Z}(g_w)$ *is a Zariski open dense subset of* $\check{Z}_M$. *In particular*, $\check{Z}(g_w)$ *is a Zariski open dense subset of* $\check{Z}_{T,T'}$.

(3) *Fix* $\check{N} \in \check{\mathcal{N}}$ *of Jordan type* λ. *Then* $\{(\boldsymbol{K}, \boldsymbol{K}') \in \check{X}_{\check{N}} \times \check{X}_{\check{N}} \mid (\check{N}, \boldsymbol{K}, \boldsymbol{K}') \in \check{Z}(g_w)\}$ *is a Zariski open dense subset of* $\check{X}_{\check{N},T} \times \check{X}_{\check{N},T'}$.

Remark G. Tesler formulated a numerical counterpart of (3) (see [36, Conjecture 7.9]). Actually his conjecture is for a wider class of objects, namely the flags in q-regular semi-primary lattices, which include the $\check{N}$-stable flags in a vector space over $\mathbb{F}_q$. [36] deals with more related problems in this context.

Proof: Since $|g_w(p, q)| = d_{pq}(w)$ (see Section 2, before Lemma 2.1), we have $\check{Z}(g_w) \subset \check{Z}_w$. In view of the final part of the proof of Proposition 4.5 again, the Zariski dense subset of $\check{Z}_w$ mentioned in the proof of Corollary 4.6 actually lies inside $\check{Z}(g_w)$. On the other hand, $\check{Z}(g_w)$ is Zariski locally closed in $\check{Z}_w$ since it is defined by a finite number of equalities on the dimensions of the subspaces $\check{N}^j(K_p \cap K'_q)$. A Zariski locally closed subset can be dense only if it is Zariski open. Hence $\check{Z}(g_w)$ is Zariski open dense in $\check{Z}_w$. This proves (1).

Let ε and M be as in (2). Since $M = g_w|_{W_\varepsilon}$, we have $\check{Z}(g_w) \subset \check{Z}_M$ by definition. (1) implies that $\check{Z}(g_w)$ is Zariski open and dense in the closure of $\check{Z}_w$ in $\check{Z}_\varepsilon$, which equals the closure of $\check{Z}_M$ in $\check{Z}_\varepsilon$ by Corollary 4.6. Hence $\check{Z}(g_w)$ is Zariski open and dense in $\check{Z}_M$. In particular, any w lies in C_{ε_0}, and $\check{Z}_M$ for $M = g_w|_{W_{\varepsilon_0}}$ is none other than $\check{Z}_{T,T'}$. This proves (2).

Since the subset described in (3), which we temporarily denote by S, is Zariski locally closed in $\check{X}_{\check{N},T} \times \check{X}_{\check{N},T'}$ by the same reason as in (1), it is enough to show that it is Zariski dense. If not, the dimension of S would be strictly smaller than that of $\check{X}_{\check{N},T} \times \check{X}_{\check{N},T'}$. Then the dimension of $\check{Z}(g_w) \cong \check{G} \times^{Z_{\check{G}}(\check{N})} S$ would be strictly smaller than that of $\check{Z}_{T,T'} \cong \check{G} \times^{Z_{\check{G}}(\check{N})} (\check{X}_{\check{N},T} \times \check{X}_{\check{N},T'})$ (where $\check{G} = GL(\check{V})$ and $Z_{\check{G}}(\check{N})$ is the centralizer of $\check{N}$ in $\check{G}$), which would contradict (2). This proves (3). $\square$

5. Irreducibility of Z_M

Let $\varepsilon \in \mathcal{E}_{2n}$ and $M \in \mathcal{M}_\varepsilon$. We continue to use the notations introduced during the arguments in Section 4, such as $\mathcal{M}_\varepsilon$ and Σ (as well as $\check{\mathcal{N}}$, $\check{X}_\varepsilon$, ι_ε, $\check{Z}_\varepsilon$, $\check{Z}_M$, once we fix a maximal isotropic subspace $\check{V}$). The following proposition is our main objective in this section.

Proposition 5.1 *Z_M is an irreducible*, *nonsingular locally closed subvariety of Z of dimension* $4n^2 - n$.

Proof: Here we outline the proof, giving over the details to the lemmas below.

In order to see the relation between $\check{Z}_M$ and Z_M, it turns out to be natural to consider the relation between $\check{Z}_\varepsilon$ and $Z_\varepsilon = \coprod_{M \in \mathcal{M}_\varepsilon} Z_M$, and then restrict to Z_M.

First note that $G = GL(2n, \mathbb{C})$ acts transitively on Y. Let us fix $\omega_0 \in Y$ for the rest of the argument. We will always take $\perp$ with respect to ω_0, and similarly omit writing ω_0 in some other occasions where formally the form should be mentioned. Let $H \cong Sp(2n, \mathbb{C})$ denote the stabilizer of ω_0. Now H acts transitively on the set of maximal isotropic subspaces of V. We also fix a maximal isotropic subspace $\check{V}$. The stabilizer $H_{\check{V}}$ of $\check{V}$ in H (see (3)) is a maximal parabolic subgroup of H. Thus, if we put

$$\dot{X} = \{\boldsymbol{V} \in X \mid \Sigma \boldsymbol{R}(\boldsymbol{V}) = \check{V}\} \supset \dot{X}_\varepsilon = \{\boldsymbol{V} \in \dot{X} \mid \boldsymbol{R}(\boldsymbol{V}) \text{ is of class } \varepsilon\}$$

and

$$\dot{Z} = \{(N, \omega, \boldsymbol{V}) \in Z \mid \omega = \omega_0,\ \boldsymbol{V} \in \dot{X}\} \supset \dot{Z}_\varepsilon = \dot{Z} \cap Z_\varepsilon \supset \dot{Z}_M = \dot{Z} \cap Z_M,$$

where $\boldsymbol{R}(\boldsymbol{V}) = \boldsymbol{R}(\omega_0, \boldsymbol{V})$, then the map $G \times \dot{Z} \to Z$, $(g, (N, \omega_0, \boldsymbol{V})) \mapsto (\mathrm{Ad}(g)N, g^*\omega_0, g \cdot \boldsymbol{V})$ is surjective, and restricts to surjections $G \times \dot{Z}_\varepsilon \to Z_\varepsilon$ and $G \times \dot{Z}_M \to Z_M$. We can write $Z \cong G \times^{H_{\check{V}}} \dot{Z}$, $Z_\varepsilon \cong G \times^{H_{\check{V}}} \dot{Z}_\varepsilon$ and $Z_M \cong G \times^{H_{\check{V}}} \dot{Z}_M$. In this case Z, Z_ε, Z_M are algebraic fiber bundles over $G/H_{\check{V}}$ with fibers $\dot{Z}$, $\dot{Z}_\varepsilon$, $\dot{Z}_M$ respectively, since both $G \to G/H$ and $H \to H/H_{\check{V}}$ admit regular sections on Zariski open subsets.

The projection $\pi_X : Z \to X$, $(N, \omega, \boldsymbol{V}) \mapsto \boldsymbol{V}$ restricts to $\dot{Z} \to \dot{X}$ and to $\dot{Z}_\varepsilon \to \dot{X}_\varepsilon$, and these pairs inherit equivariant actions of $H_{\check{V}}$. ($H_{\check{V}}$ is intransitive even on $\dot{X}_\varepsilon$.)

