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# Finite-dimensional crystals $B^{2,s}$ for quantum affine algebras of type $D_n^{(1)}$

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**Abstract** The Kirillov–Reshetikhin modules $W^{r,s}$ are finite-dimensional representations of quantum affine algebras $U_q'(\mathfrak{g})$, labeled by a Dynkin node $r$ of the affine Kac–Moody algebra $\mathfrak{g}$ and a positive integer $s$. In this paper we study the combinatorial structure of the crystal basis $B^{2,s}$ corresponding to $W^{2,s}$ for the algebra of type $D_n^{(1)}$.

**Keywords** Quantum affine algebras · Crystal bases · Kirillov-Reshetikhin crystals

**2000 Mathematics Subject Classification** Primary—17B37; Secondary—81R10

## 1. Introduction

Quantum algebras were introduced independently by Drinfeld [4] and Jimbo [8] in their study of two dimensional solvable lattice models in statistical mechanics. Since then quantum algebras have surfaced in many areas of mathematics and mathematical physics, such as the theory of knots and links, representation theory, and topological quantum field theory. Of special interest, in particular for lattice models and representation theory, are finite-dimensional representations of quantum affine algebras. The irreducible finite-dimensional $U_q'(\mathfrak{g})$-modules for an affine Kac–Moody algebra $\mathfrak{g}$ were classified by Chari and Pressley [2, 3] in terms of Drinfeld polynomials. The Kirillov–Reshetikhin modules $W^{r,s}$, labeled by a Dynkin node $r$ and a positive integer $s$, form a special class of these

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finite-dimensional modules. They naturally correspond to the weight $s\varpi_r$, where $\varpi_r$ is the $r$-th fundamental weight of the underlying finite algebra $\overline{\mathfrak{g}}$.

Kashiwara [12, 13] showed that in the limit $q \to 0$ the highest-weight representations of the quantum algebra $U_q(\mathfrak{g})$ have very special bases, called crystal bases. This construction makes it possible to study modules over quantum algebras in terms of crystals graphs, which are purely combinatorial objects. However, in general it is not yet known which finite-dimensional representations of affine quantum algebras have crystal bases and what their combinatorial structure is. Recently, Hatayama et al. [5, 6] conjectured that crystal bases $B^{r,s}$ for the Kirillov–Reshetikhin modules $W^{r,s}$ exist. For type $A_n^{(1)}$, the crystals $B^{r,s}$ are known to exist [10], and the explicit combinatorial crystal structure is also well-understood [28]. Assuming that the crystals $B^{r,s}$ exist, their structure for non-simply laced algebras can be described in terms of virtual crystals introduced in [26, 27]. The virtual crystal construction is based on the following well-known algebra embeddings of non-simply laced into simply laced types:

$$\begin{aligned} C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} &\hookrightarrow A_{2n-1}^{(1)} \\ A_{2n-1}^{(2)}, B_n^{(1)} &\hookrightarrow D_{n+1}^{(1)} \\ E_6^{(2)}, F_4^{(1)} &\hookrightarrow E_6^{(1)} \\ D_4^{(3)}, G_2^{(1)} &\hookrightarrow D_4^{(1)}. \end{aligned}$$

The main open problems in the theory of finite-dimensional affine crystals are therefore the proof of the existence of $B^{r,s}$ and the combinatorial structure of these crystals for types $D_n^{(1)}$ ($n \geq 4$) and $E_n^{(1)}$ ($n = 6, 7, 8$). In this paper, we concentrate on type $D_n^{(1)}$. For irreducible representations corresponding to multiples of the first fundamental weight (indexed by a one-row Young diagram) or any single fundamental weight (indexed by a one-column Young diagram) the crystals have been proven to exist and the structure is known [10, 18]. In [5, 6], a conjecture is presented on the decomposition of $B^{r,s}$ as a crystal for the underlying finite algebra of type $D_n$. Specifically, as a type $D_n$ classical crystal the crystals $B^{r,s}$ of type $D_n^{(1)}$ for $r \leq n-2$ decompose as

$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda),$$

where the direct sum is taken over all weights $\Lambda$ for the finite algebra corresponding to partitions obtained from an $r \times s$ rectangle by removing any number of $2 \times 1$ vertical dominoes. Here $B(\Lambda)$ is the $U_q(D_n)$-crystal associated with the highest weight representation of highest weight $\Lambda$ (see [17]). In the sequel, we consider the case $r = 2$, for which the above direct sum specializes to

$$B^{2,s} \cong \bigoplus_{k=0}^{s} B(k\varpi_2), \tag{1}$$

where once again the summands in the right hand side of the equation are crystals for the finite algebra. Our approach to study the combinatorics of $B^{2,s}$ is as follows. First, we introduce tableaux of shape $(s, s)$ to define a $U_q(D_n)$-crystal whose vertices are in bijection with the classical tableaux from the direct sum decomposition (1). Using the automorphism of the $D_n^{(1)}$ Dynkin diagram which interchanges nodes 0 and 1, we define the unique action of $\tilde{f}_0$ and $\tilde{e}_0$ which makes this crystal into a perfect crystal $\tilde{B}^{2,s}$ of level $s$ with an energy function. (See sections 2.3 and 2.4 for definitions of these terms.)

Assuming the existence of the crystal $B^{r,s}$, the main result of our paper states that our combinatorially constructed crystal $\tilde{B}^{2,s}$ is the unique perfect crystal of level $s$ with the classical decomposition (1) with a given energy function. More precisely:

**Theorem 1.1.** *If $B^{2,s}$ exists with the properties as in Conjecture 3.4, then $\tilde{B}^{2,s} \cong B^{2,s}$.*

This is the first step in confirming Conjecture 2.1 of [5], which states that as modules over the embedded classical quantum group, $W^{2,s}$ decomposes as $\bigoplus_{k=0}^{s} V(k\varpi_2)$, where $V(\Lambda)$ is the classical module with highest weight $\Lambda$, $W^{2,s}$ has a crystal basis, and this crystal is a perfect crystal of level $s$.

