Journal of Algebraic Combinatorics, 22, 317–329, 2005

Covers of Point-Hyperplane Graphs

ARJEH M. COHEN a.m.cohen@tue.nl
E.J. POSTMA e.j.postma@tue.nl
Department of Mathematics and Computer, Science, Technische Universiteit Eindhoven, Department of Mathematics and Computer Science, Den Dolech 2, Eindhoven, The Netherlands

Received November 15, 2003; Revised February 9, 2005; Accepted April 25, 2005

Abstract. A cover of the non-incident point-hyperplane graph of projective dimension 3 for fields of characteristic 2 is constructed. For fields $\mathbb{F}$ of even order larger than 2, this leads to an elementary construction of the non-split extension of $\mathrm{SL}_4(\mathbb{F})$ by $\mathbb{F}^6$.

Keywords: group extension, graph cover, special linear group, projective geometry

1. Introduction

The non-incident point-hyperplane graph $H_n(\mathbb{F})$ has as vertex set the non-incident pairs of a point and a hyperplane in the projective geometry of projective dimension n over a field $\mathbb{F}$. Two distinct vertices are adjacent if the points and hyperplanes are mutually incident. These graphs have been studied extensively, cf. Gramlich [4]. One of their properties is that $H_{n+1}(\mathbb{F})$ is locally $H_n(\mathbb{F})$ for all $\mathbb{F}$ and n, and that every connected and locally $H_n(\mathbb{F})$ graph is isomorphic to $H_{n+1}(\mathbb{F})$ whenever $n > 2$.

This property does not necessarily hold if $n \leq 2$. Indeed, if $n \in \{0, 1\}$ then it is easily seen not to hold. In Gramlich [4], a covering graph of $H_3(\mathbb{F}_2)$, constructed by means of a computer algebra computation, shows that it does not hold in dimension $n = 2$ over $\mathbb{F} = \mathbb{F}_2$. In this paper we give a computer-free construction of a covering graph of $H_3(\mathbb{F})$ for char $\mathbb{F} = 2$, thus providing counterexamples to the local recognizability of $H_3(\mathbb{F})$ for a wider class. This is the content of the main theorem:

Theorem 1.1 *Let q be a power of 2. Then there is a q^6-cover of $H_3(\mathbb{F}_q)$ which is locally $H_2(\mathbb{F}_q)$ and whose automorphism group contains an extension of $\mathrm{SL}(4, q)$ by an irreducible 6-dimensional module. This extension is split only if $q = 2$.*

The existence part of the proof (see Theorem 3.6) is based on a construction developed in Sections 2 and 3. These sections are based on the second author's Masters' thesis [8].

In Section 5 we find automorphisms of this covering graph, generating an extension of $\mathrm{SL}_4(\mathbb{F})$ by $\mathbb{F}^6$, which is non-split if $\mathbb{F} \neq \mathbb{F}_2$ (see Theorem 5). Since our module is, up to a field twist, the second exterior wedge of the natural module, by the Klein correspondence we are dealing with an extension of $O^+(6, q)$ by its natural module. Therefore, the non-split extension is the one found by Griess [5], Sah [9], and Bell [1].

1.1. Notation and conventions

We let groups act on the right. All graphs are simple and undirected. The adjacency of vertices u and v is denoted by $u \perp v$. For a graph Γ, we let $V(\Gamma)$ be the set of its vertices and $D(\Gamma)$ be the set of its *darts* or *oriented* edges; that is, the set of ordered pairs of vertices (u, v) for which $u \perp v$.

2. Voltage assignments

In this section, we discuss a general method of constructing covers of a given graph by means of voltage assignments. For a general introduction to voltage assignments, see Malnič et al. [7].

For all vertices v of a graph, we call the induced graph on the neighbourhood of v the *local graph* at v. Let Γ and Δ be two connected graphs. If a map $\alpha: \Gamma \to \Delta$ preserves adjacency and if α maps the local graph at every vertex of Γ isomorphically to the local graph at its image, we call Γ a *geometric cover* of Δ.

Let N be a group. A map $\ell: D(\Delta) \to N$ such that $\ell(u, v) = \ell(v, u)^{-1}$ is called a *voltage assignment* of Δ. We will often write $\ell_{u,v}$ for $\ell(u, v)$, or $\ell_{i,j}$ if $u = v_i$ and $v = v_j$. The *lift* of Δ with respect to ℓ is the graph with vertex set $V(\Delta) \times N$, where $(u, m) \perp (v, n)$ if and only if $u \perp v$ and $\ell(u, v) = mn^{-1}$. So N acts as an automorphism group on the lift of Δ as $(v, n)^k = (v, nk)$, where $v \in V(\Delta)$ and $n, k \in N$.

Given a walk $P = (v_0, v_1, \ldots, v_i)$, where $v_i \perp v_{i+1}$, we call $\ell_{0,1}\ell_{1,2} \ldots \ell_{n-1,n}$ the *voltage* of P, denoted by $\ell(P)$. Using induction it is immediate that for any $m \in N$, there exists exactly one walk in the lift from (v_0, m) to $(v_n, \ell(P)^{-1}m)$ that projects to P.

We start with two observations from Gross and Tucker [6]. Let Δ be a connected graph with voltage assignment $\ell: D(\Delta) \to N$. Let Γ be the lift of Δ with respect to ℓ. Then Γ is connected if and only if for every $n \in N$ and every $v_0 \in V(\Delta)$, there is an $i \in \mathbb{N}$ and a closed walk $(v_0, v_1, \ldots, v_i = v_0)$ such that $\ell_{0,1}\ell_{1,2} \ldots \ell_{i-1,i} = n$. The proof is easy, and so is the proof of the following lemma.