We want to compare $\dot{Z}_\varepsilon$, $\dot{Z}_M$ with $\check{Z}_\varepsilon$, $\check{Z}_M$. To do this, we further fix a complementary maximal isotropic subspace $\check{V}^\dagger$. Let us say that $\boldsymbol{V} = (V_k)_{k=0}^{2n} \in \dot{X}_\varepsilon$ is *split* along $(\check{V}, \check{V}^\dagger)$ if $V_k = (V_k \cap \check{V}) \oplus (V_k \cap \check{V}^\dagger)$ holds for every k, and put

$$\dot{X}_\varepsilon^0 = \{\boldsymbol{V} \in \dot{X}_\varepsilon \mid \boldsymbol{V} \text{ is split along } (\check{V}, \check{V}^\dagger)\}$$

and

$$\dot{Z}_\varepsilon^0 = \dot{Z}_\varepsilon \cap \pi_X^{-1}\big(\dot{X}_\varepsilon^0\big) \supset \dot{Z}_M^0 = \dot{Z}_M \cap \pi_X^{-1}\big(\dot{X}_\varepsilon^0\big)$$

Now we define a subgroup

$$H_1 = \{h \in H_{\check{V}} \mid h \text{ induces the identity maps on } \check{V} \text{ and } V/\check{V}\}$$

(see (4) below). Due to Lemma 5.4 and Lemma 5.6 below, $\dot{X}_\varepsilon^0$ meets every H_1-orbit on $\dot{X}_\varepsilon$ (exactly once). Hence the maps $H_1 \times \dot{Z}_\varepsilon^0 \to \dot{Z}_\varepsilon$ and $H_1 \times \dot{Z}_M^0 \to \dot{Z}_M$ given by the action of H_1 on $\dot{Z}_\varepsilon$ (preserving $\dot{Z}_M$) are surjective.

We define intermediate varieties

$$\hat{Z}_\varepsilon^0 = \big\{(\check{N}, \boldsymbol{V}) \in \check{\mathcal{N}} \times \dot{X}_\varepsilon^0 \mid \boldsymbol{R}(\boldsymbol{V}) \text{ is } \check{N}\text{-stable}\big\}$$

and

$$\hat{Z}_M^0 = \big\{(\check{N}, \boldsymbol{V}) \in \hat{Z}_\varepsilon^0 \mid \mathrm{type}_{\check{N}}(\boldsymbol{R}(\boldsymbol{V})) = M\big\},$$

so that the projections $\dot{Z}^0_\varepsilon \to \dot{X}^0_\varepsilon$ and $\dot{Z}^0_M \to \dot{X}^0_\varepsilon$ factor through $\hat{Z}^0_\varepsilon$ and $\hat{Z}^0_M$ respectively. Lemma 5.4 below implies $\dot{X}^0_\varepsilon \cong \check{X}_\varepsilon$ via $\boldsymbol{V} \mapsto \boldsymbol{R}(\boldsymbol{V})$, and hence we have $\hat{Z}^0_\varepsilon \cong \check{Z}_\varepsilon$ and $\hat{Z}^0_M \cong \check{Z}_M$.

Moreover, by Lemma 5.7, there exists a vector bundle $\mathcal{Q}$ over $\dot{X}^0_\varepsilon$ such that $\dot{Z}^0_\varepsilon \cong \mathcal{Q} \underset{\dot{X}^0_\varepsilon}{\times} \hat{Z}^0_\varepsilon$. Thus $\dot{Z}^0_\varepsilon$ is isomorphic to a vector bundle over $\hat{Z}^0_\varepsilon$, and by restriction, $\dot{Z}^0_M$ is also isomorphic to a vector bundle over $\hat{Z}^0_M$. Since $\hat{Z}^0_M \cong \check{Z}_M$ is irreducible by Proposition 4.5, $\dot{Z}^0_M$ is irreducible, and hence $Z_M = G \cdot \dot{Z}_M = G \cdot (H_1 \cdot \dot{Z}^0_M) = G \cdot \dot{Z}^0_M$ is also irreducible.

With some more work, we will see in Corollary 5.9 that $\dot{Z}_\varepsilon$ is isomorphic to an algebraic vector bundle over $\check{Z}_\varepsilon$. By restriction, $\dot{Z}_M$ is isomorphic to an algebraic vector bundle over $\check{Z}_M$. Since $\check{Z}_M$ is nonsingular by Proposition 4.5, $\dot{Z}_M$ is also nonsingular. Hence Z_M, which is an algebraic fiber bundle over $G/H_{\check{V}}$ with $\dot{Z}_M$ as fiber, is also nonsingular.

As for the dimension, we have $\dim Z_M = \dim \dot{Z}_M + \dim G/H_{\check{V}} = \dim \dot{Z}_M + 4n^2 - (n^2 + \binom{n+1}{2})$. Corollary 5.9 gives $\dim \dot{Z}_M = \dim \check{Z}_M + \binom{n+1}{2} = \dim \check{Z} + \binom{n+1}{2} = n^2 - n + \binom{n+1}{2}$, so that $\dim Z_M = 4n^2 - n$. □

We argue some more details to state and prove the Lemmas quoted in the proof of Proposition 5.1. We continue to use various notations introduced there. In particular, we continue to fix ω_0, $\check{V}$ and $\check{V}^\dagger$. Also, let (p_k, q_k) be the coordinates of the kth point of W_ε as in Section 4.

We start by giving an alternate description for the pair $(\boldsymbol{K}, \boldsymbol{K}')$ corresponding to $\boldsymbol{R}(\boldsymbol{V})$, which is useful when $\Sigma\boldsymbol{R}(\boldsymbol{V}) = \check{V}$ is specified.

Lemma 5.2 *Let $\boldsymbol{V} = (V_k)_{k=0}^{2n} \in \dot{X}_\varepsilon$, $\boldsymbol{W} = \boldsymbol{R}(\boldsymbol{V}) = (W_k)_{k=0}^{2n}$, $\iota_\varepsilon(\boldsymbol{W}) = (\boldsymbol{K}, \boldsymbol{K}')$, $\boldsymbol{K} = (K_i)_{i=0}^{2n}$ and $\boldsymbol{K}' = (K'_j)_{j=0}^{2n}$. Then we have $V_k \cap \check{V} = K_{p_k}$ and $V_k^\perp \cap \check{V} = K'_{q_k}$ (or equivalently $V_k + \check{V} = (K'_{q_k})^\perp$) for any k. In other words, we have $K_i = V_{a_i} \cap \check{V}$ and $K'_{j-1} = (V_{b_j} + \check{V})^\perp$ for any i and j.*

Proof: First, recall from the definition of $\bar{g}$ in the proof of Lemma 4.2 that

$$\sum_{k'=0}^{k} W_{k'} = K_{p_k} \quad \text{and} \quad \sum_{k'=k}^{2n} W_{k'} = K'_{q_k} \quad \text{for all} \quad 0 \leqq k \leqq 2n.$$

Now fix k, and inductively claim for $k \leqq k_1 \leqq 2n$ that $V_k \cap \sum_{k'=0}^{k_1} W_{k'} = K_{p_k}$. The case $k_1 = k$ is what we just recalled. Suppose $k_1 > k$. It is easy by induction if $W_{k_1-1} \supset W_{k_1}$. Otherwise $W_{k_1} = W_{k_1-1} \oplus \mathbb{C}v$ with some $v \notin V_{k_1-1}$ by the proof of Lemma 2.1, so that $V_k \cap \sum_{k'=0}^{k_1} W_{k'} = V_k \cap V_{k_1-1} \cap \sum_{k'=0}^{k_1} W_{k'} = V_k \cap V_{k_1-1} \cap \sum_{k'=0}^{k_1-1} W_{k'} = V_k \cap \sum_{k'=0}^{k_1-1} W_{k'} = K_{p_k}$ by induction. Putting $k_1 = 2n$, we have the first statement. The second statement follows from the first, applied to $\boldsymbol{V}^\perp$, since $\boldsymbol{R}(\boldsymbol{V}^\perp)$ is $\boldsymbol{W}$ read backwards. □

We can use ω_0 to identify $\check{V}^\dagger$ with $\check{V}^*$; namely put

$$\tau : \check{V}^\dagger \to \check{V}^*, \quad w_0(v, v') = \langle v, \tau(v') \rangle \quad ({}^\forall v \in \check{V},\ {}^\forall v' \in \check{V}^\dagger).$$

Then, for any subspace $K \subset \check{V}$, the orthogonal complement of K in $\check{V}^*$ corresponds to $K^\perp \cap \check{V}^\dagger$ by τ^{-1}. (This $K^\perp$ is taken with respect to ω_0 inside V, and note that $K \subset \check{V}$ implies $K^\perp \supset \check{V}^\perp = \check{V}$, so that $K^\perp = \check{V} \oplus (K^\perp \cap \check{V}^\dagger)$).