The paper is structured as follows. In section 2 the definition of quantum algebras, crystal bases and perfect crystals is reviewed. Section 3 is devoted to crystals and the plactic monoid of type $D_n$. The properties of $B^{2,s}$ of type $D_n^{(1)}$ as conjectured in [5] are given in Conjecture 3.4. In section 4 the set underlying $\tilde{B}^{2,s}$ is constructed in terms of tableaux of shape $(s, s)$ obeying certain conditions. It is shown that this set is in bijection with the union of sets appearing on the right hand side of (1). The branching component graph is introduced in section 5, which is used in section 6 to define $\tilde{e}_0$ and $\tilde{f}_0$ on $\tilde{B}^{2,s}$. This makes $\tilde{B}^{2,s}$ into an affine crystal. It is shown in section 7 that $\tilde{B}^{2,s}$ is perfect and that $\tilde{B}^{2,s}$ is the unique perfect crystal having the classical decomposition (1) with the appropriate energy function. This proves in particular Theorem 1.1. Finally, we end in section 8 with some open problems.

## 2. Review of quantum groups and crystal bases

### 2.1. Quantum groups

For $n \in \mathbb{Z}$ and a formal parameter $q$, we use the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = \prod_{k=1}^{n} [k]_q, \text{ and } \begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}.$$

These are all elements of $\mathbb{Q}(q)$, called the $q$-integers, $q$-factorials, and $q$-binomial coefficients, respectively.

Let $\mathfrak{g}$ be an arbitrary Kac-Moody Lie algebra with Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ and a Dynkin diagram indexed by $I$. Here $A = (a_{ij})_{i,j \in I}$ is the Cartan matrix, $P$ and $P^\vee$ are the weight lattice and dual weight lattice, respectively, $\Pi = \{\alpha_i \mid i \in I\}$ is the set of simple roots and $\Pi^\vee = \{h_i \mid i \in I\}$ is the set of simple coroots. Furthermore, let $\{s_i \mid i \in I\}$ be the entries of the diagonal symmetrizing matrix of $A$ and define $q_i = q^{s_i}$ and $K_i = q^{s_i h_i}$. Then the quantum enveloping algebra $U_q(\mathfrak{g})$ is the associative $\mathbb{Q}(q)$-algebra generated by $e_i$ and $f_i$ for $i \in I$, and $q^h$ for $h \in P^\vee$, with the following relations (see e.g. [7, Def. 3.1.1]):

(1) $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ for all $h, h' \in P^\vee$,
(2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for all $h \in P^\vee$,
(3) $q^h f_i q^{-h} = q^{\alpha_i(h)} f_i$ for all $h \in P^\vee$,
(4) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$,
(5) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for all $i \neq j$,
(6) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for all $i \neq j$.

2.2. Crystal bases

The quantum algebra $U_q(\mathfrak{g})$ can be viewed as a $q$-deformation of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Lusztig [23] showed that the integrable highest weight representations of $U(\mathfrak{g})$ can be deformed to $U_q(\mathfrak{g})$ representations in such a way that the dimension of the weight spaces are invariant under the deformation, provided $q \neq 0$ and $q^k \neq 1$ for all $k \in \mathbb{Z}$ (see also [7]). Let $M$ be a $U_q(\mathfrak{g})$-module and $R$ the subset of all elements in $\mathbb{Q}(q)$ which are regular at $q = 0$. Kashiwara [12, 13] introduced Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ as certain linear combinations of powers of $e_i$ and $f_i$. A crystal lattice $\mathcal{L}$ is a free $R$-submodule of $M$ that generates $M$ over $\mathbb{Q}(q)$, has the same weight decomposition and has the property that $\tilde{e}_i\mathcal{L} \subset \mathcal{L}$ and $\tilde{f}_i\mathcal{L} \subset \mathcal{L}$ for all $i \in I$. The passage from $\mathcal{L}$ to the quotient $\mathcal{L}/q\mathcal{L}$ is referred to as taking the crystal limit. A crystal basis is a $\mathbb{Q}$-basis of $\mathcal{L}/q\mathcal{L}$ with certain properties.

Axiomatically, we may define a $U_q(\mathfrak{g})$-crystal as a nonempty set $B$ equipped with maps $\mathrm{wt} : B \to P$ and $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{\varnothing\}$ for all $i \in I$, satisfying

$$\tilde{f}_i(b) = b' \Leftrightarrow \tilde{e}_i(b') = b \text{ if } b, b' \in B \tag{2}$$

$$\mathrm{wt}(\tilde{f}_i(b)) = \mathrm{wt}(b) - \alpha_i \text{ if } \tilde{f}_i(b) \in B \tag{3}$$

$$\langle h_i\,,\ \mathrm{wt}(b)\rangle = \varphi_i(b) - \varepsilon_i(b). \tag{4}$$

Here for $b \in B$

$$\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n(b) \neq \varnothing\}$$
$$\varphi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n(b) \neq \varnothing\}.$$

(It is assumed that $\varphi_i(b), \varepsilon_i(b) < \infty$ for all $i \in I$ and $b \in B$.) A $U_q(\mathfrak{g})$-crystal $B$ can be viewed as a directed edge-colored graph (the crystal graph) whose vertices are the elements of $B$, with a directed edge from $b$ to $b'$ labeled $i \in I$, if and only if $\tilde{f}_i(b) = b'$.