Lemma 2.1 *Let Δ be a connected graph with voltage assignment $\ell: D(\Delta) \to N$. Let Γ be the lift of Δ with respect to ℓ. For all $n \in N$ and $v \in V(\Delta)$, the local graph at (v, n) in Γ is isomorphic to the local graph at v in Δ, if and only if for every triangle u, v, w of Δ, we have $\ell_{u,v}\ell_{v,w}\ell_{w,u} = 1$.*

These two observations lead to the following straightforward lemma.

Lemma 2.2 *Let Δ be a connected graph with voltage assignment $\ell': D(\Delta) \to N'$. Let T be the normal closure of the subgroup of N' generated by the voltages of all triangles.*

Let N be the quotient of N' by T and let $\ell_{u,v} = T\ell'_{u,v}$. Let M be the subgroup of N generated by the voltages (with respect to ℓ) of all closed walks. Let Γ be the lift of Δ with respect to ℓ.

Then by the map $(v, n) \mapsto v$, each connected component of Γ is an $|M|$-fold geometric cover of Δ.

Let G be a group of automorphisms of Δ with an action on N. We will say that ℓ is *G-equivariant* if and only if for all $g \in G$ and $v \perp w \in \Delta$, we have that $\ell_{v^g,w^g} = (\ell_{v,w})^g$.

Group-equivariant voltage assignments enable the group to lift to a group of automorphisms of the lift. This is the content of the next lemma, the proof of which is again straightforward. It occurs as Proposition 19.3 in Biggs [2]. Recall that multiplication on $G \ltimes N$ is defined by $(g,k)(g',k') = (gg', k^{g'}k)$.

Lemma 2.3 *Let G be a subgroup of $\mathrm{Aut}\Delta$ such that ℓ is G-equivariant. Let Γ be the lift of Δ with respect to ℓ. Then $G \ltimes N$ acts faithfully on Γ by the action $(v,n)^{(g,k)} = (v^g, n^g k)$.*

Now suppose we have the setup of Lemma 2.2. Let M be Abelian and let ℓ be G-equivariant. Choose a vertex $v \in V(\Delta)$. For all $g \in G$, choose a walk P_g from v^g to v, and let $\lambda(g)$ be the voltage of P_g. Choose P_1 such that $\lambda(1) = 0$. Then the following lemma holds.

Lemma 2.4 *The stabilizer in $G \ltimes N$ of the connected component Γ_0 of Γ containing $(v,0)$ is $H = \{(g, \lambda(g)+m) \mid g \in G, m \in M\}$, which is an extension of G by M.*

Proof: Since $(v,0)^{(g,\lambda(g)+m)} = (v^g, \lambda(g)+m)$ and since the walk in Γ starting at $(v^g, \lambda(g)+m)$ and projecting down to P_g ends at (v,m), we have that H stabilizes Γ_0. Conversely, if an element (g,n) stabilizes Γ_0, it maps $(v,0)$ to an element (v^g,n) such that there is a walk from $(v^g, \lambda(g))$ to (v^g, n). Then the projection of that walk down to Δ is a closed walk; hence $\lambda(g) - n \in M$. So H is the full stabilizer of Γ_0.

The kernel of the projection onto the first coordinate is $\{1\} \times M$, so that is a normal subgroup. The quotient by that subgroup is G. $\square$

3. SL(V)-modules

We recall some multilinear algebra in order to be able to construct the voltage assignment in the next section.

Consider the projective geometry $\mathbb{P}_n(\mathbb{F})$ of (projective) dimension n over the field $\mathbb{F}$. We denote incidence by $\subset$ and the projective dimension by dim. Furthermore, V will be the vector space $\mathbb{F}^4$ with basis $e_1, \ldots, e_4$ and dual basis $f_1, \ldots, f_4$, so $\mathbb{P}_3(\mathbb{F}) = \mathbb{P}(V)$. Let

$$\bigwedge\nolimits^k V = V^{\otimes k} / \langle v_1 \otimes v_2 \otimes \cdots \otimes v_k \mid v_i = v_j \text{ for some } i \neq j \rangle$$

be the kth Grassmannian of V. The image of $v_1 \otimes \cdots \otimes v_k$ in $\bigwedge^k V$ is denoted $v_1 \wedge \cdots \wedge v_k$. Let G be a group with a linear action on V; this induces a natural action on $\bigwedge^k V$. Now $G \leq \mathrm{SL}(V)$ if and only if G stabilizes every element of $\bigwedge^4 V$. We will mostly be using the case where $k = 2$. We need the following elementary lemmas.

Lemma 3.1 *There is a canonical isomorphism from $(\bigwedge^2 V)^*$ to $\bigwedge^2 (V^*)$ that preserves the induced action of* GL(V).

Sketch of Proof: Let $B^*: \bigwedge^2(V^*) \times \bigwedge^2 V \to \mathbb{F}$ be defined for $\hat{h} = h_1 \wedge h_2 \in \bigwedge^2(V^*)$ and $\hat{v} = v_1 \wedge v_2 \in \bigwedge^2 V$ by

$$B^*(\hat{h}, \hat{v}) = h_1(v_1)h_2(v_2) - h_1(v_2)h_2(v_1),$$

and extended bilinearly. As B^* is nondegenerate, $\hat{h} \mapsto (\hat{v} \mapsto B^*(\hat{h}, \hat{v}))$ is an isomorphism $(\bigwedge^2 V)^* \to \bigwedge^2(V^*)$ respecting the induced action of $\mathrm{GL}(V)$. □

Because of the preceding lemma we can drop the parentheses in the future and write $\bigwedge^2 V^*$. Fix an isomorphism $\chi: \bigwedge^4 V \to \mathbb{F}$.