Lemma 5.3 *Let $\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{K}, \boldsymbol{K}'$ be as in Lemma 5.2. Then $\boldsymbol{V}$ is split along $(\check{V}, \check{V}^\dagger)$ if and only if we have $V_k = K_{p_k} \oplus (K'^{\perp}_{q_k} \cap \check{V}^\dagger)$ for all k.*

Proof: We have $K'^{\perp}_{q_k} \cap \check{V}^\dagger = (V_k + \check{V}) \cap \check{V}^\dagger$ by Lemma 5.2. If $\boldsymbol{V}$ is split, this equals $V_k \cap \check{V}^\dagger$, so that $V_k = (V_k \cap \check{V}) \oplus (V_k \cap \check{V}^\dagger) = K_{p_k} \oplus (K'^{\perp}_{q_k} \cap \check{V}^\dagger)$. Conversely if the equality in Lemma 5.3 holds, then V_k is a sum of a subspace of $\check{V}$ and a subspace of $\check{V}^\dagger$, whence $\boldsymbol{V}$ is split. $\square$

Lemma 5.4 *For $\boldsymbol{W} \in \check{X}_\varepsilon$, let $s(\boldsymbol{W})$ denote the split flag determined by Lemma 5.3 for $(\boldsymbol{K}, \boldsymbol{K}') = \iota_\varepsilon(\boldsymbol{W})$. Then s is a closed embedding of $\check{X}_\varepsilon$ into $\dot{X}_\varepsilon$, whose image equals $\dot{X}^0_\varepsilon$, and is a section of $\dot{\boldsymbol{R}}_\varepsilon = \boldsymbol{R}|_{\dot{X}_\varepsilon} : \dot{X}_\varepsilon \to \check{X}_\varepsilon$.*

Proof: The map $(\boldsymbol{K}, \boldsymbol{K}') \mapsto \boldsymbol{V} = (V_k)_{k=0}^{2n}$, $V_k = K_{p_k} \oplus (K'^{\perp}_{q_k} \cap \check{V}^\dagger)$ is a morphism of algebraic varieties $\check{X} \times \check{X} \to X$. We have $V_k \cap V_k^\perp = (K_{p_k} \cap K'_{q_k}) \oplus ((K'_{q_k} + K_{p_k})^\perp \cap \check{V}^\dagger)$, which reduces to $K_{p_k} \cap K'_{q_k} = W_k$ if $(\boldsymbol{K}, \boldsymbol{K}')$ is ε-transversal. Therefore we have $s(\boldsymbol{W}) \in \dot{X}_\varepsilon$ and $\dot{\boldsymbol{R}}_\varepsilon \circ s(\boldsymbol{W}) = \boldsymbol{W}$, so that s is a section of $\dot{\boldsymbol{R}}_\varepsilon$. Actually $s(\boldsymbol{W}) \in \dot{X}^0_\varepsilon$ since it is split and $s \circ \dot{\boldsymbol{R}}_\varepsilon|_{\dot{X}^0_\varepsilon} = \mathrm{id}_{\dot{X}^0_\varepsilon}$ due to Lemma 5.3, so that im $s = \dot{X}^0_\varepsilon$. Now $\dot{X}^0_\varepsilon$ is closed in $\dot{X}_\varepsilon$, since it is defined by the condition $\dim(V_k \cap \check{V}^\dagger) \geqq n - q_k = k - p_k$ for all k (note that $\dim(V_k \cap \check{V}^\dagger)$ cannot exceed $k - p_k$ for $\boldsymbol{V} \in \dot{X}_\varepsilon$, since $\dim(V_k \cap \check{V}) = p_k$ and $\check{V} \cap \check{V}^\dagger = \{0\}$). Hence s is a closed embedding. $\square$

We use explicit matrix representation to further analyze the situation. For any basis $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ of $\check{V}$, define a basis $\boldsymbol{v}^\dagger$ of $\check{V}^\dagger$ by

$$\boldsymbol{v}^\dagger = \left(v_n^\dagger, \ldots, v_2^\dagger, v_1^\dagger\right), \quad v_i^\dagger = \tau^{-1}(v_i^*) \quad (1 \leqq \forall i \leqq n),$$

where $\boldsymbol{v}^* = (v_1^*, v_2^*, \ldots, v_n^*)$ is the dual basis of $\boldsymbol{v}$. Fix a basis $\check{\boldsymbol{e}}$ of $\check{V}$, and let $\check{\boldsymbol{e}}^\dagger$ be as above. We will employ matrix representation with respect to the basis

$$\boldsymbol{e} = \left(\check{\boldsymbol{e}}_1, \check{\boldsymbol{e}}_2, \ldots, \check{\boldsymbol{e}}_{\boldsymbol{n}}, \check{\boldsymbol{e}}_{\boldsymbol{n}}^\dagger, \ldots \check{\boldsymbol{e}}_2^\dagger, \check{\boldsymbol{e}}_1^\dagger\right).$$

This puts us in the same situation as the previous sections, where we defined $\boldsymbol{e}$ to be the standard basis and the form ω_0 by an explicit matrix J.

Let us express $H_{\check{V}}$ and H_1 explicitly. If A is an $n \times n$ matrix, let ${}^{t'}\!A$ denote the "transpose" of A with respect to the reverse diagonal, namely ${}^{t'}\!A = J_1\, {}^t\!A J_1$ (J_1 is the matrix that appeared before the statement of Proposition 2.2 quoted in Section 1.3). Put

$$\mathfrak{s}_n = \{S \in M_n(\mathbb{C}) \mid {}^{t'}S = S\}.$$

Then $\mathfrak{s}_n$ can be regarded as the matrix representation of

$$\{\phi \in \mathrm{Hom}_{\mathbb{C}}(\check{V}^{\dagger}, \check{V}) \mid {}^t\phi = \phi\}$$

with respect to the basis $\check{\boldsymbol{e}}^{\dagger}$ and $\check{\boldsymbol{e}}$, where ${}^t\phi \in \mathrm{Hom}_{\mathbb{C}}(\check{V}^*, (\check{V}^{\dagger})^*)$ is identified with an element of $\mathrm{Hom}_{\mathbb{C}}(\check{V}^{\dagger}, \check{V})$ by using τ twice. It can also be identified with $S^2(\check{V})$, the space of symmetric tensors over $\check{V}$ of rank 2. Then $H_{\check{V}}$ and H_1 have the forms

$$H_{\check{V}} = \left\{ \begin{pmatrix} A & S \\ O & {}^{t'}A^{-1} \end{pmatrix} \middle| A \in GL(n, \mathbb{C}),\ S \in \mathfrak{s}_n \right\} \text{ and} \tag{3}$$

$$H_1 = \left\{ \begin{pmatrix} E & S \\ O & E \end{pmatrix} \middle| S \in \mathfrak{s}_n \right\} = \{\exp \tilde{S} \mid S \in \mathfrak{s}_n\}, \tag{4}$$

where $\tilde{S} = \left(\begin{smallmatrix} O & S \\ O & O \end{smallmatrix}\right)$, which satisfies $\tilde{S}^2 = O$ and $\exp \tilde{S} = E + \tilde{S}$. Hence $S \mapsto \exp \tilde{S}$ gives an isomorphism of H_1 with a vector group, and in particular H_1 is connected.

