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Construction of Graded Covariant $GL(m/n)$ Modules Using Tableaux

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Abstract. Irreducible covariant tensor modules for the Lie supergroups $GL(m/n)$ and the Lie superalgebras $gl(m/n)$ and $sl(m/n)$ are obtained through the use of Young tableaux techniques. The starting point is the graded permutation action, first introduced by Dondi and Jarvis, on $V^{\otimes l}$. The isomorphism between this group of actions and the symmetric group S_l enables the graded generalization of the Young symmetrizers, and hence of the column relations and Garnir relations, to be made. Consequently, corresponding to each partition of l an irreducible $GL(m/n)$ module may be obtained as a submodule of $V^{\otimes l}$. A basis for the module labeled by the partition λ is provided by $GL(m/n)$-standard tableaux of shape λ defined by Berele and Regev. The reduction of an arbitrary tableau to standard form is accomplished through the use of graded column relations and graded Garnir relations. The standardization procedure is algorithmic and allows matrix representations of the Lie superalgebras $gl(m/n)$ and $sl(m/n)$ to be constructed explicitly over the field of rational numbers. All the various steps of the standardization algorithm are exemplified, as well as the explicit construction of matrices representing particular elements of $gl(m/n)$ and $sl(m/n)$.

Keywords: Young tableaux, Lie superalgebras, modules

1. Introduction

With the recent success of techniques involving Young tableaux and Garnir relations in obtaining irreducible modules of the classical groups and their Lie algebras [3], [9], [10], attention is turned here to the Lie supergroups $GL(m/n)$ and Lie superalgebras $gl(m/n)$ and $sl(m/n)$. As with the classical groups, partitions have a key role [1], [2], [4], [6] in the classification of the irreducible representations (and modules) of the supergroups and superalgebras.

The partition of the positive integer l into p positive integer parts $\lambda_1, \lambda_2, \ldots, \lambda_p$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_p = l$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$ is denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$. It is understood that if $i > p$, then $\lambda_i = 0$, since two partitions are equal only if their nonzero parts are equal. It is sometimes convenient to use an index to denote a repeated part; for example, $(3, 3, 3, 2, 1, 1) \equiv (3^3, 2, 1^2)$. Partitions will always be denoted by lowercase Greek letters. Let $P(l)$ denote the set of all partitions of l. Each partition $\lambda \in P(l)$ specifies a regular Young diagram F^λ consisting of l boxes arranged in p left-adjusted rows. The number of boxes

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in the ith row is λ_i for $i = 1, 2, \ldots, p$. Let $q = \lambda_1$. Then for $j = 1, 2, \ldots, q$ let $\tilde{\lambda}_j$ be the length of the jth column of F^λ. This defines $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_q)$, the partition conjugate to λ. Consequently, the Young diagram $F^{\tilde{\lambda}}$ is obtained from F^λ by reflection in the main diagonal, that is, interchanging rows and columns. If μ and ν are partitions of u and v, respectively, then ν is said to be contained in μ, denoted $\nu \leq \mu$, if $\nu_i \leq \mu_i$ for each $i = 1, 2, \ldots$. If $\nu \leq \mu$, the skew Young diagram $F^{\mu/\nu}$ consisting of $u - v$ boxes is defined as that diagram resulting from the removal of the v boxes of F^ν from F^μ. A (skew) Young tableau results from the filling of the boxes of a (skew) Young diagram with entries, often integers, from a specified set.

If V serves as a module for the defining representation of any of the classical groups, the l-fold tensor product module $V^{\otimes l}$ is fully reducible. In the case of $GL(m)$, the inequivalent irreducible representations appearing as submodules of $V^{\otimes l}$ are naturally labeled by partitions with each denoted by $\{\lambda\}$ for some partition λ of l with $\tilde{\lambda}_1 \leq m$.

In the case of $GL(m/n)$, once more $V^{\otimes l}$ is fully reducible [4]. Each inequivalent irreducible submodule of $V^{\otimes l}$ is again labeled by a partition λ of l, and the corresponding representation is denoted by $\{\lambda\}$, but now $\tilde{\lambda}_{n+1} \leq m$. The purpose of this paper is to construct each of these irreducible covariant $GL(m/n)$ modules and the explicit matrix representations $\{\lambda\}$. It should be noted that this class of representations contains both typical and atypical cases; the former are those that are labeled by partitions λ for which $\lambda_m \geq n$ or, equivalently, $\tilde{\lambda}_n \geq m$, whereas the latter are such that $\lambda_m < n$ and $\tilde{\lambda}_n < m$ [2].

The importance of partitions in the classification of the irreducible representations of these groups derives from the representation theory of the symmetric group and the dual centralizing action of this group and the group in question on $V^{\otimes l}$ [4], [7]. The Young symmetrizer Y^λ (see [3], [9], [10]), suitably normalized, is a specific primitive idempotent of the symmetric group associated with the partition λ of l and may, in the classical case, be used to project submodules out of $V^{\otimes l}$. In this way certain symmetrized tensors, which are identified with symmetrized Young tableaux, arise [9]. These objects span M^λ, the irreducible $GL(m)$ submodule of $V^{\otimes l}$ corresponding to $\{\lambda\}$. They are not, in general, linearly independent. The most useful identities between them, the column relations and the Garnir relations, may be used to express an arbitrary symmetrized Young tableau in terms of $GL(m)$–standard tableaux, which thus serve as a basis for M^λ [8], [9].

In the case of $GL(m/n)$, the above formalism requires the role of the symmetric group to be generalized. The appropriate construction of a $\mathbb{Z}_2$-graded symmetric group action on $V^{\otimes l}$ was considered in [1], [4], [6] and is elucidated in Section 2. In Section 3 the graded analogues of the Young symmetrizers are constructed, and graded versions of the column relations and Garnir relations arise naturally. These are then shown to be sufficient to construct the irreducible graded $GL(m/n)$ modules along lines similar to those of the construction of covariant $GL(m)$ modules. The basis for each $GL(m/n)$ module is provided by the set of (m, n) semistandard tableaux of Berele and Regev [4], which, for consistency, are here referred to as $GL(m/n)$-standard tableaux. In Section 3 the algorithm used to write an arbitrary grade-symmetrized tableau in terms of $GL(m/n)$-standard tableaux is obtained as a modification of that used in the $GL(m)$ case. In general, arbitrary rational numbers arise in the decomposition. This is a consequence of the graded role of the symmetric group.

Section 4 demonstrates that the same techniques may be used to construct irreducible modules of the Lie superalgebras $gl(m/n)$ and $sl(m/n)$. It contains a number of examples of the use of the standardization algorithm presented in Section 3, as well as the construction of explicit matrices for certain elements of irreducible representations of $gl(2/2)$ and $gl(2/1)$ having dimensions 32 and 8, respectively.