Lemma 3.2 *Let V be a vector space of dimension 4 over a field $\mathbb{F}$. Then there is a canonical isomorphism $\psi: \bigwedge^2 V \to \bigwedge^2 V^*$ with inverse ϕ, which respects the natural induced group actions of $\mathrm{SL}_4(\mathbb{F})$ on $\bigwedge^2 V$ and $\bigwedge^2 V^*$.*

Sketch of Proof: We define $B: \bigwedge^2 V \times \bigwedge^2 V \to \mathbb{F}$ as follows:

$$B(v_1 \wedge v_2, w_1 \wedge w_2) = (v_1 \wedge v_2 \wedge w_1 \wedge w_2)^{\chi},$$

and extended by linearity. Then B is nondegenerate. Now let ψ map $\hat{w} \in \bigwedge^2 V$ to the linear functional that maps $\hat{v} \in \bigwedge^2 V$ to $B(\hat{v}, \hat{w})$. Then ψ is a vector space isomorphism. □

Whenever we consider it appropriate, we will omit ψ and ϕ.

Lemma 3.3 *Let h_1, h_2 be linearly independent elements of V^* and let v_1, v_2 be linearly independent elements of V such that $h_i(v_j) = 0$. Then $(v_1 \wedge v_2)^{\psi} = \alpha h_1 \wedge h_2$ for some $\alpha \in \mathbb{F}$.*

Proof: Let $K = \mathrm{Ker} h_1 \cap \mathrm{Ker} h_2 = \langle v_1, v_2 \rangle$. Put $\hat{v} = (v_1 \wedge v_2)^{\psi}$ and $\hat{h} = h_1 \wedge h_2$.

Let $w_1, w_2 \in V$ and write $\hat{w} = w_1 \wedge w_2$. We will first show that $\hat{h}(\hat{w}) = 0$ precisely if $\hat{v}(\hat{w}) = 0$. We may assume $\hat{w} \neq 0$. We can move w_1 to any projective point on the projective line $\langle w_1, w_2 \rangle$ keeping the same value for $\hat{w}$ by either switching w_1 and w_2 or replacing w_1 by $w_1 + r w_2$ for some field element r. So if $\langle w_1, w_2 \rangle$ intersects K, then we may assume that the point of intersection is w_1. Then

$$\hat{v}(\hat{w}) = v_1 \wedge v_2 \wedge w_1 \wedge w_2 = 0,$$
$$\hat{h}(\hat{w}) = h_1(w_1)h_2(w_2) - h_1(w_2)h_2(w_1) = 0.$$

Otherwise, $\langle v_1, v_2, w_1, w_2 \rangle = V$ so $\hat{v}(\hat{w}) \neq 0$. So $\hat{v}(\hat{w}) = 0$ precisely if $\langle w_1, w_2 \rangle$ intersects K, and in that case we also have $\hat{h}(\hat{w}) = 0$.

Now suppose $\hat{h}(\hat{w}) = 0$. Then $h_1(w_1)h_2(w_2) = h_1(w_2)h_2(w_1)$. Let $w = f_1(w_2)w_1 - f_1(w_1)w_2$. Then

$$f_1(w) = f_1(w_2)f_1(w_1) - f_1(w_1)f_1(w_2) = 0,$$
$$f_2(w) = f_1(w_2)f_2(w_1) - f_1(w_1)f_2(w_2) = 0.$$

So again $\hat{h}(\hat{w}) = 0$ precisely if $\langle w_1, w_2 \rangle$ intersects K.

Now let $x_1, x_2 \in V$ with $\hat{x} = x_1 \wedge x_2$. We will show that $\hat{v}(\hat{w})\hat{h}(\hat{x}) = \hat{v}(\hat{x})\hat{h}(\hat{w})$. We may assume that $\hat{h}$ and $\hat{v}$ are nonzero on both $\hat{w}$ and $\hat{x}$. So $\langle w_1, w_2\rangle$ intersects Kerh_1 and Kerh_2 in distinct projective points. We may assume that these intersection points are $\langle w_1\rangle$ and $\langle w_2\rangle$, respectively, so $h_i(w_i) = 0$. Similarly we may assume $h_i(x_i) = 0$.

Now $\langle v_1\rangle$, $\langle v_2\rangle$, $\langle w_i\rangle$ and $\langle x_i\rangle$ are four projective points in the hyperplane Kerh_i, so we write $w_i = \alpha_{i,1}v_1 + \alpha_{i,2}v_2 + \alpha_{i,3}x_i$. Then

$$\hat{v}(\hat{w})\hat{h}(\hat{x}) = -\alpha_{1,3}\alpha_{2,3}(v_1 \wedge v_2 \wedge x_1 \wedge x_2)(h_1(x_2)h_2(x_1))$$

and

$$\hat{v}(\hat{x})\hat{h}(\hat{w}) = -\alpha_{1,3}\alpha_{2,3}(v_1 \wedge v_2 \wedge x_1 \wedge x_2)(h_1(x_2)h_2(x_1)).$$

It follows that $\hat{v}$ and $\hat{h}$ differ by the same factor on all elements of shape $w_1 \wedge w_2$, and therefore on all of $\bigwedge^2 V$. □

For an arbitrary vector space Y, we let

$$S_2(Y) = (Y \otimes Y)/\langle v \otimes w - w \otimes v \mid v, w \in Y\rangle$$

be the second order symmetric tensor of Y. Then the natural action of SL(Y) on $Y \otimes Y$ induces a natural action on $S_2(Y)$. We denote the image of $v \otimes w$ in $S_2(Y)$ by vw. We will often write w^2 for ww.

Now let char $\mathbb{F} = 2$, and let $W = \bigwedge^2 V = \bigwedge^2(V^*)$ of dimension 6. Then $S_2(W)$ has dimension 21. The subspace $W^{(2)}$ of $S_2(W)$, defined as

$$W^{(2)} = \langle \hat{w}^2 \mid \hat{w} \in W\rangle,$$

has dimension 6 and is invariant under the induced action of GL(V).