Let a_i, $1 \leqq i \leqq n$, and b_j, $1 \leqq j \leqq n$, be defined with respect to ε as in Section 3.

Lemma 5.5 *Let $\boldsymbol{V} \in X$ and $\boldsymbol{W} \in \check{X}_{\varepsilon}$, $\iota_{\varepsilon}(\boldsymbol{W}) = (\boldsymbol{K}, \boldsymbol{K}')$. Fix bases $\boldsymbol{v} = (v_i)_{i=1}^n$ and $\boldsymbol{w} = (w_j)_{j=1}^n$ of $\boldsymbol{K}$ and $\boldsymbol{K}'$ respectively. Let A, B be the matrices representing $\boldsymbol{v}$ and $\boldsymbol{w}$ in terms of $\check{\boldsymbol{e}}$ respectively. (It follows that the matrix representation of the basis $\boldsymbol{w}^{\dagger}$ of $\check{V}^{\dagger}$ in terms of $\check{\boldsymbol{e}}^{\dagger}$ is ${}^{t'}B^{-1}$, and that $K_j'^{\perp} \cap \check{V}^{\dagger}$ is spanned by $w_n^{\dagger}, w_{n-1}^{\dagger}, \ldots, w_{j+1}^{\dagger}$). Then we have $\boldsymbol{V} \in \dot{\boldsymbol{R}}_{\varepsilon}^{-1}(\boldsymbol{W})$ (where $\dot{\boldsymbol{R}}_{\varepsilon} = \boldsymbol{R}|_{\dot{X}_{\varepsilon}}$) if and only if $\boldsymbol{V}$ has a basis $\boldsymbol{u} = (u_k)_{k=1}^{2n}$ whose matrix representation in terms of $\boldsymbol{e}$ has the form*

$$\begin{pmatrix} A & O \\ O & {}^{t'}B^{-1} \end{pmatrix} \begin{pmatrix} E & L \\ O & E \end{pmatrix} \dot{w}_{\varepsilon},$$

$$w_{\varepsilon} = \begin{pmatrix} 1 & 2 & \cdots & n & n+1 & \cdots & 2n-1 & 2n \\ a_1 & a_2 & \cdots & a_n & b_n & \cdots & b_2 & b_1 \end{pmatrix}^{-1} \in \mathfrak{S}_{2n}, \tag{5}$$

where the a_i and b_j correspond to ε as in Section 3, and $\dot{w}_{\varepsilon}$ denotes the permutation matrix representing w_{ε}. Two such bases span the same flag if and only if the entries of L in the positions

$$\mathcal{L}_{\nu} = \{(i, j) \mid 1 \leqq i, j \leqq n,\ j \leqq \nu_{n+1-i}\}$$

are the same.

Proof: The multiplication by $\dot{w}_{\varepsilon}$ from the right amounts to changing the order of the columns according to the permutation w_{ε}^{-1}. Therefore (5) is equivalent to saying that:

for each k, the subspace V_k is spanned by $v_1, v_2, \ldots, v_{p_k}$ and

$$w_n^{\dagger} + l_n', w_{n-1}^{\dagger} + l_{n-1}', \ldots, w_{q_k+1}^{\dagger} + l_{q_k+1}', \text{ where } l_n', l_{n-1}', \ldots, l_{q_k+1}' \in \check{V}. \tag{6}$$

We first assume $\boldsymbol{V} \in \dot{\boldsymbol{R}}_\varepsilon^{-1}(\boldsymbol{W})$ and prove (6) inductively on k, starting with the case $k=0$, which is trivial. If $k>0$ and $k=a_i$ for some i, then we have $i=p_k$, and $K_{p_k-1}=V_{k-1}\cap\check{V} \subsetneqq V_k\cap\check{V}=K_{p_k}$. Since $v_{p_k}\in K_{p_k}-K_{p_k-1}$, the subspace V_k is spanned by V_{k-1} and v_{p_k}. Due to the induction hypothesis, (6) also holds for this k. On the other hand if $k=b_j$ for some j, then we have $j=q_k+1$, and $(K'_{q_k+1})^\perp = V_{k-1}+\check{V} \subsetneqq V_k+\check{V}=(K'_{q_k})^\perp$. Since $w^\dagger_{q_k+1}\in(K'_{q_k})^\perp-(K'_{q_k+1})^\perp$, we have $V_k+\check{V}=V_{k-1}+\check{V}+\mathbb{C}w^\dagger_{q_k+1}$, namely V_k is spanned by V_{k-1} and $w^\dagger_{q_k+1}$ modulo $\check{V}$. Therefore there exists $u_k\in w^\dagger_{q_k+1}+\check{V}$ such that V_k is spanned by V_{k-1} and u_k. Due to the induction hypothesis, (6) also holds for this k.

Conversely assume (6). Then it follows that $V_k\cap\check{V}=\sum_{i=1}^{p_k}\mathbb{C}v_i=K_{p_k}$ and $V_k+\check{V}=\sum_{j=q_k+1}^{n}\mathbb{C}w_j^\dagger+\check{V}=(K'_{q_k})^\perp$ by definition. Then $V_k^\perp\cap V_k\subset(V_k\cap\check{V})^\perp\cap(V_k+\check{V})=K_{p_k}^\perp\cap(K'_{q_k})^\perp=(K_{p_k}+K'_{q_k})^\perp$, which equals $\check{V}^\perp=\check{V}$ by the ε-transversality of $(\boldsymbol{K},\boldsymbol{K}')$. Therefore $V_k\cap V_k^\perp=V_k\cap V_k^\perp\cap\check{V}=(V_k\cap\check{V})\cap(V_k^\perp\cap\check{V}^\perp)=(V_k\cap\check{V})\cap(V_k+\check{V})^\perp=K_{p_k}\cap K'_{q_k}=W_k$. Since this holds for all k, we have $\boldsymbol{V}\in\dot{\boldsymbol{R}}_\varepsilon^{-1}(\boldsymbol{W})$.

Finally let L and L' be two n by n matrices. Then (5) spans the same flag for L and L' if and only if there exists a matrix $b\in B$ (see Section 2) such that $\left(\begin{smallmatrix}E & L\\ O & E\end{smallmatrix}\right)\dot{w}_\varepsilon b=\left(\begin{smallmatrix}E & L'\\ O & E\end{smallmatrix}\right)\dot{w}_\varepsilon$, namely $\dot{w}_\varepsilon b\dot{w}_\varepsilon^{-1}=\left(\begin{smallmatrix}E & L'-L\\ O & E\end{smallmatrix}\right)$. This means that the (i,j) entry of $L'-L$ is zero unless a_i comes earlier than b_{n+1-j}. This condition is equivalent to $j>\nu_{n+1-i}$ (see figure 5), hence follows the final claim. □