2. The supergroups $GL(m/n)$

Much of the following account has been extracted from [5]. Define $B = \mathbb{F}B_L$ to be the exterior algebra of $\{\zeta_1, \zeta_2, \ldots, \zeta_L\}$ over the field $\mathbb{F}$, where L is taken to be arbitrarily large. B is known as a Grassmann algebra, and its elements are referred to as Grassmann parameters. B has dimension 2^L. Let the exterior product $\zeta_{i_1} \wedge \zeta_{i_2} \wedge \cdots \wedge \zeta_{i_l} \in B$ be denoted by $\zeta_{i_1 i_2 \ldots i_l}$. Then $\zeta_{i_2 i_1} = -\zeta_{i_1 i_2}$, and, more generally, if $\sigma \in S_l$, $\zeta_{i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(l)}} = (-1)^{\sigma} \zeta_{i_1 i_2 \ldots i_l}$. Denote by B_l the subspace of B whose basis is the set of l-fold exterior products of the generators $\{\zeta_1, \zeta_2, \ldots, \zeta_L\}$. Thus B_e has a basis $\{\zeta_{i_1 i_2 \ldots i_l} \ : \ 1 \leq i_1 < i_2 < \cdots < i_l \leq L\}$ and B_0 has basis $\{1 \in F\}$. Let $B_{\bar{0}} = B_0 \oplus B_2 \oplus B_4 \oplus \cdots$, and let $B_{\bar{1}} = B_1 \oplus B_3 \oplus B_5 \oplus \cdots$. Then $B = B_{\bar{0}} \oplus B_{\bar{1}}$ with both $B_{\bar{0}}$ and $B_{\bar{1}}$ having dimension 2^{L-1}. The following properties are immediate from the definitions:

$$f^0 \wedge g^0 = g^0 \wedge f^0, \ f^0 \wedge g^1 = g^1 \wedge f^0, \ f^1 \wedge g^1 = -g^1 \wedge f^1 \tag{2.1}$$

for all $f^0, g^0 \in B_{\bar{0}}$ and $f^1, g^1 \in B_{\bar{1}}$.

The properties of the Grassmann algebra are typical of a structure with a $\mathbb{Z}_2$ grading. In view of (2.1), the $\mathbb{Z}_2$-graded subspaces of B, $B_{\bar{0}}$ and $B_{\bar{1}}$, are known as the even and odd subspaces, respectively. Their elements are known as even or odd Grassmann parameters, respectively. Each element $b \in B$ may be written $b = b_{\bar{0}} + b_{\bar{1}}$, where $b_{\bar{0}} \in B_{\bar{0}}$ is even and $b_{\bar{1}} \in B_{\bar{1}}$ is odd. If $b \neq 0$ and either $b_{\bar{0}} = 0$ or $b_{\bar{1}} = 0$, then b is said to be a homogeneous element of B. In such a case the degree of b, denoted deg b, is defined to be

$$\deg b = \begin{cases} 0 & \text{if } b_{\bar{1}} = 0, \\ 1 & \text{if } b_{\bar{0}} = 0. \end{cases} \tag{2.2}$$

Let $B^{m,n}$ be the vector space $B_{\bar{0}}^{\oplus m} \oplus B_{\bar{1}}^{\oplus n}$. A typical element of $B^{m,n}$ is $X = (X_0, X_1) = (X_0^1, X_0^2, \ldots, X_0^m, X_1^{m+1}, \ldots, X_1^{m+n})$, where each $X_0^i \in B_{\bar{0}}$ and each $X_{\bar{1}}^i \in B_{\bar{1}}$. It is convenient to define the index sets $\mathcal{I}_{\bar{0}} = \{1, 2, \ldots, m\}$, $\mathcal{I}_{\bar{1}} = \{m+1, m+2, \ldots, m+n\}$, and $\mathcal{I} = \mathcal{I}_{\bar{0}} \cup \mathcal{I}_{\bar{1}}$. $B^{m,n}$ is naturally $\mathbb{Z}_2$ graded because it has a $\mathbb{Z}_2$ graded basis $\{e_i : i \in \mathcal{I}\}$. A typical element of $B^{m,n}$ may thus be expressed as $X = (X_0, X_1) = \sum_{i \in I} X^i e_i$, where deg $X^i = 0$ if $i \in \mathcal{I}_{\bar{0}}$ and deg $X^i = 1$ if $i \in \mathcal{I}_{\bar{1}}$.

In view of the above, the following notation is useful for the grading of an index:

$$\text{grad } i = (i) = \begin{cases} 0 & \text{if } i \in \mathcal{I}_{\bar{0}}, \\ 1 & \text{if } i \in \mathcal{I}_{\bar{1}}. \end{cases} \tag{2.3}$$

Thus, if $X = \sum_{i \in \mathcal{I}} X^i e_i \in B^{m,n}$, then $X^i \in B_{\overline{(i)}}$. A further useful notation assigns to the symbol

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1\lambda_1} \\ a_{21} & a_{22} & \cdots & \\ \vdots & \vdots & \ddots & \\ a_{\tilde{\lambda}_1 1} & & & \end{bmatrix} \tag{2.4}$$

the value $(-1)^{((a_{11})+(a_{21})+\cdots+(a_{\tilde{\lambda}_1 1}))((a_{12})+\cdots+(a_{\tilde{\lambda}_2 2}))\cdots((a_{1\lambda_1})\cdots)}$, that is, -1 to the power of the product of the column sums of the respective gradings (which may all be taken mod 2). By using this notation, (2.1) may be written $f^a \wedge g^b = [a\, b] g^b \wedge f^a$, where $f^a \in B_{\overline{(a)}}$ and $g^b \in B_{\overline{(b)}}$.

The supergroup $GL(m/n)$ is the group of invertible endomorphisms of $B^{m,n}$. Thus $A \in GL(m/n)$ may be realized by a matrix of the form

$$A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \tag{2.5}$$

where P, Q, R, and S are submatrices of sizes $m \times m$, $m \times n$, $n \times m$, and $n \times n$, respectively, with $P^i{}_j \in B_{\bar{0}}$, $Q^i{}_k \in B_{\bar{1}}$, $R^l{}_j \in B_{\bar{1}}$, and $S^l{}_k \in B_{\bar{0}}$ for $1 \le i, j \le m$ and $1 \le k, l \le n$. In the notation defined above, $A \in GL(m/n)$ if $A^i{}_j \in B_{\overline{(i)+(j)}}$, where the sum is taken mod 2. Note that

$$A^i{}_j A^k{}_l = \begin{bmatrix} i & k \\ j & l \end{bmatrix} A^k{}_l A^i{}_j.$$

Let $V = B^{m,n}$. Its l-fold tensor product $V^{\otimes l}$ has a $\mathbb{Z}_2$-graded basis $\{e_{i_1 i_2 \ldots i_l} : i_k \in \mathcal{I}$ for $k = 1, 2, \ldots, l\}$, where $e_{i_1 i_2 \cdots i_l} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_l}$ and

$$\deg e_{i_1 i_2 \cdots i_l} = (i_1) + (i_2) + \cdots + (i_l) \bmod 2. \tag{2.6}$$

If $A \in GL(m/n)$, then A has a corresponding diagonal action $\tilde{A} \in \text{End}\, V^{\otimes l}$ given by

$$\begin{aligned} \tilde{A} : e_{i_1 i_2 \cdots i_l} &= A^{j_1}{}_{i_1} e_{j_1} \otimes A^{j_2}{}_{i_2} e_{j_2} \otimes \cdots \otimes A^{j_l}{}_{i_l} e_{j_l} \\ &= \prod_{1 \le a \le b \le l} [j_a j_b][j_a i_b] A^{j_1}{}_{i_1} A^{j_2}{}_{i_2} \cdots A^{j_l}{}_{i_l} e_{j_1 j_2 \cdots j_l}. \end{aligned} \tag{2.7}$$

This defines $V^{\otimes l}$ as a $GL(m/n)$ module.