Let $B_1$ and $B_2$ be $U_q(\mathfrak{g})$-crystals. The Cartesian product $B_2 \times B_1$ can also be endowed with the structure of a $U_q(\mathfrak{g})$-crystal. The resulting crystal is denoted by $B_2 \otimes B_1$ and its elements $(b_2, b_1)$ are written $b_2 \otimes b_1$. (The reader is warned that our convention is opposite to that of Kashiwara [14]). For $i \in I$ and $b = b_2 \otimes b_1 \in B_2 \otimes B_1$, we have $\mathrm{wt}(b) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,

$$\tilde{f}_i(b_2 \otimes b_1) = \begin{cases} \tilde{f}_i(b_2) \otimes b_1 & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1) \\ b_2 \otimes \tilde{f}_i(b_1) & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1) \end{cases} \tag{5}$$

and

$$\tilde{e}_i(b_2 \otimes b_1) = \begin{cases} \tilde{e}_i(b_2) \otimes b_1 & \text{if } \varepsilon_i(b_2) > \varphi_i(b_1) \\ b_2 \otimes \tilde{e}_i(b_1) & \text{if } \varepsilon_i(b_2) \leq \varphi_i(b_1). \end{cases} \tag{6}$$

Combinatorially, this action of $\tilde{f}_i$ and $\tilde{e}_i$ on tensor products can be described by the signature rule. The $i$-signature of $b$ is the word consisting of the symbols $+$ and $-$ given by

$$\underbrace{-\cdots-}_{\varphi_i(b_2)\text{ times}} \quad \underbrace{+\cdots+}_{\varepsilon_i(b_2)\text{ times}} \quad \underbrace{-\cdots-}_{\varphi_i(b_1)\text{ times}} \quad \underbrace{+\cdots+}_{\varepsilon_i(b_1)\text{ times}}.$$

The reduced $i$-signature of $b$ is the subword of the $i$-signature of $b$, given by the repeated removal of adjacent symbols $+-$ (in that order); it has the form

$$\underbrace{-\cdots-}_{\varphi \text{ times}} \quad \underbrace{+\cdots+}_{\varepsilon \text{ times}}.$$

If $\varphi = 0$ then $\tilde{f}_i(b) = \varnothing$; otherwise $\tilde{f}_i$ acts on the tensor factor corresponding to the rightmost symbol $-$ in the reduced $i$-signature of $b$. Similarly, if $\varepsilon = 0$ then $\tilde{e}_i(b) = \varnothing$; otherwise $\tilde{e}_i$ acts on the leftmost symbol $+$ in the reduced $i$-signature of $b$. From this it is clear that

$$\varphi_i(b_2 \otimes b_1) = \varphi_i(b_2) + \max(0, \varphi_i(b_1) - \varepsilon_i(b_2)),$$
$$\varepsilon_i(b_2 \otimes b_1) = \varepsilon_i(b_1) + \max(0, -\varphi_i(b_1) + \varepsilon_i(b_2)).$$

## 2.3. Perfect crystals

Of particular interest is a class of crystals called perfect crystals, which are crystals for affine algebras satisfying a set of very special properties. These properties ensure that perfect crystals can be used to construct the path realization of highest weight modules [11]. To define them, we need a few preliminary definitions.

Recall that $P$ denotes the weight lattice of a Kac-Moody algebra $\mathfrak{g}$; for the remainder of this section, $\mathfrak{g}$ is of affine type. The center of $\mathfrak{g}$ is one-dimensional and is generated by the canonical central element $c = \sum_{i \in I} a_i^\vee h_i$, where the $a_i^\vee$ are the numbers on the nodes of the Dynkin diagram of the algebra dual to $\mathfrak{g}$ given in Table Aff of [9, section 4.8]. Moreover, the imaginary roots of $\mathfrak{g}$ are nonzero integral multiples of the null root $\delta = \sum_{i \in I} a_i \alpha_i$, where the $a_i$ are the numbers on the nodes of the Dynkin diagram of $\mathfrak{g}$ given in Table Aff of [9]. Define $P_{\text{cl}} = P/\mathbb{Z}\delta$, $P_{\text{cl}}^+ = \{\lambda \in P_{\text{cl}} | \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I\}$, and $U_q'(\mathfrak{g})$ to be the quantum enveloping algebra with the Cartan datum $(A, \Pi, \Pi^\vee, P_{\text{cl}}, P_{\text{cl}}^\vee)$.

Define the set of level $\ell$ weights to be $(P_{\text{cl}}^+)_\ell = \{\lambda \in P_{\text{cl}}^+ | \langle c, \lambda \rangle = \ell\}$. For a crystal basis element $b \in B$, define

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i,$$

where $\Lambda_i$ is the $i$-th fundamental weight of $\mathfrak{g}$. Finally, for a crystal basis $B$, we define $B_{\min}$ to be the set of crystal basis elements $b$ such that $\langle c, \varepsilon(b) \rangle$ is minimal over $b \in B$.

*Definition 2.1.* A crystal $B$ is a perfect crystal of level $\ell$ if:

(1) $B \otimes B$ is connected;
(2) there exists $\lambda \in P_{\text{cl}}$ such that $\text{wt}(B) \subset \lambda + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i$ and $\#(B_\lambda) = 1$;
(3) there is a finite-dimensional irreducible $U_q'(\mathfrak{g})$-module $V$ with a crystal base whose crystal graph is isomorphic to $B$;
(4) for any $b \in B$, we have $\langle c, \varepsilon(b) \rangle \geq \ell$;
(5) the maps $\varepsilon$ and $\varphi$ from $B_{\min}$ to $(P_{\text{cl}}^+)_\ell$ are bijective.

We use the notation $lev(B)$ to indicate the level of the perfect crystal $B$.

### 2.4. Energy function

The existence of an affine crystal structure usually provides an energy function. Let $B_1$ and $B_2$ be finite $U_q'(\mathfrak{g})$-crystals. Then following [11, Section 4] we have:

(1) There is a unique isomorphism of $U_q'(\mathfrak{g})$-crystals $R = R_{B_2,B_1} : B_2 \otimes B_1 \to B_1 \otimes B_2$.
(2) There is a function $H = H_{B_2,B_1} : B_2 \otimes B_1 \to \mathbb{Z}$, unique up to global additive constant, such that $H$ is constant on classical components and, for all $b_2 \in B_2$ and $b_1 \in B_1$, if $R(b_2 \otimes b_1) = b_1' \otimes b_2'$, then

$$H(\tilde{e}_0(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } \varepsilon_0(b_2) > \varphi_0(b_1) \text{ and } \varepsilon_0(b_1') > \varphi_0(b_2') \\ 1 & \text{if } \varepsilon_0(b_2) \leq \varphi_0(b_1) \text{ and } \varepsilon_0(b_1') \leq \varphi_0(b_2') \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

We shall call the maps $R$ and $H$ the local isomorphism and local energy function on $B_2 \otimes B_1$, respectively. The pair $(R, H)$ is called the combinatorial $R$-matrix.