Lemma 3.4 *Let $w, x, y, z \in V$ be such that $w \wedge x \wedge y \wedge z = 1$. Then the vector*

$$U = (w \wedge x)(y \wedge z) + (w \wedge y)(z \wedge x) + (w \wedge z)(x \wedge y)$$

does not depend on the choice of w, x, y, z and is fixed by SL(V).

Proof: The map

$$\Delta\colon (w, x, y, z) \mapsto (w \wedge x)(y \wedge z) + (w \wedge y)(z \wedge x) + (w \wedge z)(x \wedge y)$$

is 4-linear and alternating. There is only one such map, up to scalar multiples: the determinant. Hence for tuples of vectors such that det$(w, x, y, z) = w \wedge x \wedge y \wedge z = 1$, we find that Δ must be constant.

Since the image of $\Delta(w, x, y, z)$ under an element of SL(V) is $\Delta(w', x', y', z')$ for some tuple satisfying $w' \wedge x' \wedge y' \wedge z' = 1$, the element U is fixed by SL(V). □

4. The graph and its voltage assignment

Following Gramlich [4], we define the graph $H_3(\mathbb{F})$ to have vertex set

$$\{(x, X) \mid x, X \in \mathbb{P}_3(\mathbb{F}), \dim x = 0, \dim X = 2, x \not\subset X\}$$

and adjacency defined by

$$(x, X) \perp (y, Y) \Leftrightarrow x \subset Y \quad \text{and} \quad y \subset X.$$

We require char $\mathbb{F} = 2$ and we retain V, W, $W^{(2)}$ and U as in the previous section.

The computer algebra computations of Gramlich [4] indicated that it might be possible to find a 6-dimensional module to extend $\mathrm{SL}_4(\mathbb{F}_2)$ with and obtain the automorphism group of the cover of $H_3(\mathbb{F}_2)$. It seemed natural that this module would be $\bigwedge^2 V$. On the other hand, the vertices of the graph could be modelled as projective points in $V \otimes V^*$ with edges corresponding to projective points in $S_2(V \otimes V^*)$. By the composition of natural maps

$$S_2(V \otimes V^*) \to \bigwedge^2 V \otimes \bigwedge^2 V^* = \bigwedge^2 V \otimes \bigwedge^2 V \to S_2(\bigwedge^2 V),$$

we could map an edge into an $\mathrm{SL}_4(\mathbb{F})$-module containing a twisted copy of $\bigwedge^2 V$, viz. $W^{(2)}$. The composition of these maps gives the setting for our voltage assignment.

We will often represent a vertex (x, X) of $H_3(\mathbb{F})$ by a pair (v, h) of a nonzero vector v in x and a functional h with kernel X. Let $(v_1, h_1) \perp (v_2, h_2)$ be two adjacent vertices in $H_3(\mathbb{F})$. We let $\ell\colon D(H_3(\mathbb{F})) \to S_2(W)$ assign the voltage

$$h_1(v_1)^{-1} h_2(v_2)^{-1} (v_1 \wedge v_2)(h_1 \wedge h_2)^{\phi} \tag{1}$$

to the dart from (v_1, h_1) to (v_2, h_2). Note that this is independent of the representatives v_i and h_i. We will often choose v and h such that $h(v) = 1$.

We will sometimes regard $S_2(W)$ as a group only, so the subgroups are the subspaces over $\mathbb{F}_2$—not necessarily over $\mathbb{F}$. We denote the $\mathbb{F}_2$-linear span of $v_0, \ldots, v_k$ by $\langle v_0, \ldots, v_k \rangle_{\mathbb{F}_2}$. Let $\ell^U\colon D(H_3(\mathbb{F})) \to S_2(W)/\langle U \rangle_{\mathbb{F}_2}$ be the composition of ℓ with the natural projection of $S_2(W)$ to $S_2(W)/\langle U \rangle_{\mathbb{F}_2}$. Note that both ℓ and ℓ^U are $\mathrm{SL}_4(\mathbb{F})$-equivariant.

The following theorem shows the existence of a q^6-fold geometric cover of $H_3(\mathbb{F}_q)$ if q is even, and so provides infinitely many counterexamples to the extension of Theorem 1.3.21 of Gramlich [4] for $H_{n+1}(\mathbb{F}_q)$ to $n = 2$. This proves the existence part of Theorem 1.1.

Theorem 4.1 *Let char $\mathbb{F} = 2$. Let Γ be the lift of $H_3(\mathbb{F})$ with respect to ℓ^U. Then every connected component of Γ is an $|\mathbb{F}^6|$-fold geometric cover of $H_3(\mathbb{F})$.*

For proving Theorem 4.1. we need some auxiliary lemmas which are of interest in their own right, since they provide information on the geometric covers of $H_3(\mathbb{F})$ in general. We will use the words triangle, quadrangle and pentagon to mean closed walks of the obvious lengths consisting of different vertices. A quadrangle will be called *special of type A* if the number of distinct (projective) points, or the number of distinct hyperplanes, occurring in its vertices, is two; it will be called *special of type B* if the number of distinct points and the number of distinct hyperplanes occurring are both three.

Consider the following chain complex over $\mathbb{F}_2$. We let C_0 and C_1 be the free modules spanned by the vertices and edges of $H_3(\mathbb{F})$, respectively. The boundary map $\partial_1 : C_1 \to C_0$ maps an edge to the sum of its vertices. We let C_2 be the trivial module and therefore ∂_2 is the zero map. In this manner we have an explicit description of the homology group $H_1 = H_1(C_*, \mathbb{F}_2) = \ker \partial_1$. Let Q be the submodule of H_1 spanned by triangles and special quadrangles of type A. In Proposition 4.4 we will show that each closed walk is an element of Q; that is, $Q = H_1$.