Schur (see [7]) has shown that the irreducible modules occurring as submodules of $V^{\otimes l}$ may be obtained by considering the algebra which centralizes this module action. To this end it is necessary to define [4], [6] the graded action of the symmetric group S_l on $V^{\otimes l}$. First, the action of the symmetric group S_l on the basis of $V^{\otimes l}$ is defined, for $\sigma \in S_l$, by $\sigma : e_{i_1 i_2 \cdots i_l} = e_{i_{\sigma^{-1}(1)} i_{\sigma^{-1}(2)} \cdots i_{\sigma^{-1}(l)}}$. The graded action $\tilde{\sigma}$ on $V^{\otimes l}$ is then defined by (see [6])

$$\begin{aligned} \tilde{\sigma} : e_{i_1 i_2 \cdots i_l} &= \prod_{\substack{1 \le a < b \le l \\ \sigma(a) > \sigma(b)}} [i_a i_b] \sigma : e_{i_1 i_2 \cdots i_l} \\ &= \prod_{\substack{1 \le a < b \le l \\ \sigma(a) > \sigma(b)}} [i_a i_b] e_{i_{\sigma^{-1}(1)} i_{\sigma^{-1}(2)} \cdots i_{\sigma^{-1}(l)}}. \end{aligned} \tag{2.8}$$

The set of actions $\tilde{\sigma}$ for $\sigma \in S_l$ constitutes a group which will be called the graded symmetric group and will be denoted $\tilde{S}_l$. This defines $V^{\otimes l}$, by linear extension, to be an $\tilde{S}_l$ module and also a $\mathbb{Z}\tilde{S}_l$ module. It is not difficult to show that if $\rho, \sigma \in S_l$ and $\tau = \rho\sigma$, then $\tilde{\tau} = \tilde{\rho}\tilde{\sigma}$. Therefore, $\mathbb{Z}\tilde{S}_l$ is isomorphic to $\mathbb{Z}S_l$ in its action on $V^{\otimes l}$. Thus, like $\mathbb{Z}S_l$, $\mathbb{Z}\tilde{S}_l$ may be written as a direct sum of simple two-sided ideals labeled by $P(l)$, the set of partitions of l.

LEMMA 2.1. *$GL(m/n)$ and $\mathbb{Z}\tilde{S}_l$ commute in their actions on $V^{\otimes l}$.*

Proof. Let $A \in GL(m/n)$ and $\tilde{\sigma} \in \tilde{S}_l$. Then

$$\begin{aligned} \tilde{A}\tilde{\sigma} : e_{i_1 i_2 \cdots i_l} &= \tilde{A} : \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} [i_a, i_b]\, e_{i_{\sigma^{-1}(1)} i_{\sigma^{-1}(2)} \cdots i_{\sigma^{-1}(l)}} \\ &= \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} [i_a i_b] \prod_{a \le b} \left[j_{\sigma^{-1}(a)} j_{\sigma^{-1}(b)} \right] \left[j_{\sigma^{-1}(a)} i_{\sigma^{-1}(b)} \right] \\ &\quad \times A^{j_{\sigma^{-1}(1)}}{}_{i_{\sigma^{-1}(1)}} A^{j_{\sigma^{-1}(2)}}{}_{i_{\sigma^{-1}(2)}} \cdots A^{j_{\sigma^{-1}(l)}}{}_{i_{\sigma^{-1}(l)}} e_{j_{\sigma^{-1}(1)} j_{\sigma^{-1}(2)} \cdots j_{\sigma^{-1}(l)}} \\ &= \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} [i_a i_b] \prod_{\sigma(a) \le \sigma(b)} [j_a j_b][j_a i_b] \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} \begin{bmatrix} i_a & i_b \\ j_a & j_b \end{bmatrix} \\ &\quad \times A^{j_1}{}_{i_1} A^{j_2}{}_{i_2} \cdots A^{j_l}{}_{i_l} e_{j_{\sigma^{-1}(1)} j_{\sigma^{-1}(2)} \cdots j_{\sigma^{-1}(l)}}. \end{aligned}$$

On the other hand,

$$\tilde{\sigma}\tilde{A} : e_{i_1 i_2 \ldots i_l} = \tilde{\sigma} : \prod_{a \leq b} [j_a\, j_b][j_a\, i_b] A^{j_1}{}_{i_1} A^{j_2}{}_{i_2} \ldots A^{j_l}{}_{i_l} e_{j_1 j_2 \ldots j_l}$$

$$= \prod_{a \leq b} [j_a\, j_b][j_a\, i_b] \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} [j_a\, j_b] A^{j_1}{}_{i_1} A^{j_2}{}_{i_2} \ldots A^{j_l}{}_{i_l} e_{j_{\sigma^{-1}(1)} j_{\sigma^{-1}(2)} \ldots j_{\sigma^{-1}(l)}}.$$

Then, since

$$\begin{aligned}
\prod_{a \leq b} [j_a\, j_b][j_a\, i_b] \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} [j_a\, j_b] &= \prod_{\substack{a \leq b \\ \sigma(a)\leq\sigma(b)}} [j_a\, j_b][j_a\, i_b] \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} [j_a\, i_b] \\
&= \prod_{\substack{a \geq b \\ \sigma(a)\geq\sigma(b)}} [j_a\, j_b][i_a\, j_b] \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} [j_a\, i_b] \\
&= \prod_{\sigma(a)\geq\sigma(b)} [j_a\, j_b][i_a\, j_b] \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} [j_a\, j_b][i_a\, j_b][j_a\, i_b] \\
&= \prod_{\sigma(a)\leq\sigma(b)} [j_a\, j_b][j_a\, i_b] \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} [i_a\, i_b] \left[\begin{matrix} i_a & i_b \\ j_a & j_b \end{matrix} \right],
\end{aligned}$$

the theorem is proved. □

This theorem implies, since irreducible representations of S_l, and hence of $\tilde{S}_l$, are labeled by partitions λ of l, that, as in the case of $GL(m)$, the finite-dimensional irreducible representations of $GL(m/n)$ may be labeled by partitions. In fact, [4] provides a stronger statement:

THEOREM 2.2. *The irreducible covariant representations of $GL(m/n)$ may be labeled by the set $P(m, n, l) \subset P(l)$, where $\lambda \in P(m, n, l)$ if λ is a partition of l such that $\lambda_{m+1} \leq n$.*

This theorem is known in [4] as the "hook theorem" because, if $\lambda \in P(m, n, l)$, the Young diagram F^λ lies in a hook with leg width n and arm width m.

The irreducible covariant $GL(m/n)$ modules are obtained by means of the graded analogue of the Young symmetrizer $\tilde{Y}^\lambda$, which generates the simple ideal of $\mathbb{Z}\tilde{S}_l$ corresponding to the partition λ of l.

3. Irreducible covariant $GL(m/n)$ modules

This section concerns the construction of the irreducible $GL(m/n)$ modules in terms of Young tableaux and Garnir relations in a direct extension of the techniques presented in [8] and [9] for the construction of irreducible covariant $GL(m)$ modules.