Remark This endows $\dot{X}_\varepsilon$ with a vector bundle structure over $\check{X}_\varepsilon$, which depends on the choice of $\check{V}^\dagger$, and of which $\dot{X}_\varepsilon^0$ is the zero section. In fact, let $\check{\mathcal{S}}$ denote the set of all bases of $\check{V}$, and let $\check{p}:\check{\mathcal{S}}\times\check{\mathcal{S}}\to\check{X}\times\check{X}$ denote the map $(\boldsymbol{v},\boldsymbol{w})\mapsto(\mathrm{Fl}(\boldsymbol{v}),\ \mathrm{Fl}(\boldsymbol{w}))$. Then $\check{X}\times\check{X}$ can be covered by open sets U on each of which $\check{p}$ admits a regular section $\xi\mapsto(\boldsymbol{v}(\xi),\boldsymbol{w}(\xi))$. If $\mathcal{H}=M_n(\mathbb{C})$ and $\mathcal{H}_{\bar\nu}=\{L=(l_{ij})\in\mathcal{H}\mid l_{ij}=0\text{ for }(i,j)\in\mathcal{L}_\nu\}$, then this argument gives an isomporphism (as varieties) $\dot{\boldsymbol{R}}_\varepsilon^{-1}(U)\cong U\times(\mathcal{H}/\mathcal{H}_{\bar\nu})$ commuting with projections onto U. If U and U' are two such open sets, with regular sections $(\boldsymbol{v},\boldsymbol{w})$ and $(\boldsymbol{v}',\boldsymbol{w}')$, the transition function $U\cap U'\to GL(\mathcal{H}/\mathcal{H}_{\bar\nu})$ is given by $\xi\mapsto(L \bmod \mathcal{H}_{\bar\nu}\mapsto\sigma(\xi)L\,{}^{t'}\tau(\xi) \bmod \mathcal{H}_{\bar\nu})$, where $\boldsymbol{v}(\xi)=\boldsymbol{v}'(\xi)\sigma(\xi)$ and $\boldsymbol{w}(\xi)=\boldsymbol{w}'(\xi)\tau(\xi)$ on $U\cap U'$ (note that $\check{B}\mathcal{H}_{\bar\nu}{}^{t'}\check{B}\subset\mathcal{H}_{\bar\nu}$, where $\check{B}$ is the group of the invertible $n\times n$ upper triangular matrices).

Lemma 5.6 *The action of H_1 on $\dot{X}_\varepsilon$ respects each fiber of $\dot{\boldsymbol{R}}_\varepsilon=\boldsymbol{R}|_{\dot{X}_\varepsilon}:\dot{X}_\varepsilon\to\check{X}_\varepsilon$, $\boldsymbol{V}\mapsto\boldsymbol{R}(\boldsymbol{V})$, and is transitive on each fiber.*

Proof: The group $H_{\check{V}}$ acts on $\dot{X}_\varepsilon$ and $\check{X}_\varepsilon$, and $\dot{\boldsymbol{R}}_\varepsilon$ is $H_{\check{V}}$-equivariant. Since the subgroup H_1 acts trivially on $\check{X}_\varepsilon$, it preserves each fiber of $\dot{\boldsymbol{R}}_\varepsilon$. Fix $\boldsymbol{W}\in\check{X}_\varepsilon$, and let $\boldsymbol{v}$, $\boldsymbol{w}$, A, B be as in Lemma 5.5. Then $s(\boldsymbol{W})\in\dot{X}_\varepsilon^0\cap\dot{\boldsymbol{R}}_\varepsilon^{-1}(\boldsymbol{W})$ (see Lemma 5.4) can be represented by the basis $\left(\begin{smallmatrix}A & O\\ O & {}^{t'}B^{-1}\end{smallmatrix}\right)\dot{w}_\varepsilon$. An element of H_1 carries this flag to the one corresponding to

$$\begin{pmatrix}E & S\\ O & E\end{pmatrix}\begin{pmatrix}A & O\\ O & {}^{t'}B^{-1}\end{pmatrix}\dot{w}_\varepsilon=\begin{pmatrix}A & O\\ O & {}^{t'}B^{-1}\end{pmatrix}\begin{pmatrix}E & A^{-1}S\,{}^{t'}B^{-1}\\ O & E\end{pmatrix}\dot{w}_\varepsilon.$$

Now $\boldsymbol{v}$ and $\boldsymbol{w}$ satisfy the assumption for Lemma 4.4, so the entries of $A^{-1}S\,{}^{t'}B^{-1}$ in $\mathcal{L}_\nu$ provide all elements of $\mathbb{C}^{|\nu|}$ as S varies, since the (i,j)-entry of $A^{-1}S\,{}^{t'}B^{-1}$ is the value of $v_i^*\otimes w_{n+1-j}^*$ at S. Hence H_1 acts transitively on each fiber of $\dot{\boldsymbol{R}}_\varepsilon$. □

Remark This shows that the vector bundle $\dot{X}_\varepsilon$ over $\check{X}_\varepsilon$ is a quotient of the trivial vector bundle $\check{X}_\varepsilon \times \mathfrak{s}_n$, via the "action" map $(\boldsymbol{W}, S) \mapsto (\exp \tilde{S}) \cdot s(\boldsymbol{W})$.

Lemma 5.7 *There exists a vector bundle $\mathcal{Q}$ over $\dot{X}^0_\varepsilon \cong \check{X}_\varepsilon$ of rank $\binom{n+1}{2} - |\nu|$ such that $\dot{Z}^0_\varepsilon \cong \mathcal{Q} \times_{\dot{X}^0_\varepsilon} \hat{Z}^0_\varepsilon$.*

Proof: For $S \in \mathfrak{s}_n$, let $\phi_S \in \operatorname{Hom}_{\mathbb{C}}(\check{V}^\dagger, \check{V})$ denote the map represented by S in the bases $\check{\boldsymbol{e}}^\dagger$ of $\check{V}^\dagger$ and $\check{\boldsymbol{e}}$ of $\check{V}$. Let $(\check{N}, \boldsymbol{V}) \in \hat{Z}^0_\varepsilon$ (so that $\boldsymbol{V}$ splits along $(\check{V}, \check{V}^\dagger)$), and let $(N, \boldsymbol{V}) \in \dot{Z}_\varepsilon$ be such that $N|_{\check{V}} = \check{N}$. We have $G_{\omega_0} \cap G_{\boldsymbol{V}} \subset G_{\omega_0} \cap G_{\check{V}} = H_{\check{V}}$, and

$$\operatorname{Lie} H_{\check{V}} = \left\{ \begin{pmatrix} A & S \\ O & -{}^{t'}A \end{pmatrix} \middle| A \in \mathfrak{gl}(n, \mathbb{C}),\ S \in \mathfrak{s}_n \right\}$$

by (3). Since $N \in \operatorname{Lie}(G_{\omega_0} \cap G_{\boldsymbol{V}})$ (see the paragraph after the proof of Proposition 2.2), we can write N in this form. Then $N|_{\check{V}} = \check{N}$ if and only if A represents $\check{N}$ in the basis $\check{\boldsymbol{e}}$. Note that N is automatically nilpotent if N has this form and $N|_{\check{V}}$ is nilpotent. Since $\boldsymbol{V}$ splits and we have $V_k \cap \check{V} = K_{p_k}$ and $V_k \cap \check{V}^\dagger = {K'_{q_k}}^\perp \cap \check{V}^\dagger$ for all k (where $(\boldsymbol{K}, \boldsymbol{K}') = \iota_\varepsilon(\boldsymbol{R}(\boldsymbol{V}))$), the condition $N \in \operatorname{Lie} G_{\boldsymbol{V}}$ breaks up into $\check{N}K_i \subset K_i$, $\check{N}K'_j \subset K'_j$ for all i and j (already fulfilled due to the definition of $\hat{Z}^0_\varepsilon$) and

$$\phi_S({K'_{q_k}}^\perp \cap \check{V}^\dagger) \subset K_{p_k}, \quad 0 \leqq \forall k \leqq 2n.$$

Since this condition for ϕ_S only depends on $\boldsymbol{V}$, we can define $\mathcal{Q} \subset \dot{X}^0_\varepsilon \times \mathfrak{s}_n$ by this condition and have $\dot{Z}^0_\varepsilon \cong \mathcal{Q} \times_{\dot{X}^0_\varepsilon} \hat{Z}^0_\varepsilon$.