Let $u(B_1)$ and $u(B_2)$ be extremal vectors of $B_1$ and $B_2$, respectively (see [15] for a definition of extremal vectors). Then

$$R(u(B_2) \otimes u(B_1)) = u(B_1) \otimes u(B_2).$$

It is convenient to normalize the local energy function $H$ by requiring that

$$H(u(B_2) \otimes u(B_1)) = 0.$$

With this convention it follows by definition that

$$H_{B_1,B_2} \circ R_{B_2,B_1} = H_{B_2,B_1}$$

as operators on $B_2 \otimes B_1$.

We wish to define an energy function $D_B : B \to \mathbb{Z}$ for tensor products of perfect crystals of the form $B^{r,s}$ [5, Section 3.3]. Let $B = B^{r,s}$ be perfect. Then there exists a unique element $b^\natural \in B$ such that $\varphi(b^\natural) = lev(B)\Lambda_0$. Define $D_B : B \to \mathbb{Z}$ by

$$D_B(b) = H_{B,B}(b \otimes b^\natural) - H_{B,B}(u(B) \otimes b^\natural). \tag{8}$$

The intrinsic energy $D_B$ for the $L$-fold tensor product $B = B_L \otimes \cdots \otimes B_1$ where $B_j = B^{r_j,s_j}$ is given by

$$D_B = \sum_{1 \leq i < j \leq L} H_i R_{i+1} R_{i+2} \cdots R_{j-1} + \sum_{j=1}^{L} D_{B_j} R_1 R_2 \cdots R_{j-1},$$

where $H_i$ and $R_i$ are the local energy function and $R$-matrix on the $i$-th and $i+1$-th tensor factor, respectively.

## 3. Crystals and plactic monoid of type $D$

From now on we restrict our attention to the finite Lie algebra of type $D_n$ and the affine Kac-Moody algebra of type $D_n^{(1)}$. Denote by $I = \{0, 1, \ldots, n\}$ the index set of the Dynkin diagram for $D_n^{(1)}$ and by $J = \{1, 2, \ldots, n\}$ the Dynkin diagram for type $D_n$.

### 3.1. Dynkin data

For type $D_n$, the simple roots are

$$\begin{aligned}\alpha_i &= \varepsilon_i - \varepsilon_{i+1} \qquad \text{for } 1 \le i < n\\ \alpha_n &= \varepsilon_{n-1} + \varepsilon_n\end{aligned} \tag{9}$$

and the fundamental weights are

$$\begin{aligned}\varpi_i &= \varepsilon_1 + \cdots + \varepsilon_i \qquad \text{for } 1 \le i \le n-2\\ \varpi_{n-1} &= (\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)/2\\ \varpi_n &= (\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n)/2\end{aligned}$$

where $\varepsilon_i \in \mathbb{Z}^n$ is the $i$-th unit standard vector. The central element for $D_n^{(1)}$ is given by

$$c = h_0 + h_1 + 2h_2 + \cdots + 2h_{n-2} + h_{n-1} + h_n.$$

### 3.2. Classical crystals

Kashiwara and Nakashima [17] described the crystal structure of all classical highest weight crystals $B(\Lambda)$ of highest weight $\Lambda$ explicitly. For the special case $B(k\varpi_2)$ as occuring in (1) each crystal element can be represented by a tableau of shape $\lambda = (k, k)$ on the partially ordered alphabet

$$1 < 2 < \cdots < n-1 < \begin{matrix} n \\ \bar{n} \end{matrix} < \overline{n-1} < \cdots \bar{2} < \bar{1}$$

such that the following conditions hold [7, page 202]:

**Criterion 3.1.**

1. *If $ab$ is in the filling, then $a \le b$;*
2. *If $\begin{smallmatrix} a \\ b \end{smallmatrix}$ is in the filling, then $b \not\le a$;*
3. *No configuration of the form $\begin{smallmatrix} a & a \\ & \bar{a} \end{smallmatrix}$ or $\begin{smallmatrix} a & \\ \bar{a} & \bar{a} \end{smallmatrix}$ appears;*
4. *No configuration of the form $\begin{smallmatrix} n-1 \\ n \end{smallmatrix} \cdots \begin{smallmatrix} n \\ \overline{n-1} \end{smallmatrix}$ or $\begin{smallmatrix} n-1 \\ \bar{n} \end{smallmatrix} \cdots \begin{smallmatrix} \bar{n} \\ \overline{n-1} \end{smallmatrix}$ appears;*
5. *No configuration of the form $\begin{smallmatrix} 1 \\ \bar{1} \end{smallmatrix}$ appears.*

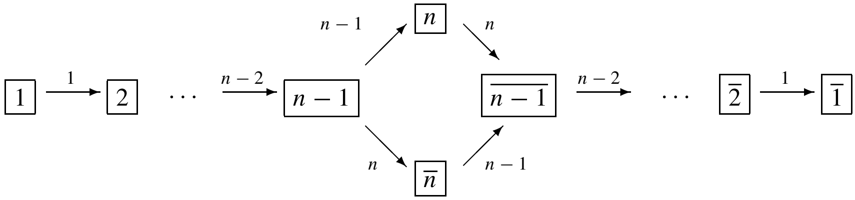
Note that for $k \ge 2$, condition 5 follows from conditions 1 and 3.