Lemma 4.2 *The sum of edges of a quadrangle is in Q.*

Proof: Consider the quadrangle $(v_0, h_0), \ldots, (v_3, h_3)$. We may assume that these representatives have been chosen such that $h_i(v_i) = 1$. Suppose that for some i, we have $h_i(v_{i+2}) = h_{i+2}(v_i) = 0$; say for $i = 0$. Then the quadrangle consists of two triangles as depicted in figure 1(a).

Now consider a quadrangle where for all i either $h_i(v_{i+2})$ or $h_{i+2}(v_i)$ is nonzero; we may assume that $h_0(v_2) = h_1(v_3) = 1$. Then we can split the quadrangle into four special quadrangles by adding the vertices (v_2, h_0) and (v_3, h_1), as depicted in figure 1(b) and specified in the table below.

Vertices				Type
(v_0, h_0)	(v_1, h_1)	(v_2, h_0)	(v_3, h_1)	A
(v_1, h_1)	(v_2, h_2)	(v_3, h_1)	(v_2, h_0)	B
(v_2, h_2)	(v_3, h_3)	(v_2, h_0)	(v_3, h_1)	A
(v_3, h_3)	(v_0, h_0)	(v_3, h_1)	(v_2, h_0)	B

If the added vertices coincide with vertices of the quadrangle, then instead the quadrangle is split into two special quadrangles or the quadrangle itself is special.

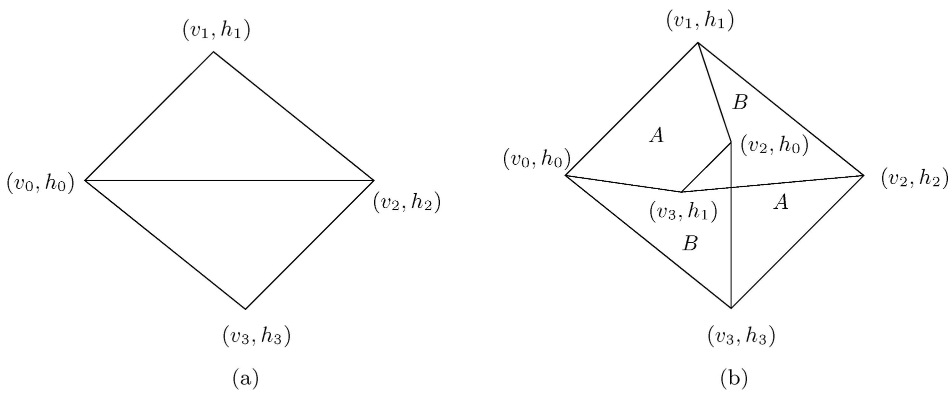

Figure 1. (a) A quadrangle consisting of two triangles. (b) Splitting a quadrangle into four special quadrangles.

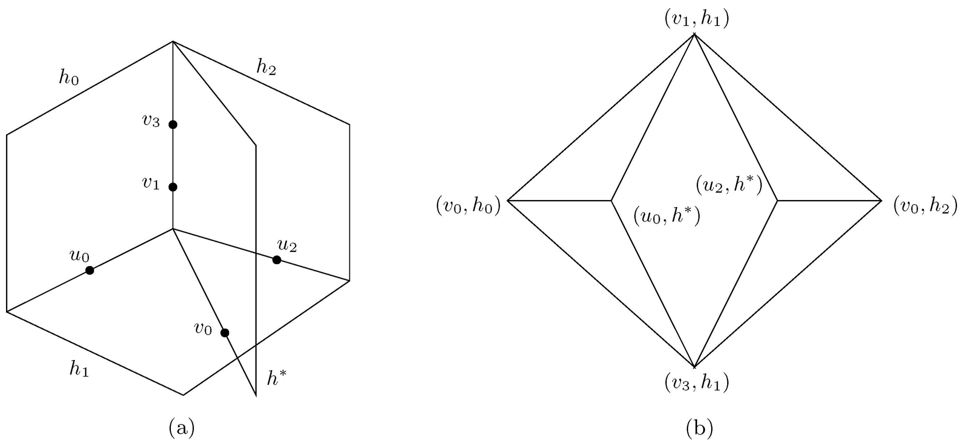

Figure 2. (a) Finding the vertices (u_0, h^*) and (u_2, h^*). (b) A quadrangle of type B is a quadrangle of type A plus four triangles.

So let us consider a special quadrangle of type B. It may be assumed to have vertices $(v_0, h_0), (v_1, h_1), (v_0, h_2), (v_3, h_1)$. The projective geometric relations between v_i and h_j are depicted in figure 2(a). Let h^* be the hyperplane containing v_0, v_1 and v_3, and pick two points u_0 and u_2 such that u_i is in the intersection of h_i and h_1, but not in h^*. Then (u_i, h^*) are vertices of $H_3(\mathbb{F})$, and their neighbours are as depicted in figure 2(b). We see that a quadrangle of type B is the sum of a quadrangle of type A and four triangles. □

Lemma 4.3 *The sum of edges of a pentagon is in Q.*

Proof: Let us take a pentagon $(v_0, h_0), \ldots, (v_4, h_4)$. We choose the representatives such that $h_i(v_i) = 1$ for all i.

Choose an index i and consider the index set $\{i-2, i, i+2\}$ (modulo 5). Now suppose that the common null space $\mathcal{N}_i$ of h_{i-2}, h_i and h_{i+2} is *not* contained in $\mathcal{V}_i = \langle v_{i-2}, v_i, v_{i+2}\rangle$. Then take some $v \in \mathcal{N}_i \setminus \mathcal{V}_i$, and some $h \in V^*$ such that h is zero on $\mathcal{V}_i$, but not on v. Then the vertex (v, h) is adjacent to (v_{i-2}, h_{i-2}), (v_i, h_i) and (v_{i+2}, h_{i+2}). Hence the pentagon is the sum of two quadrangles and a triangle, as in figure 3.