It remains to show that $\mathcal{Q}$ is a vector bundle over $\dot{X}^0_\varepsilon \cong \check{X}_\varepsilon$. For $\boldsymbol{W} \in \check{X}_\varepsilon$, let $Q(\boldsymbol{W})$ denote the set of S satisfying the above condition for $(\boldsymbol{K}, \boldsymbol{K}') = \iota_\varepsilon(\boldsymbol{W})$, or equivalently

$$\begin{aligned} Q(\boldsymbol{W}) &= \left\{ S \in \mathfrak{s}_n \mid \phi_S\left({K'_{n-j}}^\perp \cap \check{V}^\dagger\right) \subset K_{n-\nu'_j} (1 \leqq \forall j \leqq l(\nu')) \right\} \\ &= \left\{ S \in \mathfrak{s}_n \mid \left\langle \phi_s\left(w^\dagger_{n+1-j}\right), v^*_{n+1-i} \right\rangle = 0 (\forall (j, i) \in \nu') \right\}, \end{aligned}$$

where $\boldsymbol{v}$, $\boldsymbol{w}$ are arbitary bases of $\boldsymbol{K}, \boldsymbol{K}'$ respectively, since ${K'_{n-j}}^\perp \cap \check{V}^\dagger$ is spanned by $w^\dagger_n$, $w^\dagger_{n-1}, \ldots, w^\dagger_{n-j+1}$. Shifting to the symmetric tensors over $\check{V}$, we have

$$\cong \{ S \in S^2(\check{V}) \mid \langle S, v^*_{n+1-i} \otimes w^*_{n+1-j} \rangle = 0 \, (\forall (i, j) \in \nu) \}.$$

Let $\check{X}_\varepsilon \times S^2(\check{V}) \to \check{X}_\varepsilon$ be the trivial vector bundle with fiber $S^2(\check{V})$. By Lemma 4.4, the forms $v^*_{n+1-i} \otimes w^*_{n+1-j}$, $(i, j) \in \nu$, remain linearly independent when restricted to $S^2(\check{V})$ so long as $(\operatorname{Fl}(\boldsymbol{v}), \operatorname{Fl}(\boldsymbol{w}))$ is ε-transversal. If $U \subset \check{X}_\varepsilon$ is an open subset admitting a regular section $\xi \mapsto (\boldsymbol{v}(\xi), \boldsymbol{w}(\xi))$ of $\check{p}$ over $\iota_\varepsilon(U)$ (see remark after Lemma 5.6), this section defines $|\nu|$ linear independent regular sections of the dual bundle $U \times (S^2(\check{V}))^* \to U$, and $Q(\boldsymbol{W})$ is the space of solutions of their values at $\boldsymbol{W}$. By a general argument in Lemma 5.8

below, $\bigcup_{\boldsymbol{W}\in U}\{\boldsymbol{W}\}\times Q(\boldsymbol{W})\subset U\times S^2(\check{V})$ is a subbundle of rank $\binom{n+1}{2}-|\nu|$, and hence so is $\bigcup_{\boldsymbol{W}\in \check{X}_\varepsilon}\{\boldsymbol{W}\}\times Q(\boldsymbol{W})\subset \check{X}_\varepsilon\times S^2(\check{V})$. □

Although the following general claim, used in the proof of Lemma 5.7, is elementary, let us include a proof for convenience.

Lemma 5.8 *Let U be a variety over $\mathbb{C}$, W a finite-dimensional vector space over $\mathbb{C}$ and d its dimension, and let $U\times W\to U$ be the trivial vector bundle over U with fiber W. Suppose $\alpha_1,\alpha_2,\dots,\alpha_e\in\mathcal{O}(U)\otimes W^*$ are regular sections of the dual bundle which are everywhere linearly independent, and put $Q(u)=\bigcap_{p=1}^{e}\ker\alpha_p(u)\subset W$ for all $u\in U$. Then $\bigcup_{u\in U}\{u\}\times Q(u)\subset U\times W$ is a subbundle of rank $d-e$.*

Proof: Fix a basis $(s_q)_{q=1}^{d}$ of W, and let $A=(a_{pq})\in M_{e,d}(\mathcal{O}(U))$ be defined by $\alpha_p=\sum_q a_{pq}s_q^*$ for all p, where (s_q^*) is the dual basis of (s_q). Then $Q(u)=\{b\mid A(u)b=0\}$, where b is the column vector representing an element of W in terms of (s_q). For each $\boldsymbol{q}\in\binom{\{1,2,\dots,d\}}{e}$, let $U_{\boldsymbol{q}}\subset U$ be the set of points where the columns of A indexed by the elements of $\boldsymbol{q}$ are linearly independent. Then $U=\bigcup_{\boldsymbol{q}}U_{\boldsymbol{q}}$ is an open covering. If $\check{\boldsymbol{q}}$ denotes the complement of $\boldsymbol{q}$, then inside $U_{\boldsymbol{q}}$ there are $|\check{\boldsymbol{q}}|=d-e$ linearly independent solutions b_p, one for each $p\in\check{\boldsymbol{q}}$, of the form $b_p=s_p+\sum_{q\in\boldsymbol{q}}b_{pq}s_q$. The coefficients b_{pq}, $q\in\boldsymbol{q}$, are polynomials in the entries of A and the inverse of the full minor of A consisting of the columns indexed by $\boldsymbol{q}$, hence belong to $\mathcal{O}(U_{\boldsymbol{q}})$. Namely they give $d-e$ regular sections of W over $U_{\boldsymbol{q}}$. Hence $\bigcup_{u\in U}\{u\}\times Q(u)\subset U\times W$ is a subbundle of rank $d-e$. □

Let us remark here a little more on the structure of $\dot{Z}_\varepsilon$.

Corollary 5.9 *$\dot{Z}_\varepsilon$ is isomorphic to an algebraic vector bundle over $\check{Z}_\varepsilon$ of rank $\binom{n+1}{2}$.*

Proof: By an argument similar to Lemma 5.8, remark after Lemma 5.6 implies that $\check{X}_\varepsilon$ can be covered by open sets U for which (1) one can choose a $|\nu|$-dimensional subspace $P_U\subset\mathfrak{s}_n$ such that, for every $\boldsymbol{W}\in U$, the map $P_U\ni S\mapsto(\exp\tilde{S})\cdot s(\boldsymbol{W})$ gives a linear isomorphism $P_U\cong\dot{\boldsymbol{R}}_\varepsilon^{-1}(\boldsymbol{W})$. Note that the collection of these isomorphisms constitutes a trivialization of the vector bundle $\dot{\boldsymbol{R}}_\varepsilon:\dot{X}_\varepsilon\to\check{X}_\varepsilon$ over U, which we denote by $\alpha:\dot{\boldsymbol{R}}_\varepsilon^{-1}(U)\cong U\times P_U$. Combining with Lemma 5.7, one can impose one more condition on U. For simplicity, we put $\tilde{U}=\{(\check{N},\boldsymbol{W})\in\check{Z}_\varepsilon\mid\boldsymbol{W}\in U\}$ for each open set $U\subset\check{X}_\varepsilon$. We require that (2) the vector bundle $\dot{Z}_\varepsilon^0\to\hat{Z}_\varepsilon^0\cong\check{Z}_\varepsilon$ is trivial over $\tilde{U}$. Let us denote this projection by π for now, and let $\beta:\pi^{-1}(\tilde{U})\cong\tilde{U}\times Q_U$ be a trivialization, with some vector space Q_U.