The irreducible covariant $GL(m/n)$ module M^λ is constructed by means of graded Young symmetrizer $\tilde{Y}^\lambda$ associated with the partition λ. Fill the Young diagram F^λ with the integers $1, 2, \ldots, l$ placed consecutively in the boxes of F^λ first passing down the leftmost column and then down subsequent columns taken in turn from left to right. Designate the Young tableau so obtained by t^λ. For example, if $\lambda = (4, 3, 1)$, then

$$t^\lambda = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 6 & 8 \\ \hline 2 & 5 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \; . \tag{3.1}$$

Define the column group $\mathcal{C}^\lambda \subset S_l$ as that group which preserves the column of each integer within t^λ. Similarly, the row group $\mathcal{R}^\lambda \subset S_l$ is that which preserves the row of each integer within t^λ. The graded Young symmetrizer $\tilde{Y}^\lambda \in \mathbb{Z}\tilde{S}_l$ is then defined as

$$\tilde{Y}^\lambda = \sum_{\rho \in \mathcal{R}^\lambda} \sum_{\sigma \in \mathcal{C}^\lambda} (-1)^\sigma \tilde{\rho}\tilde{\sigma}, \tag{3.2}$$

where $(-1)^\sigma$ is the parity or signature of the permutation $\sigma \in S_l$.

Each basis element $w = e_{i_1 i_2 \ldots i_l}$ of $V^{\otimes l}$ is conveniently identified with the tableau T_w^λ obtained from t^λ by replacing each integer entry k by i_k for $k = 1, \ldots, l$. For example, with $\lambda = (4, 3, 1)$ as before and $w = e_{41234325}$,

$$T_w^\lambda = \begin{array}{|c|c|c|c|} \hline 4 & 3 & 3 & 5 \\ \hline 1 & 4 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \; . \tag{3.3}$$

Following from the action of $\tilde{S}_l$ on w, $\tilde{S}_l$ acts on T_w^λ by place permutation together with a sign factor associated with the grading of $\tilde{S}_l$. For each tableau T_w^λ, denote by $\{T_w^\lambda\}^\sim$ the grade-symmetrized tableau $\{T_w^\lambda\}^\sim = \tilde{Y}^\lambda T_w^\lambda$ which is identified with the grade-symmetrized tensor $\tilde{Y}^\lambda w$. The $GL(m/n)$ module M^λ is then defined as the span of $\{T^\lambda\}^\sim$, where T^λ is any tableau of shape F^λ filled with entries from the set $\mathcal{I} = \{1, 2, \ldots, m + n\}$.

The analysis of [4] leads to

THEOREM 3.1. *Each irreducible covariant tensor $GL(m/n)$ module may be labeled by a partition λ for which $\lambda_{m+1} \leq n$. The $GL(m/n)$ module M^λ is irreducible.*

The grade-symmetrized tableaux $\{T^\lambda\}^\sim$ are not linearly independent since there exist graded analogues of the column relations and the Garnir relations. The first of these takes the form:

LEMMA 3.2. *For any tableau T^λ and $\tau \in \mathcal{C}^\lambda$,*

$$\{T^\lambda\}^\sim = (-1)^\tau \{\tilde{\tau} T^\lambda\}^\sim. \tag{3.4}$$

Proof. By the definition of $\tilde{Y}^\lambda$ in (3.2),

$$\begin{aligned} \tilde{Y}^\lambda \tilde{\tau} = \sum_{\rho\in\mathcal{R}^\lambda} \sum_{\sigma\in\mathcal{C}^\lambda} (-1)^\sigma \tilde{\rho}\tilde{\sigma}\tilde{\tau} &= \sum_{\rho\in\mathcal{R}^\lambda} \sum_{\sigma\in\mathcal{C}^\lambda} (-1)^\sigma \tilde{\rho}\widetilde{\sigma\tau} \\ &= (-1)^\tau \sum_{\rho\in\mathcal{R}^\lambda} \sum_{\sigma\in\mathcal{C}^\lambda} (-1)^\sigma \tilde{\rho}\tilde{\sigma}, \end{aligned}$$

where the isomorphism between $\mathbb{Z}\tilde{S}_l$ and $\mathbb{Z}S_l$ has been used. Therefore, $\tilde{Y}^\lambda\tilde{\tau} = (-1)^\tau \tilde{Y}^\lambda$, which proves Lemma 3.2. □

Lemma 3.2 implies that if T^λ has an entry from the set $\mathcal{I}_{\bar{0}}$ repeated in any column, then $\{T^\lambda\}^\sim$ vanishes. However, because of the grading property, this is not the case for a repeated entry from the set $\mathcal{I}_{\bar{1}}$. Nevertheless, (3.4) allows $\{T^\lambda\}^\sim$ to be expressed as $\pm\{T'^\lambda\}$ for some tableau T'^λ in which the entries are nondecreasing down each column and strictly increasing on the set $\mathcal{I}_{\bar{0}}$. Such a tableau is termed column superstrict. To illustrate the use of Lemma 3.2, consider the $GL(2/2)$ module $M^{(2,2,1)}$ where

$$\left\{\begin{matrix} 1 & 1 \\ 2 & 4 \\ 2 & \end{matrix}\right\}^\sim = 0, \quad \left\{\begin{matrix} 4 & 1 \\ 3 & 2 \\ 3 & \end{matrix}\right\}^\sim = \left\{\begin{matrix} 3 & 1 \\ 3 & 2 \\ 4 & \end{matrix}\right\}^\sim, \quad \left\{\begin{matrix} 1 & 4 \\ 4 & 2 \\ 3 & \end{matrix}\right\}^\sim = -\left\{\begin{matrix} 1 & 2 \\ 3 & 4 \\ 4 & \end{matrix}\right\}^\sim. \tag{3.5}$$

The Garnir relations have the following graded analogue:

LEMMA 3.3. *For $i < j$ let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of the entries in the ith and jth columns, respectively, of t^λ such that $\#(\mathcal{X} \cup \mathcal{Y}) > \tilde{\lambda}_i$. Let $S(\mathcal{X})$, $S(\mathcal{Y})$, and $S(\mathcal{X} \cup \mathcal{Y})$ be the subgroups of S_l preserving $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{X} \cup \mathcal{Y}$, respectively. Then, if $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ is a set of right coset representatives for $S(\mathcal{X}) \times S(\mathcal{Y})$ in $S(\mathcal{X} \cup \mathcal{Y})$,*

$$\sum_{\eta\in\mathcal{G}(\mathcal{X},\mathcal{Y})} (-1)^\eta \{\tilde{\eta} T^\lambda\}^\sim = 0. \tag{3.6}$$

Proof. Let $G^\lambda_{\mathcal{X},\mathcal{Y}} = \sum_{\eta\in\mathcal{G}(\mathcal{X},\mathcal{Y})} (-1)^\eta \eta$. The ungraded Garnir relation (see [8], [9]) implies that $Y^\lambda G^\lambda_{\mathcal{X},\mathcal{Y}} = 0$, whereupon, once more, the isomorphism between $\mathbb{Z}\tilde{S}_l$ and $\mathbb{Z}S_l$ proves the lemma. □