Now suppose that for all indices i, we have that $\mathcal{N}_i \subseteq \mathcal{V}_i$. Since $\mathcal{N}_i$ has a positive vector space dimension, there is a nonzero vector v in $\mathcal{V}_i$ on which h_{i-2}, h_i and h_{i+2} are all zero. Let

$$v = \lambda_{i-2} v_{i-2} + \lambda_i v_i + \lambda_{i+2} v_{i+2}.$$

If $\lambda_i = 0$, then also

$$0 = h_{i\pm 2}(\lambda_{i\pm 2} v_{i\pm 2} + \lambda_{i\mp 2} v_{i\mp 2}) = \lambda_{i\pm 2},$$

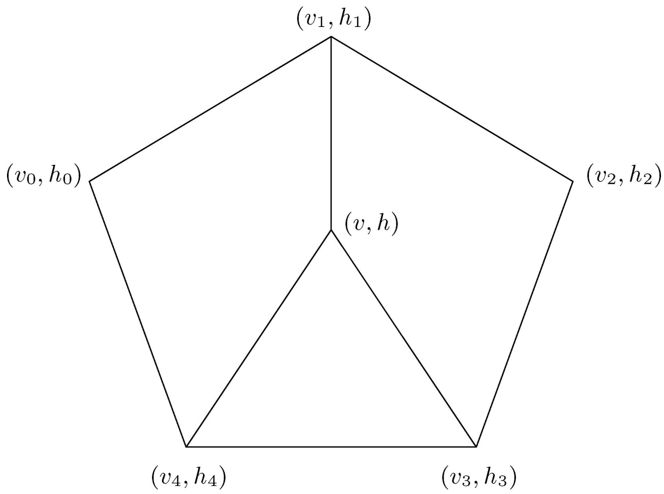

Figure 3. Splitting a pentagon into a triangle and two quadrangles.

contradicting $v \neq 0$. So we may assume $\lambda_i = 1$. Then

$$\begin{aligned} 0 = h_i(v) &= \lambda_{i-2} h_i(v_{i-2}) + 1 + \lambda_{i+2} h_i(v_{i+2}); \\ 0 = h_{i-2}(v) &= \lambda_{i-2} + h_{i-2}(v_i); \\ 0 = h_{i+2}(v) &= \lambda_{i+2} + h_{i+2}(v_i). \end{aligned}$$

So we find $\lambda_{i\pm 2} = h_{i\pm 2}(v_i)$ and hence

$$h_{i-2}(v_i) h_i(v_{i-2}) + h_{i+2}(v_i) h_i(v_{i+2}) = 1. \tag{2}$$

If we sum Eq. (2) over all i the right hand side is 1. But every term on the left hand side occurs twice, so the left hand side is 0. Contradiction. □

Proposition 4.4 $Q = H_1$.

Proof: It is sufficient to show that the sum of edges of any closed walk is in Q. Lemmas 4.2 and 4.3 tell us that the statement holds for all closed walks of length at most 5. Let $c = (v_0, v_1, \ldots, v_n = v_0)$ be a shortest closed walk not in Q; so $n \geq 6$. By Lemma 1.3.5 of Gramlich [4], the diameter of $H_3(\mathbb{F})$ is two, so there is a path of length at most 2 from v_0 to v_3. Let us call this path P. This gives us two new closed walks: the first one is formed by $v_0, \ldots, v_3$, followed by the reverse of P—the length of which is at most 5, whence it is in Q; the second one is formed by P, followed by $v_4, \ldots, v_n$—the length of which is at most $n - 1$, whence it is also in Q. Therefore c is also in Q. Contradiction. □

Note that this proposition can also be shown to hold for homology over $\mathbb{Z}$.

Lemma 4.5 *The voltage of a triangle with respect to ℓ is U.*

Proof: Let (v_1, h_1), (v_2, h_2), (v_3, h_3) be a triangle in $H_3(\mathbb{F})$. We assume $h_i(v_i) = 1$. Then $\{v_i\}$ and $\{h_i\}$ are both linearly independent sets. Hence the intersection of the null spaces of $\{h_i\}$ has dimension 1; choose u nonzero in it. Then $\{v_1, v_2, v_3, u\}$ form a basis for V. We may assume that $v_1 \wedge v_2 \wedge v_3 \wedge u = 1$.

Since h_1 and h_2 both vanish on v_3 and u, we have $\alpha(h_1 \wedge h_2)^\phi = v_3 \wedge u$ for some nonzero $\alpha \in \mathbb{F}$ by Lemma 3.3. Now $1 = v_1 \wedge v_2 \wedge v_3 \wedge u = \alpha(h_1(v_1)h_2(v_2) + h_1(v_2)h_2(v_1)) = \alpha$. Hence $(h_1 \wedge h_2)^\phi = v_3 \wedge u$; similarly we obtain $(h_1 \wedge h_3)^\phi = v_2 \wedge u$ and $(h_2 \wedge h_3)^\phi = v_1 \wedge u$. So if $\{i, j, k\} = \{1, 2, 3\}$, then the voltage of the dart from (v_i, h_i) to (v_j, h_j) is $(v_i \wedge v_j)(v_k \wedge u)$. The sum of the voltages is then

$$(v_1 \wedge v_2)(v_3 \wedge u) + (v_1 \wedge v_3)(v_2 \wedge u) + (v_2 \wedge v_3)(v_1 \wedge u) = U.$$

□

Lemma 4.6 *The voltage of a closed walk in $H_3(\mathbb{F})$ with respect to ℓ is in $W^{(2)} \oplus \langle U \rangle_{\mathbb{F}_2}$.*

Proof: By Proposition 4.4, the voltage of any closed walk can be written as the sum of voltages of triangles and special quadrangles of type A. By Lemma 4.5, the voltage of a triangle is U.