Also, observe that the conditions given in [7] apply only to adjacent columns, not to non-adjacent columns as in condition 4 above. However, Criterion 3.1 is unchanged by replacing condition 4 with the following:

(4a) *No configuration of the form* $\begin{smallmatrix} n-1 \\ n \end{smallmatrix} \begin{smallmatrix} n \\ \overline{n-1} \end{smallmatrix}$ *or* $\begin{smallmatrix} n-1 \\ \bar{n} \end{smallmatrix} \begin{smallmatrix} \bar{n} \\ \overline{n-1} \end{smallmatrix}$ *appears.*

To see this equivalence, observe that by conditions 1 and 2 the only columns that can appear between $\begin{smallmatrix} n-1 \\ n \end{smallmatrix}$ and $\begin{smallmatrix} n \\ \overline{n-1} \end{smallmatrix}$ are $\begin{smallmatrix} n-1 \\ n \end{smallmatrix}$, $\begin{smallmatrix} n-1 \\ \overline{n-1} \end{smallmatrix}$, and $\begin{smallmatrix} n \\ \overline{n-1} \end{smallmatrix}$, and they must appear in that order from left to right. If a column of the form $\begin{smallmatrix} n-1 \\ \overline{n-1} \end{smallmatrix}$ appears, we have a configuration of the form $\begin{smallmatrix} n-1 & n-1 \\ & \overline{n-1} \end{smallmatrix}$, which is forbidden by condition 3. On the other hand, if no column of the form $\begin{smallmatrix} n-1 \\ \overline{n-1} \end{smallmatrix}$ appears, the columns $\begin{smallmatrix} n-1 \\ n \end{smallmatrix}$ and $\begin{smallmatrix} n \\ \overline{n-1} \end{smallmatrix}$ are adjacent, which is disallowed by condition 4a.

The crystal $B(\varpi_1)$ is described pictorially by the crystal graph:




For a tableau $T = \begin{smallmatrix} a_1 & \cdots & a_k \\ b_1 & & b_k \end{smallmatrix} \in B(k\varpi_2)$, the action of the Kashiwara operators $\tilde{f}_i$ and $\tilde{e}_i$ is defined as follows. Consider the column word $w_T = b_1a_1 \cdots b_ka_k$ and view this word as an element in $B(\varpi_1)^{\otimes 2k}$. Then $\tilde{f}_i$ and $\tilde{e}_i$ act by the tensor product rule as defined in section 2.2.

*Example 3.2.* Let $n = 4$. Then the tableau

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & \bar{3} & \bar{3} \\ \hline 3 & \bar{4} & \bar{4} & \bar{2} & \bar{1} \\ \hline \end{array}$$

has column word $w_T = 31\bar{4}2\bar{4}4\bar{2}\bar{3}\bar{1}\bar{3}$. The 2-signature of $T$ is $+-+--$, derived from the subword $32\bar{2}\bar{3}\bar{3}$, and the reduced 2-signature is a single $-$. Therefore,

$$\tilde{f}_2(T) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & \bar{3} & \bar{2} \\ \hline 3 & \bar{4} & \bar{4} & \bar{2} & \bar{1} \\ \hline \end{array},$$

since the rightmost—in the reduced 2-signature of $T$ comes from the northeastmost $\bar{3}$. The 4-signature of $T$ is $-++-++$, derived from the subword $3\bar{4}\bar{4}4\bar{3}\bar{3}$, and the reduced 4-signature is $-+++$, from the subword $3\bar{4}\bar{3}\bar{3}$. This tells us that

$$\tilde{f}_4(T) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & \bar{3} & \bar{3} \\ \hline \bar{4} & \bar{4} & \bar{4} & \bar{2} & \bar{1} \\ \hline \end{array} \quad \text{and} \quad \tilde{e}_4(T) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{4} & \bar{2} & \bar{1} \\ \hline \end{array}.$$

### 3.3. Dual crystals

Let $\omega_0$ be the longest element in the Weyl group of $D_n$. The action of $\omega_0$ on the weight lattice $P$ of $D_n$ is given by

$$\omega_0(\varpi_i) = -\varpi_{\tau(i)}$$
$$\omega_0(\alpha_i) = -\alpha_{\tau(i)}$$

where $\tau : J \to J$ is the identity if $n$ is even and interchanges $n-1$ and $n$ and fixes all other Dynkin nodes if $n$ is odd.

There is a unique involution $* : B \to B$, called the dual map, satisfying

$$\mathrm{wt}(b^*) = \omega_0 \mathrm{wt}(b)$$
$$\tilde{e}_i(b)^* = \tilde{f}_{\tau(i)}(b^*)$$
$$\tilde{f}_i(b)^* = \tilde{e}_{\tau(i)}(b^*).$$

The involution $*$ sends the highest weight vector $u \in B(\Lambda)$ to the lowest weight vector (the unique vector in $B(\Lambda)$ of weight $\omega_0(\Lambda)$). We have

$$(B_1 \otimes B_2)^* \cong B_2 \otimes B_1$$

with $(b_1 \otimes b_2)^* \mapsto b_2^* \otimes b_1^*$.

Explicitly, on $B(\varpi_1)$ the involution $*$ is given by

$$i \longleftrightarrow \bar{i}$$

except for $i = n$ with $n$ odd in which case $n \leftrightarrow n$ and $\bar{n} \leftrightarrow \bar{n}$. For $T \in B(\Lambda)$ the dual $T^*$ is obtained by applying the $*$ map defined for $B(\varpi_1)$ to each of the letters of $w_T^{\mathrm{rev}}$ (the reverse column word of $T$), and then rectifying the resulting word.