As an example, consider the grade-symmetrized tableau $\{T_w^\lambda\}^\sim$ in the $GL(2/3)$ module M^λ, where T_w^λ and λ are as in (3.3). Then, with $i = 1$, $j = 2$, $\mathcal{X} = \{2, 3\}$, and $\mathcal{Y} = \{4, 5\}$, an appropriate set of coset representatives produces the identity

$$\begin{Bmatrix} 4 & 3 & 3 & 5 \\ 1 & 4 & 2 & \\ 2 & & & \end{Bmatrix}^\sim - \begin{Bmatrix} 4 & 1 & 3 & 5 \\ 3 & 4 & 2 & \\ 2 & & & \end{Bmatrix}^\sim - \begin{Bmatrix} 4 & 2 & 3 & 5 \\ 1 & 4 & 2 & \\ 3 & & & \end{Bmatrix}^\sim + \begin{Bmatrix} 4 & 3 & 3 & 5 \\ 4 & 1 & 2 & \\ 2 & & & \end{Bmatrix}^\sim$$
$$+ \begin{Bmatrix} 4 & 3 & 3 & 5 \\ 1 & 2 & 2 & \\ 4 & & & \end{Bmatrix}^\sim + \begin{Bmatrix} 4 & 1 & 3 & 5 \\ 3 & 2 & 2 & \\ 4 & & & \end{Bmatrix}^\sim = 0. \tag{3.7}$$

It should not be assumed that the occurrence of identical entries in the same column implies that the term vanishes. For example, the fourth term in (3.4) is not identically zero.

For each irreducible covariant representation of $GL(m/n)$ a favored set of tableaux has been introduced by Berele and Regev [4]:

Definition 3.4. Let $\mathcal{I} = \mathcal{I}_{\bar{0}} \cup \mathcal{I}_{\bar{1}}$, where $\mathcal{I}_{\bar{0}} = \{1, 2, \ldots, m\}$ and $\mathcal{I}_{\bar{1}} = \{m + 1, \ldots, m + n\}$. T^λ is $GL(m/n)$-standard if and only if

(i) each entry is taken from the set $\mathcal{I}$;
(ii) the entries from the set $\mathcal{I}_{\bar{0}}$ form a tableau T^μ, for some $\mu \leq \lambda$, within T^λ;
(iii) the entries from the set $\mathcal{I}_{\bar{0}}$ are strictly increasing from top to bottom down each column of T^μ;
(iv) the entries from the set $\mathcal{I}_{\bar{1}}$ are nondecreasing from top to bottom down each column of $T^{\lambda/\mu}$;
(v) the entries from the set $\mathcal{I}_{\bar{0}}$ are nondecreasing from left to right across each row of T^μ;
(vi) the entries from the set $\mathcal{I}_{\bar{1}}$ are strictly increasing from left to right across each row of $T^{\lambda/\mu}$.

As will be seen, the systematic application of the column identities in (3.4), and the Garnir relations in (3.6) permit an arbitrary grade-symmetrized tableau $\{T^\lambda\}^\sim$ to be reduced to a linear combination of grade-symmetrized $GL(m/n)$-standard tableaux. That this procedure is algorithmic may be shown by using an ordering on the set of all tableaux. Such an ordering is provided by

Definition 3.5. Let t_u^b be the sum of the entries in the bth column of T_u^λ for $b = 1, 2, \ldots, q$, where $q = \lambda_1$. Define $|T_u^\lambda|$ to be the equivalence class of all tableaux which have their sequences of column sums identical to that of T_u^λ; that is, $T_v^\lambda \in |T_u^\lambda|$ if $t_v^b = t_u^b$ for $b = 1, 2, \ldots, q$. A total order on the set of equivalence classes of tableaux is defined by $|T_u^\lambda| > |T_v^\lambda|$ if, for some $k \leq q$, $t_u^k < t_v^k$ with $t_u^b = t_v^b$ for each $b = 1, 2, \ldots, k-1$. It is convenient to write $T_u^\lambda > T_v^\lambda$ when this strict inequality is true of the equivalence classes to which T_u^λ and T_v^λ belong and to say in such a case that T_u^λ is higher than T_v^λ.

The following standardization algorithm is a grade-symmetrized version of that used in the construction of irreducible convariant $GL(m)$ modules by James and Kerber [8] and reproduced and used in [9]. Rearrange the entries of a grade-symmetrized tableau by using Lemma 3.2 to form $\{T^\lambda\}^\sim$, where T^λ is column superstrict. If T^λ is not $GL(m/n)$-standard, then either condition (v) or condition (vi) of Definition 3.4 is violated and, in particular, is violated by a neighboring pair of entries. With $T_{i,j}^\lambda$ the entry in the ith row and jth column of T^λ, let a and b be such that this neighboring pair is $T_{a,b}^\lambda$ and $T_{a,b+1}^\lambda$. Then $T_{a,b}^\lambda \geq T_{a,b+1}^\lambda$ with equality implying that $T_{a,b}^\lambda \in \mathcal{I}_{\bar{1}}$. Let $\mathcal{X}$ be the set of positions below and including $T_{a,b}^\lambda$ in the bth column, and let $\mathcal{Y}$ be the set of positions above and including $T_{a,b+1}^\lambda$ in the $(b+1)$th column. The relevant entries of T^λ are then as follows :

$$\begin{array}{ccc} \vdots & & T^\lambda_{1,b+1} \\ \vdots & & \wedge\!| \\ \vdots & & \vdots \\ \vdots & & \wedge\!| \\ \vdots & & T^\lambda_{a-1,b+1} \\ \vdots & & \wedge\!| \\ T^\lambda_{a,b} & \geq & T^\lambda_{a,b+1} \\ \wedge\!| & & \\ T^\lambda_{a+1,b} & & \\ \wedge\!| & & \\ \vdots & & \\ \wedge\!| & & \\ T^\lambda_{\tilde{\lambda}_b,b}. & & \end{array} \qquad (3.8)$$

Since, with $\mathcal{X}$ and $\mathcal{Y}$ so defined, $\#(\mathcal{X} \cup \mathcal{Y}) = \tilde{\lambda}_b + 1$, Lemma 3.3 may be used to express $\{T^\lambda\}^\sim$ in terms of other tableaux.

Consider first the case for which $T_{a,b}^\lambda > T_{a,b+1}^\lambda$. With $\eta \in \mathcal{G}(\mathcal{X}, \mathcal{Y})$ and $\eta \notin S(\mathcal{X}) \times S(\mathcal{Y})$, $T_\eta^\lambda = \eta T^\lambda$ has necessarily been formed from T^λ by swapping the columns of at least one pair of elements from $\mathcal{X} \cup \mathcal{Y}$. Since the entries at positions $\mathcal{Y}$ are all smaller than those at positions $\mathcal{X}$, $T_\eta^\lambda > T^\lambda$. Hence, in this case the algorithm allows $\{T^\lambda\}^\sim$ to be written in terms of higher tableaux, the coefficients of which are all integral. To illustrate this case, let

$$T^{(2,2,2)} = \begin{matrix} 1 & 2 \\ 5 & 3 \\ 5 & 4 \end{matrix} \tag{3.9a}$$

Then, in the $GL(3/2)$ module $M^{(2,2,2)}$ the following identity arises when the above procedure is used with $\mathcal{X} = \{2, 3\}$ and $\mathcal{Y} = \{4, 5\}$:

$$\begin{Bmatrix} 1\,2 \\ 5\,3 \\ 5\,4 \end{Bmatrix}^\sim + \begin{Bmatrix} 1\,5 \\ 2\,3 \\ 5\,4 \end{Bmatrix}^\sim - \begin{Bmatrix} 1\,5 \\ 5\,3 \\ 2\,4 \end{Bmatrix}^\sim + \begin{Bmatrix} 1\,2 \\ 3\,5 \\ 5\,4 \end{Bmatrix}^\sim - \begin{Bmatrix} 1\,2 \\ 5\,5 \\ 3\,4 \end{Bmatrix}^\sim + \begin{Bmatrix} 1\,5 \\ 2\,5 \\ 3\,4 \end{Bmatrix}^\sim = 0. \tag{3.9b}$$

By rearranging, using (3.4), and collecting terms, this yields

$$\begin{Bmatrix} 1\,2 \\ 5\,3 \\ 5\,4 \end{Bmatrix}^\sim = 2\begin{Bmatrix} 1\,3 \\ 2\,4 \\ 5\,5 \end{Bmatrix}^\sim - 2\begin{Bmatrix} 1\,2 \\ 3\,4 \\ 5\,5 \end{Bmatrix}^\sim - \begin{Bmatrix} 1\,4 \\ 2\,5 \\ 3\,5 \end{Bmatrix}^\sim, \tag{3.9c}$$

where each of the tableaux on the right side is higher than that on the left.