Now consider a special quadrangle of type A. We may, by duality, assume its vertices are (v_0, h_0), (v_1, h_1), (v_0, h_2) and (v_1, h_3) with $h_i(v_i) = h_{i+2}(v_i) = 1$. By Lemma 3.3, its voltage is

$$\begin{aligned}&(v_0 \wedge v_1)(h_0 \wedge h_1 + h_1 \wedge h_2 + h_2 \wedge h_3 + h_3 \wedge h_0)^\phi \\ &\quad = (v_0 \wedge v_1)((h_0 + h_2) \wedge (h_1 + h_3))^\phi = \alpha(v_0 \wedge v_1)^2 \in W^{(2)},\end{aligned}$$

where α is some field element. □

Lemma 4.7 *For all $w \in W^{(2)}$ there is a closed walk in $H_3(\mathbb{F})$ with voltage w with respect to ℓ.*

Proof: It is sufficient to show that a set of generators of $W^{(2)}$ occurs as voltages of closed walks. Note that an $\mathbb{F}$-basis is not necessarily sufficient—we need an $\mathbb{F}_2$-basis.

Consider the special quadrangle of type A with vertices (e_3, f_3), (e_4, f_4), $(e_3, \lambda f_2 + f_3)$ and $(e_4, f_1 + f_4)$. Its voltage is $\lambda e_{1,2}^2$. By permuting the base vectors and by choosing different values for λ, we obtain an $\mathbb{F}_2$-basis for $W^{(2)}$. □

Proof of Theorem 4.1: We apply Lemma 2.2. Lemma 4.5 gives us $T = \langle U \rangle_{\mathbb{F}_2}$; then Lemmas 4.6 and 4.7 imply that $M = W^{(2)}$. Hence every connected component of Γ is a $|W^{(2)}|$-fold geometric cover of Δ. Since $W^{(2)} \cong \mathbb{F}^6$, we have finished the proof. □

Notice that Γ is a cover of $H_3(\mathbb{F})$ in the sense of 2-dimensional simplicial complexes whose 2-simplices are the triangles of the graphs. Since closed walks in Γ correspond to closed walks in $H_3(\mathbb{F})$ with voltage 0, the cover is simply connected if and only if every closed walk of voltage 0 in $H_3(\mathbb{F})$ is a sum of triangles. We conjecture that Γ is not simply connected in this sense. For $\mathbb{F} = \mathbb{F}_2$ this is true because a computer computation shows that the walk with the vertices in the list below and with voltage 0 is not a sum of triangles in that case:

$$(e_1, f_1), \quad (e_2, f_2 + f_4), \quad (e_1, f_1 + f_3), \quad (e_2, f_2), \quad (e_1 + e_3, f_1), \quad (e_2 + e_4, f_2),$$

where $\{e_i\}$ and $\{f_i\}$ are bases of V and V^*, respectively.

5. A group of automorphisms

Let Γ be the graph of Theorem 4.1, so char $\mathbb{F} = 2$. Set $N = S_2(W)/\langle U\rangle_{\mathbb{F}_2}$ and $M = (W^{(2)} + \langle U\rangle_{\mathbb{F}_2})/\langle U\rangle_{\mathbb{F}_2}$. When writing elements of N and M, we will often omit the added $\langle U\rangle_{\mathbb{F}_2}$. The group $\mathrm{SL}_4(\mathbb{F})$ acts on $H_3(\mathbb{F})$ as follows. A group element g maps a vertex (v, h) to (v^g, h^g), where $h^g(w) = h(w^{g^{-1}})$. According to Lemma 2.3, the group $\mathrm{SL}_4(\mathbb{F}) \ltimes N$ acts on Γ. By Lemma 2.4 and the results of Section 4, an extension E of $\mathrm{SL}_4(\mathbb{F})$ by M acts on a connected component of Γ. The content of Theorem 5.1. below is that this extension is nonsplit unless $|\mathbb{F}| = 2$. The existence of this nonsplit extension was known by Bell [1], Griess [5], and Sah [9]. The theorem proves the automorphism group part of Theorem 1.1.

An extension of a group by an Abelian group corresponds to a 2-cocycle in the standard chain complex of the group that is being extended. The extension is nonsplit exactly if the cocycle is not a 2-coboundary, see Brown [3]. In this section we find an explicit cocycle that defines this extension.

We let i and π denote the natural maps in the following diagram:

By Brown [3], a 2-cocycle is determined by a section $s\colon \mathrm{SL}_4(\mathbb{F}) \to E$ of π. It is a map $f\colon G \times G \to M$ such that

$$s(g)s(h) = s(gh)i(f(g, h)), \qquad f(g, 1) = f(1, g) = 0. \tag{3}$$

This is the right-action version of (3.3.3) of [3]. The group law on the set $\mathrm{SL}_4(\mathbb{F}) \times M$ that makes it into a group isomorphic with E is

$$[g_1, m_1][g_2, m_2] = \left[g_1 g_2, m_1^{g_1} + m_2 + f(g_1, g_2)\right]. \tag{4}$$

So in order to describe the cocycle, we need to define the section s. The elements of E are most easily described as elements of $\mathrm{SL}_4(\mathbb{F}) \ltimes N$. Therefore we construct a map $\lambda\colon \mathrm{SL}_4(\mathbb{F}) \to N$ such that $s(g) = (g, \lambda(g)) \in E$ and $\lambda(1) = 0$. Then f is determined by Eq. (3) as

$$f(g, h) = -\lambda(gh) + \lambda(g)^h + \lambda(h).$$

We can choose λ as in Lemma 2.4. The construction of λ is then coordinate-dependent. In order to compute it, choose a basis $\{e_i\}$ for $\mathbb{F}^4$ and a dual basis $\{f_i\}$. We need to fix a vertex of $H_3(\mathbb{F})$, say (e_1, f_1), and then for each $g \in \mathrm{SL}_4(\mathbb{F})$ we need to choose a walk from (e_1^g, f_1^g) to (e_1, f_1). The voltage along this walk is then $\lambda(g)$.