*Example 3.3.* If

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \bar{3} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \in B(2\varpi_1 + \varpi_2)$$

we have

$$T^* = \begin{array}{|c|c|c|}\hline 3 & \bar{1} & \bar{1} \\ \hline \bar{2} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}.$$

## 3.4. Plactic monoid of type $D$

The plactic monoid for type $D$ is the free monoid generated by $\{1, \ldots, n, \bar{n}, \ldots, \bar{1}\}$, modulo certain relations introduced by Lecouvey [22]. Note that we write our words in the reverse order compared to [22]. A column word $C = x_L x_{L-1} \cdots x_1$ is a word such that $x_{i+1} \not\leq x_i$ for $i = 1, \ldots, L-1$. Note that the letters $n$ and $\bar{n}$ are the only letters that may appear more than once in $C$. Let $z \leq n$ be a letter in $C$. Then $N(z)$ denotes the number of letters $x$ in $C$ such that $x \leq z$ or $x \geq \bar{z}$. A column $C$ is called admissible if $L \leq n$ and for any pair $(z, \bar{z})$ of letters in $C$ with $z \leq n$ we have $N(z) \leq z$. The Lecouvey $D$ equivalence relations are given by:

(1) If $x \neq \bar{z}$, then

$$xzy \equiv zxy \text{ for } x \leq y < z \text{ and } yzx \equiv yxz \text{ for } x < y \leq z.$$

(2) If $1 < x < n$ and $x \leq y \leq \bar{x}$, then

$$(x-1)\overline{(x-1)}y \equiv \bar{x}xy \text{ and } y\bar{x}x \equiv y(x-1)\overline{(x-1)}.$$

(3) If $x \leq n-1$, then

$$\begin{cases} n\bar{x}\bar{n} \equiv n\bar{n}\bar{x} \\ \bar{n}\bar{x}n \equiv \bar{n}n\bar{x} \end{cases} \text{and} \begin{cases} xn\bar{n} \equiv nx\bar{n} \\ x\bar{n}n \equiv \bar{n}xn \end{cases}.$$

(4)

$$\begin{cases} \bar{n}\bar{n}n \equiv \bar{n}(n-1)(\overline{n-1}) \\ nn\bar{n} \equiv n(n-1)(\overline{n-1}) \end{cases} \text{and} \begin{cases} (n-1)(\overline{n-1})\bar{n} \equiv n\bar{n}\bar{n} \\ (n-1)(\overline{n-1})n \equiv \bar{n}nn \end{cases}.$$

(5) Consider $w$ a non-admissible column word each strict factor of which is admissible. Let $z$ be the lowest unbarred letter such that the pair $(z, \bar{z})$ occurs in $w$ and $N(z) > z$. Then $w \equiv \tilde{w}$ is the column word obtained by erasing the pair $(z, \bar{z})$ in $w$ if $z < n$, by erasing a pair $(n, \bar{n})$ of consecutive letters otherwise.

This monoid gives us a bumping algorithm similar to the Schensted bumping algorithm. It is noted in [22] that a general type $D$ sliding algorithm, if one exists, would be very complicated. However, for tableaux with no more than two rows, these relations provide us with the following relations on subtableaux:

(1) If $x \neq \bar{z}$, then

$$\begin{array}{cc} & y \\ x & z \end{array} \equiv \begin{array}{cc} x & y \\ & z \end{array} \equiv \begin{array}{cc} x & y \\ z & \end{array} \quad \text{for } x \leq y < z,$$

and

$$\begin{array}{cc} & x \\ y & z \end{array} \equiv \begin{array}{cc} x & \\ y & z \end{array} \equiv \begin{array}{cc} x & z \\ y & \end{array} \quad \text{for } x < y \leq z.$$

(2) If $1 < x < n$ and $x \leq y \leq \bar{x}$, then

$$\begin{array}{cc} & y \\ x-1 & \overline{x-1} \end{array} \equiv \begin{array}{cc} x-1 & y \\ & \overline{x-1} \end{array} \equiv \begin{array}{cc} x & y \\ \bar{x} & \end{array}$$

and

$$\begin{array}{cc} & x \\ y & \bar{x} \end{array} \equiv \begin{array}{cc} x-1 & \\ y & \overline{x-1} \end{array} \equiv \begin{array}{cc} x-1 & \overline{x-1} \\ y & \end{array}.$$

(3) If $x \leq n-1$, then

$$\begin{cases} \begin{array}{cc} & \bar{n} \\ n & \bar{x} \end{array} \equiv \begin{array}{cc} \bar{n} & \\ n & \bar{x} \end{array} \equiv \begin{array}{cc} \bar{n} & \bar{x} \\ n & \end{array} \\ \begin{array}{cc} & n \\ \bar{n} & \bar{x} \end{array} \equiv \begin{array}{cc} n & \\ \bar{n} & \bar{x} \end{array} \equiv \begin{array}{cc} n & \bar{x} \\ \bar{n} & \end{array} \end{cases}$$

and

$$\begin{cases} \begin{array}{cc} & \bar{n} \\ x & n \end{array} \equiv \begin{array}{cc} x & \bar{n} \\ & n \end{array} \equiv \begin{array}{cc} x & \bar{n} \\ n & \end{array} \\ \begin{array}{cc} & n \\ x & \bar{n} \end{array} \equiv \begin{array}{cc} x & n \\ & \bar{n} \end{array} \equiv \begin{array}{cc} x & n \\ \bar{n} & \end{array} \end{cases}.$$

(4)

$$\left\{\begin{array}{l}
\begin{array}{cc} & n \\ \bar n & \bar n \end{array} \equiv \begin{array}{cc} n-1 & \\ \bar n & \overline{n-1} \end{array} \equiv \begin{array}{cc} n-1 & \overline{n-1} \\ \bar n & \end{array} \\[2ex]
\begin{array}{cc} & \bar n \\ n & n \end{array} \equiv \begin{array}{cc} n-1 & \\ n & \overline{n-1} \end{array} \equiv \begin{array}{cc} n-1 & \overline{n-1} \\ n & \end{array}
\end{array}\right.$$

and

$$\left\{\begin{array}{l}
\begin{array}{cc} & \bar n \\ n-1 & \overline{n-1} \end{array} \equiv \begin{array}{cc} n-1 & \bar n \\ & \overline{n-1} \end{array} \equiv \begin{array}{cc} \bar n & \bar n \\ n & \end{array} \\[2ex]
\begin{array}{cc} & n \\ n-1 & \overline{n-1} \end{array} \equiv \begin{array}{cc} n-1 & n \\ & \overline{n-1} \end{array} \equiv \begin{array}{cc} n & n \\ \bar n & \end{array}
\end{array}\right. .$$

If a word is composed entirely of barred letters or entirely of unbarred letters, only relation (1) (the Knuth relation) applies, and the type *A jeu de taquin* may be used.