Let p denote the projection $\dot{Z}_\varepsilon\ni(N,\boldsymbol{V})\mapsto(N|_{\check{V}},\boldsymbol{R}(\boldsymbol{V}))\in\check{Z}_\varepsilon$. Then we have an isomorphism $\gamma:p^{-1}(\tilde{U})\ni(N,\boldsymbol{V})\mapsto((N|_{\check{V}},\boldsymbol{R}(\boldsymbol{V})),(S,q))\in\tilde{U}\times(P_U\oplus Q_U)$, where S is the P_U-component of $\alpha(\boldsymbol{V})$ and q is the Q_U-component of $\beta((\exp\tilde{S})^{-1}\cdot(N,\boldsymbol{V}))$. Note that $(\exp\tilde{S})^{-1}\cdot(N,\boldsymbol{V})\in\dot{Z}_\varepsilon^0$. Suppose U' is another such open set, and let α',β',γ' be the counterparts of α,β,γ for U', respectively. Remark after Lemma 5.6 [resp. Lemma 5.7] shows that the transition map from α to α' [resp. β to β'] is a linear isomorphism (which we denote by A [resp. B]) with coefficients in $\mathcal{O}(\tilde{U}\cap\tilde{U}')$. We claim that the transition map from γ to γ' is also a linear isomorphism, $P_U \oplus Q_U \to P_{U'} \oplus Q_{U'}$, whose coefficients are in $\mathcal{O}(\tilde{U} \cap \tilde{U}')$. This will conclude our argument.

Let $(\check{N}, \boldsymbol{W}) \in \tilde{U} \cap \tilde{U}' \subset \check{Z}_\varepsilon$ and $(N, \boldsymbol{V}) \in p^{-1}((\check{N}, \boldsymbol{W})) \subset p^{-1}(\tilde{U} \cap \tilde{U}') \subset \dot{Z}_\varepsilon$. Put $\gamma(N, \boldsymbol{V}) = ((\check{N}, \boldsymbol{W}), (S, q))$ and $\gamma'(N, \boldsymbol{V}) = ((\check{N}, \boldsymbol{W}), (S', q'))$. By definition, we have $S' = A(S)$. Also we have

$$\beta((\exp \tilde{S})^{-1} \cdot (N, \boldsymbol{V})) = \beta(\mathrm{Ad}(\exp \tilde{S})^{-1}(N), s(\boldsymbol{W})) = ((\check{N}, s(\boldsymbol{W})), q)$$

and

$$\beta'((\exp \tilde{S}')^{-1} \cdot (N, \boldsymbol{V})) = \beta'(\mathrm{Ad}(\exp \tilde{S}')^{-1}(N), s(\boldsymbol{W})) = ((\check{N}, s(\boldsymbol{W})), q')$$

We want to clarify the relation between q and q'. As an intermediary, we put $\beta((\exp \tilde{S}')^{-1} \cdot (N, \boldsymbol{V})) = ((\check{N}, s(\boldsymbol{W})), q'')$. Then we have $q' = B(q'')$. Writing $\mathrm{Ad}(\exp \tilde{S})^{-1}(N) = \left(\begin{smallmatrix} \check{N} & P \\ O & -{}^t\check{N} \end{smallmatrix}\right)$ in the basis $\boldsymbol{e}$, we have

$$\begin{aligned} \mathrm{Ad}(\exp \tilde{S}')^{-1}(N) &= \mathrm{Ad}(E + \widetilde{(S - A(S))}) \begin{pmatrix} \check{N} & P \\ O & -{}^t\check{N} \end{pmatrix} \\ &= \begin{pmatrix} \check{N} & P + (A(S) - S){}^t\check{N} + \check{N}(A(S) - S) \\ O & -{}^t\check{N} \end{pmatrix}. \end{aligned}$$

If we define $\tilde{A}_{\check{N}} : P_U \to \mathfrak{s}_n$ by $S \mapsto (A(S) - S)\,{}^t\check{N} + \check{N}(A(S) - S)$, then this is a linear map whose coefficients are in $\mathcal{O}(\tilde{U} \cap \tilde{U}')$. We have $q'' = q + \beta_{\boldsymbol{W}} \circ \tilde{A}_{\check{N}}(S)$, where $\beta_{\boldsymbol{W}} : Q(\boldsymbol{W}) \to Q_U$ is the restriction of β to the fibers of $\boldsymbol{W}$. Note that the composition $\beta_{\boldsymbol{W}} \circ \tilde{A}_{\check{N}}$ is also a linear map with coefficients in $\mathcal{O}(\tilde{U} \cap \tilde{U}')$. We then have $q' = B(q) + B \circ \beta_{\boldsymbol{W}} \circ \tilde{A}_{\check{N}}(S)$, which is a linear map $P_U \oplus Q_U \to Q'_U$ again with coefficients in $\mathcal{O}(\tilde{U} \cap \tilde{U}')$. Therefore the transition map from γ to γ', $P_U \oplus Q_U \ni (S, q) \mapsto (A(S), B(q) + B \circ \beta_{\boldsymbol{W}} \circ \tilde{A}_{\check{N}}(S)) \in P_{U'} \oplus Q_{U'}$, is a linear isomorphism depending regularly on $(\check{N}, \boldsymbol{W}) \in \tilde{U} \cap \tilde{U}'$. □

Corollary 5.10 *The subvarieties $\overline{Z_M}$, $M \in \mathcal{M}_{2n}$, give all irreducible components of Z.*

Proof: This follows from Proposition 3.1 and Proposition 5.1 by the general argument reviewed in the third paragraph of Section 1.2. □

6. Coincidence with the combinatorial correspondence

We have seen that the irreducible components of the variety Z are parametrized by the Brauer diagrams on $2n$ points (Proposition 2.5) as well as the updown tableaux of degree $2n$ (Proposition 5.1 and Corollary 5.10). Thus the relation $\overline{Z_d} = \overline{Z_M}$ for $d \in \mathcal{D}_{2n}$ and $M \in \mathcal{M}_{2n}$ determines a bijection $\mathfrak{M}_{geom} : \mathcal{D}_{2n} \to \mathcal{M}_{2n}$. On the other hand, there is a "combinatorial" bijection between these sets reviewed in Section 1.3, which we denote by $\mathfrak{M}_{comb}$.

Our objective is to show that these two bijections are the same. The following is the essense of its proof.

Proposition 6.1 *Let $d \in \mathcal{D}_{2n}$, and let $M \in \mathcal{M}_{2n}$ be the updown tableau produced from d by the combinatorial correspondence reviewed in Section* 1.3. *If $(\omega, \boldsymbol{V}) \in \mathcal{O}_d$, then the nilpotent elements N in* $\mathrm{Lie}G_{(\omega,\boldsymbol{V})}$ *form a vector space, in which the ones such that $(N, \omega, \boldsymbol{V}) \in Z_M$ form a Zariski open and dense subset.*

Proof: By Proposition 2.2, we may assume that $\boldsymbol{V} = \boldsymbol{V}^{w_d}$ and $\omega = \omega_0$. Then, as in the proof of Lemma 2.3, Rad $(\omega_0 \mid V_i)$ is spanned by the $\boldsymbol{e}_{w_d(k)}$, $k \in \{1, 2, \ldots, i\} \cap i_d(\{i+1, i+2, \ldots, 2n\})$. Let I_i denote the set of k satisfying this condition, namely the set of position labels of the left-end vertices of the edges in d that connect one of the i vertices from the left with one of the $2n-i$ vertices from the right. Note that $w_d(I_i) \subset \{1, 2, \ldots, n\}$ since these are left-end vertices. Since $\Sigma \boldsymbol{R}(\omega_0, \boldsymbol{V}^{w_d})$ is a maximal isotropic subspace, this implies that it coincides with $\sum_{i=1}^{n} \mathbb{C}\, \boldsymbol{e}_i$, which here we denote by $\check{V}$.