For the case for which $T^\lambda_{a,b} = T^\lambda_{a,b+1} \in \mathcal{I}_{\bar{1}}$ the same technique produces a similar sum of terms. However, as may be seen by considering the coset containing the permutation which swaps the two identical entries, the original grade-symmetrized tableau is repeated in this identity. Since both of these entries are of odd grade, the signs of these two terms are the same and thus *do not cancel*. The possibility that the entries immediately below $T^\lambda_{a,b}$ or immediately above $T^\lambda_{a,b+1}$ are identical to these two is not excluded. If this entry occurs c times in the bth column and d times in the $(b+1)$th column, then, by considering coset representatives which permute these entries among themselves, it can be seen that the original grade-symmetrized tableau occurs with a multiplicity of $(c+d)!/c!d!$ in the Garnir identity resulting from the selection of $\mathcal{X}$ and $\mathcal{Y}$ given above. Again, all of these terms have the same sign. The previous argument shows, once more, that the remaining terms in the expression are higher than the original. Therefore, in this case $\{T^\lambda\}^\sim$ may be expressed in terms of higher tableaux, the coefficient of each being rational. This case is exemplified by the following example in the $GL(2/2)$ module $M^{(2,2,1)}$:

$$\begin{Bmatrix} 1\,2 \\ 3\,3 \\ 4 \end{Bmatrix}^\sim + \begin{Bmatrix} 1\,3 \\ 2\,3 \\ 4 \end{Bmatrix}^\sim + \begin{Bmatrix} 1\,2 \\ 3\,3 \\ 4 \end{Bmatrix}^\sim + \begin{Bmatrix} 1\,3 \\ 2\,4 \\ 3 \end{Bmatrix}^\sim + \begin{Bmatrix} 1\,2 \\ 3\,4 \\ 3 \end{Bmatrix}^\sim + \begin{Bmatrix} 1\,3 \\ 2\,4 \\ 3 \end{Bmatrix}^\sim = 0, \tag{3.10a}$$

whereupon

$$\left\{\begin{matrix} 1 & 2 \\ 3 & 3 \\ 4 & \end{matrix}\right\}^{\sim} = -\left\{\begin{matrix} 1 & 3 \\ 2 & 4 \\ 3 & \end{matrix}\right\}^{\sim} - \frac{1}{2}\left\{\begin{matrix} 1 & 2 \\ 3 & 4 \\ 3 & \end{matrix}\right\}^{\sim} - \frac{1}{2}\left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 4 & \end{matrix}\right\}^{\sim}. \qquad (3.10b)$$

As a further example, consider the $GL(2/2)$ module $M^{(2^3,1^6)}$, where the above process results in

$$84\left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 3 & 3 \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 4 & \end{matrix}\right\}^{\sim} + 36\left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 3 & \end{matrix}\right\}^{\sim} = 0, \text{ implying } \left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 3 & 3 \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 4 & \end{matrix}\right\}^{\sim} = -\frac{3}{7}\left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 3 & \end{matrix}\right\}^{\sim}. \qquad (3.11)$$

As with (3.9), a single application of the above procedure may result in further nonstandard terms. However, the process may be iterated until solely $GL(m/n)$–standard tableaux result. That this procedure terminates is guaranteed by the ordering on the set of all tableaux of shape F^λ and their finite number. Since the number of $GL(m/n)$–standard tableaux equals the dimension of the irreducible representation $\{\lambda\}$, the existence of this reduction process implies

THEOREM 3.6. *$\{\{T^\lambda\}^\sim : T^\lambda$ is a $GL(m/n)$-standard tableau$\}$ is a basis for the irreducible $GL(m/n)$ module M^λ. If λ is a partition of l, then M^λ is isomorphic to $V^{\otimes l}$ modulo the relations (3.4) and (3.6).*

4. Lie superalgebras and explicit representations of $gl(m/n)$ and $sl(m/n)$

This section applies the techniques developed in Section 3 to the Lie superalgebras $gl(m/n)$ and $sl(m/n)$. However, the relationship between Lie supergroups and Lie superalgebras is more subtle than that in the classical case, and, indeed, some Lie supergroups do not possess a corresponding Lie superalgebra. In view of this, the notion of a Lie superalgebra will be briefly outlined.

A general Lie superalgebra L_s is a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$ with a generalized product satisfying

$$[a, b] \in L_s, \tag{4.1a}$$

$$[a, b] = -(-1)^{\deg a \deg b}[b, a], \tag{4.1b}$$

$$[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c], \tag{4.1c}$$

and

$$(-1)^{\deg a \deg c}[a, [b, c]] + (-1)^{\deg a \deg b}[b, [c, a]] + (-1)^{\deg b \deg c}[c, [a, b]] = 0 \tag{4.1d}$$

for all homogeneous $a, b, c \in L_s$ and $\alpha, \beta \in \mathbb{C}$. A representation of L_s assigns to each $a \in L_s$ a square matrix $\Gamma(a)$ for which

$$\Gamma([a, b]) = \Gamma(a)\Gamma(b) - (-1)^{\deg a \deg b}\Gamma(b)\Gamma(a) \tag{4.2}$$

for each homogeneous $a, b \in L_s$. These matrices then satisfy (4.1).

As in Section 3, let $V = B^{m,n}$ be a $\mathbb{Z}_2$-graded vector space with $\mathbb{Z}_2$-graded basis $\{e_1, e_2, \ldots, e_{m+n}\}$ so that $\deg e_i = (i)$. For $a, b \in \mathcal{I}$ let $E_a{}^b$ be the operator for which

$$E_a{}^b e_c = \delta_c^b e_a. \tag{4.3}$$

$E_a{}^b$ may then be realized as an $(m + n) \times (m + n)$ matrix with a 1 at the intersection of the ath row and bth column and zeros elsewhere. In view of (4.3), $\deg E_a{}^b = (a) + (b) \bmod 2$, and, in accordance with (4.2),

$$[E_a{}^b, E_c{}^d] = \delta_c^b E_a{}^d - \begin{bmatrix} a & c \\ b & d \end{bmatrix} \delta_a^d E_c{}^b. \tag{4.4}$$

The elements $E_a{}^b$ for $a, b = 1, 2, \ldots, m + n$ form a basis for the $(m + n)^2$-dimensional Lie superalgebra $gl(m/n)$. By taking linear combinations of these elements with the coefficient of each element a Grassmann parameter of the same grading, a Lie algebra is generated by virtue of (4.4). It is this Lie algebra which forms the tangent space at the identity of $GL(m/n)$. The diagonal action of $GL(m/n)$ on $V^{\otimes l}$ given in (2.7) then leads to the following diagonal action of the basis elements of $gl(m/n)$ on $V^{\otimes l}$:

$$E_a{}^b : e_{i_1 i_2 \ldots i_l} = \sum_{j=1}^{l} \begin{bmatrix} a & i_1 \\ b & i_2 \\ & \vdots \\ & i_{j-1} \end{bmatrix} \delta_{i_j}^b e_{i_1 \ldots i_{j-1} a i_{j+1} \ldots i_l}. \tag{4.5}$$

Lemma 2.1 then implies that this action commutes with that of the graded symmetric group $\tilde{S}_l$. This enables the action of $E_a{}^b$ on the basis elements $\{T^\lambda\}^\sim$ of M^λ to be found.