### 3.5. Properties of $B^{2,s}$

As mentioned in the introduction, it was conjectured in [5, 6] that there are crystal bases $B^{r,s}$ associated with Kirillov–Reshetikhin modules $W^{r,s}$. In addition to the existence, Hatayama et al. [5] conjectured certain properties of $B^{r,s}$ which we state here in the specific case of $B^{2,s}$ of type $D_n^{(1)}$.

*Conjecture 3.4 ([5]).* If the crystal $B^{2,s}$ of type $D_n^{(1)}$ exists, it has the following properties:

(1) As a classical crystal $B^{2,s}$ decomposes as $B^{2,s} \cong \bigoplus_{k=0}^{s} B(k\varpi_2)$.
(2) $B^{2,s}$ is perfect of level $s$.
(3) $B^{2,s}$ is equipped with an energy function $D_{B^{2,s}}$ such that $D_{B^{2,s}}(b) = k - s$ if $b$ is in the component of $B(k\varpi_2)$ (in accordance with (8)).

## 4. Classical decomposition of $\tilde{B}^{2,s}$

In this section we begin our construction of the crystal $\tilde{B}^{2,s}$ mentioned in Theorem 1.1. We do this by defining a $U_q(D_n)$-crystal with vertices labeled by the set $\mathcal{T}(s)$ of tableaux of shape $(s, s)$ which satisfy conditions 1, 2, and 4 of Criterion 3.1. We will construct a bijection between $\mathcal{T}(s)$ and the vertices of $\bigoplus_{i=0}^{s} B(i\varpi_2)$, so that $\mathcal{T}(s)$ may be viewed as a $U_q(D_n)$-crystal with the classical decomposition (1). In section 6 we will define $\tilde{f}_0$ and $\tilde{e}_0$ on $\mathcal{T}(s)$ to give it the structure of a perfect $U_q'(D_n^{(1)})$-crystal. This crystal will be $\tilde{B}^{2,s}$.

The reader may note in later sections that the main result of the paper does not depend on this explicit labeling of the vertices of $\tilde{B}^{2,s}$. We have included it here because a description of the crystal in terms of tableaux will be needed to obtain a bijection with rigged configurations. It is through such a bijection that we anticipate being able to prove the $X = M$ conjecture for type $D$, as has already been done for special cases in [25, 29, 30].

**Proposition 4.1.** *Let $T \in \mathcal{T}(s) \setminus B(s\varpi_2)$ with $T \neq \frac{1}{\bar 1} \cdots \frac{1}{\bar 1}$, and define $\bar{\bar{i}} = i$ for $1 \le i \le n$. Then there is a unique $a \in \{1, \ldots, n, \bar n\}$ and $m \in \mathbb{Z}_{>0}$ such that $T$ contains one of the* *following configurations (called an a-configuration):*

$$\begin{matrix} a & a & \cdots & a & c_1 \\ b_1 & \bar{a} & & \bar{a} & d_1 \end{matrix}, \quad \text{(with } m \text{ columns } {}^{a}_{\bar a}\text{)} \quad \textit{where } b_1 \neq \bar{a}, \textit{ and } c_1 \neq a \textit{ or } d_1 \neq \bar{a};$$

$$\begin{matrix} b_2 & a & \cdots & a & d_2 \\ c_2 & \bar{a} & & \bar{a} & \bar{a} \end{matrix}, \quad \text{(with } m \text{ columns } {}^{a}_{\bar a}\text{)} \quad \textit{where } d_2 \neq a, \textit{ and } b_2 \neq a \textit{ or } c_2 \neq \bar{a};$$

$$\begin{matrix} b_3 & a & \cdots & a & d_3 \\ c_3 & \bar{a} & & \bar{a} & e_3 \end{matrix}, \quad \text{(with } m+1 \text{ columns } {}^{a}_{\bar a}\text{)} \quad \textit{where } b_3 \neq a \textit{ and } e_3 \neq \bar{a}.$$

**Proof:** If $s = 1$, the set $\mathcal{T}(s) \setminus B(s\varpi_2)$ contains only $\frac{1}{\bar{1}}$, so that the statement of the proposition is empty. Hence assume that $s \geq 2$. The existence of an $a$-configuration for some $a \in \{1, \ldots, n, \bar{n}\}$ follows from the fact that $T$ violates condition 3 of Criterion 3.1. The conditions on $b_i, c_i, d_i$ for $i = 1, 2, 3$ and $e_3$ can be viewed as stating that $m$ is chosen to maximize the size of the $a$-configuration. Condition 1 of Criterion 3.1 and the conditions on the parameters $b_i, c_i, d_i, e_3$ imply that there can be no other $a$-configurations in $T$. □

The map $D_{2,s} : \mathcal{T}(s) \to \bigoplus_{k=0}^{s} B(k\varpi_2)$, called the height-two drop map, is defined as follows for $T \in \mathcal{T}(s)$. If $T = \frac{1}{\bar{1}} \cdots \frac{1}{\bar{1}}$, then $D_{2,s}(T) = \varnothing \in B(0)$. If $T \in B(s\varpi_2)$, $D_{2,s}(T) = T$. Otherwise by Proposition 4.1, $T$ contains a unique $a$-configuration, and $D_{2,s}(T)$ is obtained from $T$ by removing $\underbrace{\begin{matrix} a & \cdots & a \\ \bar{a} & & \bar{a} \end{matrix}}_{m}$.

**Theorem 4.2.** *Let $T \in \mathcal{T}(s)$. Then $D_{2,s}(T)$ satisfies Criterion 3.1, and is therefore a tableau in $\bigoplus_{k=0}^{s} B(k\varpi_2)$.*

**Proof:** Condition 1 is satisfied since the relation $\leq$ on our alphabet is transitive. Conditions 2 and 5 are automatically satisfied, since the columns that remain are not changed. Condition 3 is satisfied since by Proposition 4.1, there can be no more than one $a$-configuration in $T$. Condition 4 is satisfied since $D_{2,s}$ does not remove any columns of the form $\begin{smallmatrix} n-1 \\ n \end{smallmatrix}$, $\begin{smallmatrix} n-1 \\ \bar{n} \end{smallmatrix}$, $\frac{n}{n-1}$, or $\frac{\bar{n}}{n-1}$. □

In Proposition 4.5, we will show that $D_{2,s}$ is a bijection by constructing its inverse.