Theorem 5.1 *Let char* $\mathbb{F} = 2$ *and* $|\mathbb{F}| > 2$. *Then the stabilizer* E *in* $\mathrm{SL}_4(\mathbb{F}) \ltimes N$ *of a connected component is a non-split extension of* $\mathrm{SL}_4(\mathbb{F})$ *by* $\mathbb{F}^6$.

Proof: Let $\alpha \in \mathbb{F}$ be an element outside the ground field. Let F denote the additive subgroup $\langle 1, \alpha \rangle_{\mathbb{F}_2}$ of $\mathbb{F}$ of order 4. For $x \in F$, let A_x be the element of $\mathrm{SL}_4(\mathbb{F})$) fixing e_2 and e_4, and mapping e_1 and e_3 to $e_1 + xe_2$ and $e_3 + xe_4$, respectively. We will show that the subgroup $A = \{A_x \mid x \in F\}$ does not lift to a subgroup of E.

We define a basis for W, and write down the images under A_x:

$$
\begin{aligned}
w_1 &= e_1 \wedge e_2 = (f_3 \wedge f_4)^\phi \mapsto w_1;\\
w_2 &= e_1 \wedge e_3 = (f_2 \wedge f_4)^\phi \mapsto w_2 + xw_3 + xw_4 + x^2 w_5;\\
w_3 &= e_1 \wedge e_4 = (f_2 \wedge f_3)^\phi \mapsto w_3 + xw_5;\\
w_4 &= e_2 \wedge e_3 = (f_1 \wedge f_4)^\phi \mapsto w_4 + xw_5;\\
w_5 &= e_2 \wedge e_4 = (f_1 \wedge f_3)^\phi \mapsto w_5;\\
w_6 &= e_3 \wedge e_4 = (f_1 \wedge f_2)^\phi \mapsto w_6.
\end{aligned}
$$

Let $v_x = (e_1 + xe_2, f_1)$ and let $u = (e_3, f_3)$. Then $v_x \perp u$ for all $x \in F$. The voltage along the dart between u and v_x is

$$((e_1 + xe_2) \wedge e_3)(f_1 \wedge f_3) = w_2 w_5 + x w_4 w_5.$$

We choose λ as in Lemma 2.4 and take $v = v_0$, so $v^{A_x} = v_x$. For all $x \in F$, we choose a walk P_{A_x} as (v_x, u, v). Then $\lambda(A_x) = xw_4w_5$. Hence

$$f(A_x, A_y) = \lambda(A_{x+y}) + \lambda(A_x)^{A_y} + \lambda(A_y) = xyw_5^2 \in M.$$

Now suppose that A lifts to a subgroup of E, that is, there is a function $c: F \to M$ such that $\{[x, c(x)] \mid x \in F\}$ with multiplication as in Eq. (4) is a group isomorphic to A. Then $[1, c(1)]$, $[\alpha, c(\alpha)]$, and $[\alpha + 1, c(\alpha + 1)]$ need to have order 2.

Now $[x, m]^2 = [0, x^2 w_5^2 + m^{A_x} + m]$, so $[x, m]$ has order two if and only if $m^{A_x} + m = x^2 w_5^2$. By elementary linear algebra we find that this is true for $x \neq 0$ if and only if $m \in w_3^2 + S$, where $S = \langle w_1^2, w_3^2 + w_4^2, w_5^2, w_6^2 \rangle_{\mathbb{F}}$. Note that S is A-invariant.

Let $s(x) = w_3^2 + c(x)$. Then $s(x) \in S$. Since $[1, c(1)][\alpha, c(\alpha)] = [\alpha + 1, c(\alpha + 1)]$, we have $s(\alpha + 1) = w_3^2 + \alpha w_5^2 + c(1)^{A_\alpha} + c(\alpha) = w_3^2 + (\alpha + \alpha^2) w_5^2 + s(1) + s(\alpha) \notin S$, a contradiction.

Since A does not lift to a subgroup of E, neither does $\mathrm{SL}_4(\mathbb{F})$. In other words, the extension of $\mathrm{SL}_4(\mathbb{F})$ by M is non-split. □

References

1. G. W. Bell, "On the cohomology of the finite special linear groups," *J. Algebra* **54** (1978), 216–259.
2. N. Biggs, *Algebraic Graph Theory, Cambridge*, Cambridge University Press, 1974.

3. K.S. Brown, *Cohomology of Groups*, New York, Springer-Verlag, 1982.
4. R. Gramlich, *On Graphs, Geometries, and Groups of Lie Type*, Ph.D. thesis, Technische Universiteit Eindhoven, 2002.
5. R.L. Griess, Jr, "Automorphisms of extra special groups and nonvanishing degree 2 cohomology," *Pacific J. Math.* **48** (1973), 403–422.
6. J. L. Gross and Th. W. Tucker, *Topological Graph Theory*, New York, John Wiley & Sons, 1987.
7. A. Malnič, R. Nedela, and M. Škoviera, "Lifting graph automorphisms by voltage assignments," *Europ. J. Combinatorics* **21** (2000), 927–947.
8. E. J. Postma, "Covers of graphs related to the general linear group," M.Sc. thesis, Technische Universiteit Eindhoven, 2003.
9. C.-H. Sah, "Cohomology of split group extensions, II," *J. Algebra* **45** (1977), 17–68.