Let T^λ be a $GL(m/n)$-standard tableau and, for $j = 1, 2, \ldots l$, let T_j^λ be the entry of T^λ at the position corresponding to j of t^λ. Let p be the number of times the index b occurs in T^λ, and form p distinct tableaux T'^λ_j by replacing a single index b in position j of T^λ with a for all appropriate positions j of T^λ. Then the action of $E_a{}^b$ on $\{T^\lambda\}^\sim$ results in

$$\tilde{E}_a{}^b\{T^\lambda\}^\sim = \sum_{\{j:T_j^\lambda=b\}} \begin{bmatrix} a & T_1^\lambda \\ b & T_2^\lambda \\ & \vdots \\ & T_{j-1}^\lambda \end{bmatrix} \{T'^\lambda_j\}^\sim. \tag{4.6}$$

Any nonstandard tableau appearing on the right-hand side may then be expressed as a linear combination of $GL(m/n)$-standard tableaux by using the techniques described in Section 3.

In precisely this way, the action of $E_a{}^b \in gl(m/n)$ on each grade-symmetrized $GL(m/n)$-standard tableau $\{T_u^\lambda\}^\sim$ gives

$$\tilde{E}_a{}^b\{T_u^\lambda\}^\sim = \sum_{\{T_v^\lambda : T_v^\lambda \ GL(m/n) \text{ standard}\}} \{\lambda\}(E_a{}^b)_{vu}\{T_v^\lambda\}^\sim. \tag{4.7}$$

The coefficients $\{\lambda\}(E_a{}^b)_{vu}$, which are rational numbers, are the matrix elements of $E_a{}^b$ in the representation $\{\lambda\}$.

As an example, consider the $gl(2/2)$ odd generator $E_3{}^2$ in the 32-dimensional $gl(2/2)$ module M^λ with $\lambda = (2^3, 1)$ and the $gl(2/2)$-standard tableau

$$T^\lambda = \begin{matrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & \end{matrix}.$$

By virtue of (2.6), T^λ and $\{T^\lambda\}^\sim$ are of even grade. By (4.6), $E_3{}^2$ acts on the basis element $\{T^\lambda\}^\sim$ of M^λ, according to

$$\tilde{E}_3{}^2 \left\{\begin{matrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & \end{matrix}\right\}^\sim = \begin{bmatrix} 2 & 1 \\ 3 & \end{bmatrix} \left\{\begin{matrix} 1 & 2 \\ 3 & 3 \\ 3 & 4 \\ 4 & \end{matrix}\right\}^\sim + \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ & 3 \\ & 4 \end{bmatrix} \left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \\ 4 & \end{matrix}\right\}^\sim$$

$$= \begin{Bmatrix} 1\ 2 \\ 3\ 3 \\ 3\ 4 \\ 4 \end{Bmatrix}^{\sim} + \begin{Bmatrix} 1\ 3 \\ 2\ 3 \\ 3\ 4 \\ 4 \end{Bmatrix}^{\sim}.$$

Using (3.6) with $\mathcal{X} = \{2, 3, 4\}$ and $\mathcal{Y} = \{5, 6\}$ gives the identity

$$\begin{Bmatrix} 1\ 2 \\ 3\ 3 \\ 3\ 4 \\ 4 \end{Bmatrix}^{\sim} - \begin{Bmatrix} 1\ 3 \\ 2\ 3 \\ 3\ 4 \\ 4 \end{Bmatrix}^{\sim} + \begin{Bmatrix} 1\ 3 \\ 3\ 3 \\ 2\ 4 \\ 4 \end{Bmatrix}^{\sim} - \begin{Bmatrix} 1\ 4 \\ 3\ 3 \\ 3\ 4 \\ 2 \end{Bmatrix}^{\sim} + \begin{Bmatrix} 1\ 2 \\ 3\ 3 \\ 3\ 4 \\ 4 \end{Bmatrix}^{\sim}$$

$$+ \begin{Bmatrix} 1\ 2 \\ 3\ 3 \\ 3\ 4 \\ 4 \end{Bmatrix}^{\sim} + \begin{Bmatrix} 1\ 2 \\ 3\ 4 \\ 3\ 4 \\ 3 \end{Bmatrix}^{\sim} - \begin{Bmatrix} 1\ 3 \\ 2\ 3 \\ 3\ 4 \\ 4 \end{Bmatrix}^{\sim} - \begin{Bmatrix} 1\ 3 \\ 2\ 4 \\ 3\ 4 \\ 3 \end{Bmatrix}^{\sim} + \begin{Bmatrix} 1\ 3 \\ 3\ 4 \\ 2\ 4 \\ 3 \end{Bmatrix}^{\sim} = 0,$$

which, by using (3.4) and collecting terms, gives

$$3\begin{Bmatrix} 1\ 2 \\ 3\ 3 \\ 3\ 4 \\ 4 \end{Bmatrix}^{\sim} - 3\begin{Bmatrix} 1\ 3 \\ 2\ 3 \\ 3\ 4 \\ 4 \end{Bmatrix}^{\sim} - 3\begin{Bmatrix} 1\ 3 \\ 2\ 4 \\ 3\ 4 \\ 3 \end{Bmatrix}^{\sim} + \begin{Bmatrix} 1\ 2 \\ 3\ 4 \\ 3\ 4 \\ 3 \end{Bmatrix}^{\sim} = 0,$$

so that

$$\begin{Bmatrix} 1\ 2 \\ 3\ 3 \\ 3\ 4 \\ 4 \end{Bmatrix}^{\sim} = \begin{Bmatrix} 1\ 3 \\ 2\ 3 \\ 3\ 4 \\ 4 \end{Bmatrix}^{\sim} + \begin{Bmatrix} 1\ 3 \\ 2\ 4 \\ 3\ 4 \\ 3 \end{Bmatrix}^{\sim} - \frac{1}{3}\begin{Bmatrix} 1\ 2 \\ 3\ 4 \\ 3\ 4 \\ 3 \end{Bmatrix}^{\sim}.$$

Hence,

$$\tilde{E}_3{}^2 \left\{\begin{matrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & \end{matrix}\right\}^{\sim} = 2\left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \\ 4 & \end{matrix}\right\}^{\sim} + \left\{\begin{matrix} 1 & 3 \\ 2 & 4 \\ 3 & 4 \\ 3 & \end{matrix}\right\}^{\sim} - \frac{1}{3}\left\{\begin{matrix} 1 & 2 \\ 3 & 4 \\ 3 & 4 \\ 3 & \end{matrix}\right\}^{\sim}.$$