By Lemma 2.4, the nilpotent elements in Lie $G_{(\omega_0, \boldsymbol{V}^{w_d})}$ form a Lie subalgebra $\mathfrak{u}_d$, which is a vector space. Let $\check{\mathfrak{u}}_d$ denote the space of $n \times n$ matrices whose nonzero entries are only in positions (p, q) satisfying $1 \leqq p, q \leqq n$, $w_d^{-1}(p) < w_d^{-1}(q)$ (or equivalently $p < q$), and $w_d^{-1}(q') < w_d^{-1}(p')$. Then Lemma 2.4 says that the map $N \mapsto N|_{\check{V}}$, which takes the top-left quadrant, maps $\mathfrak{u}_d$ onto $\check{\mathfrak{u}}_d$. Suppose that the entries of $N|_{\check{V}}$ in the above mentioned positions are algebraically independent over $\mathbb{Q}$. Note that these N form a Zariski dense subset of $\mathfrak{u}_d$. Now for each i put $\check{N}_i = \check{N} \mid_{\mathrm{Rad}(\omega_0|V_i)}$. Then the matrix representation of $\check{N}_i$ is the submatrix of $N|_{\check{V}}$ consisting of the rows and columns indexed by $w_d(I_i)$. Let $\Pi_i(d)$ denote the poset consisting of the elements of $w_d(I_i)$ and in which p and q have the order relation $p \prec q$ if and only if $w_d^{-1}(p) \leqq w_d^{-1}(q)$ (or equivalently $p \leqq q$) and $w_d^{-1}(q') \leqq w_d^{-1}(p')$. Then $\check{N}_i$ is a generic matrix of the poset $\Pi_i(d)$ (see Section 1.2). By a theorem of Gansner and Saks (again see Section 1.2), the Jordan type of $\check{N}_i$ is equal to the Greene-Kleitman invariant of the poset $\Pi_i(d)$.

Now consider the tableau $T^{(i)}$ produced from d in the combinatorial correspondence. The entries of $T^{(i)}$ are the elements of $\Pi_i(d)$, namely the $1 \leqq p \leqq n$ such that the label p appears among the leftmost i vertices and such that the label p' appears among the rightmost $2n-i$ vertices of d. If $p_1 < p_2 < \cdots < p_s$ are the elements of $\Pi_i(d)$ in the increasing order (which is the same as the order of their appearance as labels of d from left to right), and if $w \in \mathfrak{S}_r$ denotes the permutation such that the corresponding primed numbers appear from right to left in the order $(p_{w(1)})', (p_{w(2)})', \ldots, (p_{w(s)})'$, then the poset $\Pi_i(d)$ is isomorphic to $\Pi(w)$ (see the final paragraph of Section 1.2). Since the rules of row insertion are only concerned with the relative magnitudes of the letters involved, we know that $T^{(i)}$ is obtained from $P(w)$ (see the second paragraph of Section 1.1) by replacing each entry a with p_a, so that the shape of $T^{(i)}$ equals the Greene-Kleitman invariant of $\Pi(w) \cong \Pi_i(d)$, which equals the Jordan type of $\check{N}_i$ as we saw in the previous paragraph.

Hence the elements $N \in \mathfrak{u}_d$ such that $(N, \omega_0, \boldsymbol{V}^{w_d}) \in Z_M$ form a Zariski dense subset of $\mathfrak{u}_d$. By Proposition 3.1, this subset is Zariski locally closed. Since a Zariski locally closed subset can be dense only if it is Zariski open, this subset is Zariski open and dense. □

In summary, we have shown the following:

Theorem 6.2 *Let $\mathfrak{M}_{comb} : \mathcal{D}_{2n} \to \mathcal{M}_{2n}$ denote the combinatorial bijection reviewed in Section* 1.3, *and let $\mathfrak{M}_{geom} : \mathcal{D}_{2n} \to \mathcal{M}_{2n}$ denote the bijection through the labeling of* *the irreducible components of Z, namely, for $d \in \mathcal{D}_{2n}$, we put $\mathfrak{M}_{geom}(d) = M$ for the unique $M \in \mathcal{M}_{2n}$ satisfying $\overline{Z_M} = \overline{Z_d}$, whose existence as well as uniqueness is assured by Proposition* 2.5, *Proposition* 5.1 *and Corollary* 5.10. *Then we have $\mathfrak{M}_{comb} = \mathfrak{M}_{geom}$.*

Proof: Let $d \in \mathcal{D}_{2n}$, and let $M = \mathfrak{M}_{comb}(d)$. Then Proposition 6.1 shows that a Zariski dense subset of Z_d is contained in Z_M. This implies $Z_d \subset \overline{Z_M}$, and hence $\overline{Z_d} \subset \overline{Z_M}$. Since both sides are irreducible components of Z, we actually have $\overline{Z_d} = \overline{Z_M}$, whence $M = \mathfrak{M}_{geom}(d)$.

Since both $\mathfrak{M}_{comb}$ and $\mathfrak{M}_{geom}$ are bijections, this is enough to conclude that they coincide. $\square$

Remark If d and M are related as above, then $Z_d \cap Z_M$ is Zariski open and dense in $\overline{Z_d} = \overline{Z_M}$ (and hence in Z_d and in Z_M). In fact, Z_d (resp. Z_M) is Zariski locally closed in Z by Proposition 2.5 (resp. Proposition 3.1), and hence in $\overline{Z_d} = \overline{Z_M}$. Since Z_d (resp. Z_M) is Zariski dense in $\overline{Z_d} = \overline{Z_M}$ by definition, it is Zariski open by the same argument as in the final part of the proof of Proposition 6.1. The intersection of two Zariski open dense subsets of a variety is also Zariski open and dense.

7. Discussions

1. Updown tableaux appear as the "recording tableaux" in Berele's correspondence [2], which gives the character-level decomposition of $(\mathbb{C}^{2n})^{\otimes f}$ under $Sp(2n, \mathbb{C})$. On the other hand, there are "semistandard" versions of updown tableaux (see [10]), and certain semistandard updown tableaux encode the $Sp(2n)$-tableaux (see [35]; a more straightforward encoding is embedded in [19]; a more delicate version in [1]), which also appear in Berele's correspondence. Moreover, there are generalizations of the bijection discussed in this paper for semistandard updown tableaux (see [10], [22] and [18]). Is there a geometric explanation of Berele's correspondence?

2. Are there geometric interpretations of other Robinson-Schensted-type correspondences? For example, can one find an interpretation analogous to Steinberg's for shifted tableaux (see [24])? Can one relate the Edelman-Greene correspondence (see [6]) or its shifted analogue (see [13, 15]) with geometry?

Notes

1. After submitting the first version of this paper, P. Trapa informed us of the variety Z_θ defined by Springer and also investigated by himself (see [31], and the variety M in [37]). This seems to provide another ground for the interpretation duscussed in the present paper, and the comparison will be made elsewhere. We thank him for the information and comments. We also thank H. Ochiai for related remarks on Steinberg's variety Z. [31] and [37] were added to the bibliography.
2. Remark after Corollary 4.6 may be regarded a corrected version of an auxiliary result in a preliminary version of this paper, which mistakenly claimed that $\tilde{\iota}_\varepsilon^{-1}(\check{Z}_{T,T'})$ coincides with $\check{Z}_M$ even before taking closures in $\check{Z}_\varepsilon$, if M and (T, T') are related as in the remark.

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