The calculation need not be so involved, as the following two examples make clear:

$$\tilde{E}_3{}^2 \left\{\begin{matrix} 1 & 2 \\ 3 & 4 \\ 3 & 4 \\ 4 & \end{matrix}\right\}^{\sim} = \left[\begin{matrix} 2 & 1 \\ 3 & 3 \\ & 3 \\ & 4 \end{matrix}\right] \left\{\begin{matrix} 1 & 3 \\ 3 & 4 \\ 3 & 4 \\ 4 & \end{matrix}\right\}^{\sim} = -\left\{\begin{matrix} 1 & 3 \\ 3 & 4 \\ 3 & 4 \\ 4 & \end{matrix}\right\}^{\sim},$$

$$\tilde{E}_3{}^2 \left\{\begin{matrix} 1 & 1 \\ 3 & 4 \\ 3 & 4 \\ 3 & \end{matrix}\right\}^{\sim} = 0.$$

Similar calculations, when carried out for each of the 32 $GL(2/2)$–standard tableaux in M^{λ}, yield the following explicit representation $\{2^3, 1\}(E_3{}^2)$ of $E_3{}^2$:

$$\left(\begin{array}{*{32}{c}}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & -\frac{2}{3} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & 1 & \cdot & -\frac{1}{4} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\end{array}\right),$$

where each zero has been replaced by a dot. The three calculations carried out above give the entries in the 13th, 23rd, and 28th columns of this matrix, respectively. Notice that this matrix has the block diagonal nature associated with an odd grading. This structure is ensured by the odd grading of the element ${E_3}^2$ and the adoption of an ordering of the $GL(2/2)$-standard tableaux such that all those of even grade occur first. The algorithmic nature of the above construction lends itself to computer implementation. The above matrix has been produced in this way, together with similar matrices for the remaining generators of $gl(2/2)$ in the same irreducible representation $\{2^3, 1\}$. As a check of the calculations it has been confirmed that the resulting matrices satisfy the commutation relations (4.4).

As a second example, consider the eight-dimensional $GL(2/1)$ module $M^{(2,1)}$

for which the $GL(2/1)$–standard tableaux are

$$\begin{matrix} 2\ 3 \\ 3 \end{matrix}\ ,\ \begin{matrix} 1\ 3 \\ 3 \end{matrix}\ ,\ \begin{matrix} 1\ 2 \\ 2 \end{matrix}\ ,\ \begin{matrix} 1\ 1 \\ 2 \end{matrix}\ ,\ \begin{matrix} 1\ 3 \\ 2 \end{matrix}\ ,\ \begin{matrix} 2\ 2 \\ 3 \end{matrix}\ ,\ \begin{matrix} 1\ 2 \\ 3 \end{matrix}\ ,\ \begin{matrix} 1\ 1 \\ 3 \end{matrix}\ ,$$

where they have been ordered in such a way that the four even tableaux are first. Calculations involving the use of relations (3.4) and (3.6) give, by means of (4.7), the following set of explicit representation matrices for the basis elements of $gl(2/1)$:

$$E_1{}^2 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}, E_1{}^3 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, E_2{}^3 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$E_2{}^1 \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, E_3{}^1 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot & -2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, E_3{}^2 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

along with the diagonal elements

$$E_1{}^1 \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&2&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&2 \end{pmatrix}, E_2{}^2 \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&2&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&2&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, E_3{}^3 \begin{pmatrix} 2&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&2&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}.$$

Notice that, once more, these matrices possess the block diagonal structure associated with the gradings of the elements of $gl(2/1)$ that they represent.

The Lie superalgebra $gl(m/n)$ is not simple, having $sl(m/n)$ as a graded ideal. If $n > 0$, let

$$H_i{}^i = \begin{cases} E_i{}^i + E_{m+n}{}^{m+n} & \text{for } i = 1, \ldots, m, \\ E_i{}^i - E_{m+n}{}^{m+n} & \text{for } i = m+1, \ldots, m+n-1. \end{cases} \tag{4.8}$$

Then the $(m+n)^2-1$ dimensional Lie superalgebra $sl(m/n)$ has a basis consisting of the elements $H_i{}^i$ for $i = 1, \ldots, m+n-1$ and the elements $E_i{}^j$ for $i, j = 1, \ldots, m+n$ and $i \neq j$.

Consider $gl(2/1)$ once more. In accordance with (4.8), let $H_1{}^1 = E_1{}^1 + E_3{}^3$ and $H_2{}^2 = E_2{}^2 + E_3{}^3$. Then $H_1{}^1$ and $H_2{}^2$ form a basis for the Cartan subalgebra of the often studied eight-dimensional simple basic Lie superalgebra $sl(2/1)$; the other basis elements may be taken to be $E_1{}^2$, $E_2{}^1$, $E_1{}^3$, $E_3{}^1$, $E_2{}^3$, and $E_3{}^2$, as above. $M^{(2,1)}$ then serves as an $sl(2/1)$ module and the corresponding representation matrices are obtained directly from those given for $gl(2/1)$.

Notice that the highest-weight vector of the $sl(2/1)$ module $M^{(2,1)}$ is

$$\left\{\begin{matrix} 1 & 1 \\ 2 & \end{matrix}\right\}^{\sim},$$

for which

$$H_1{}^1 \left\{\begin{matrix} 1 & 1 \\ 2 & \end{matrix}\right\}^{\sim} = 2 \left\{\begin{matrix} 1 & 1 \\ 2 & \end{matrix}\right\}^{\sim} \text{ and } H_2{}^2 \left\{\begin{matrix} 1 & 1 \\ 2 & \end{matrix}\right\}^{\sim} = \left\{\begin{matrix} 1 & 1 \\ 2 & \end{matrix}\right\}^{\sim},$$

demonstrating that its highest weight is (2, 1).

5. Conclusion

The techniques presented here extend those previously used in the explicit construction of irreducible tensor modules of the classical groups to the case of irreducible covariant tensor modules of $GL(m/n)$, $gl(m/n)$, and $sl(m/n)$. Significantly, this class of modules encompasses both typical and atypical cases.

Like that of classical cases, the construction is based on the natural combinatorial objects appearing in the theory of tensor representations of $GL(m/n)$: the $GL(m/n)$-standard tableaux. The one major deviation from the classical cases is the appearance of arbitrary rational numbers in the reduction of an arbitrary symmetrized Young tableau to standard form. Despite this, the algorithm readily lends itself to computer implementation, and the explicit matrices of the representation may be generated; the rational numbers appear as entries in these matrices.

Although only purely covariant tensor $GL(m/n)$ modules are dealt with in this paper, the techniques described may equally be applied to the fully reducible contravariant $GL(m/n)$ module $(V^*)^{\otimes l}$ to obtain the purely contravariant tensor $GL(m/n)$ modules. However, the mixed tensor $GL(m/n)$ module $(V^*)^{\otimes v} \otimes V^{\otimes u}$ is totally different in that it is not fully reducible. Therefore, the techniques presented in [9] do not lead to irreducible mixed tensor $GL(m/n)$ modules analogous to those constructed for $GL(m)$.

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