*Example 4.3.* We have

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline \bar{4} & \bar{2} & \bar{2} & \bar{1} \\ \hline \end{array}, \qquad D_{2,4}(T) = \begin{array}{|c|c|c|} \hline 1 & 3 & 3 \\ \hline \bar{4} & \bar{2} & \bar{1} \\ \hline \end{array}.$$

The inverse of $D_{2,s}$ is the height-two fill map $F_{2,s} : \bigoplus_{k=0}^{s} B(k\varpi_2) \to \mathcal{T}(s)$. Let $t = \begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix} \cdots \begin{smallmatrix} a_k \\ b_k \end{smallmatrix} \in B(k\varpi_2)$. If $k = s$, $F_{2,s}(t) = t$. If $k < s$, then $F_{2,s}(t)$ is obtained by finding a subtableau $\begin{smallmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{smallmatrix}$ in $t$ such that

**Criterion 4.4.**

$$b_i \leq \bar{a}_i \leq b_{i+1} \textit{ or } a_i \leq \bar{b}_{i+1} \leq a_{i+1}.$$

(Recall that $\bar{\bar{i}} = i$ for $i \in \{1, \ldots, n\}$.) Note that the first pair of inequalities imply that $a_i$ is unbarred, and the second pair of inequalities imply that $b_{i+1}$ is barred. We may therefore insert between columns $i$ and $i+1$ of $t$ either the configuration $\underbrace{\begin{matrix} a_i & \cdots & a_i \\ \bar{a}_i & & \bar{a}_i \end{matrix}}_{s-k}$ or $\underbrace{\begin{matrix} \bar{b}_{i+1} & \cdots & \bar{b}_{i+1} \\ b_{i+1} & & b_{i+1} \end{matrix}}_{s-k}$, depending on which part of Criterion 4.4 is satisfied. We say that $i$ is the filling location of $t$. If no such subtableau exists, then $F_{2,s}$ will either append $\underbrace{\begin{matrix} a_k & \cdots & a_k \\ \bar{a}_k & & \bar{a}_k \end{matrix}}_{s-k}$ to the end of $t$, or prepend $\underbrace{\begin{matrix} \bar{b}_1 & \cdots & \bar{b}_1 \\ b_1 & & b_1 \end{matrix}}_{s-k}$ to $t$. In these cases the filling locations are $k$ and 0, respectively.

**Proposition 4.5.** *The map $F_{2,s}$ is well-defined on $\bigoplus_{i=0}^{s} B(i\varpi_2)$.*

The proof of this proposition follows from the next three lemmas.

**Lemma 4.6.** *Suppose that $t \in \bigoplus_{k=0}^{s-1} B(k\varpi_2)$ has no subtableaux $\begin{smallmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{smallmatrix}$ satisfying Criterion 4.4. Then exactly one of either appending $\begin{smallmatrix} a_k \\ \bar{a}_k \end{smallmatrix} \cdots \begin{smallmatrix} a_k \\ \bar{a}_k \end{smallmatrix}$ or prepending $\begin{smallmatrix} \bar{b}_1 \\ b_1 \end{smallmatrix} \cdots \begin{smallmatrix} \bar{b}_1 \\ b_1 \end{smallmatrix}$ to $t$ will produce a tableau in $\mathcal{T}(s) \setminus B(s\varpi_2)$.*

**Proof:** Suppose $t = \begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix} \cdots \begin{smallmatrix} a_k \\ b_k \end{smallmatrix} \in B(k\varpi_2)$ is as above for $k < s$. We will show that if prepending $\begin{smallmatrix} \bar{b}_1 \\ b_1 \end{smallmatrix} \cdots \begin{smallmatrix} \bar{b}_1 \\ b_1 \end{smallmatrix}$ to $t$ does not produce a tableau in $\mathcal{T}(s) \setminus B(s\varpi_2)$, then appending $\begin{smallmatrix} a_k \\ \bar{a}_k \end{smallmatrix} \cdots \begin{smallmatrix} a_k \\ \bar{a}_k \end{smallmatrix}$ to $t$ will produce a tableau in $\mathcal{T}(s) \setminus B(s\varpi_2)$. There are two reasons we might not be able to prepend $\begin{smallmatrix} \bar{b}_1 \\ b_1 \end{smallmatrix} \cdots \begin{smallmatrix} \bar{b}_1 \\ b_1 \end{smallmatrix}$; $b_1$ may be unbarred, or we may have $a_1 < \bar{b}_1$.

First, suppose $b_1$ is unbarred. If $b_k$ is also unbarred, then $b_k$ is certainly less than $\bar{a}_k$, so we may append $\begin{smallmatrix} a_k \\ \bar{a}_k \end{smallmatrix} \cdots \begin{smallmatrix} a_k \\ \bar{a}_k \end{smallmatrix}$ to $t$. Hence, suppose that $b_k$ is barred. We will show that $a_k$ is unbarred and $\bar{a}_k > b_k$.

We know that $t$ has a subtableau of the form $\begin{smallmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{smallmatrix}$ such that $b_i$ is unbarred and $b_{i+1}$ is barred. It follows that $a_i$ is unbarred, and therefore $\bar{a}_i > b_i$. Since $\begin{smallmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{smallmatrix}$ does not satisfy Criterion 4.4, this means that $\bar{a}_i > b_{i+1}$, which is equivalent to $\bar{b}_{i+1} > a_i$. Once again observing that $\begin{smallmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{smallmatrix}$ does not satisfy Criterion 4.4, this implies that $\bar{b}_{i+1} > a_{i+1}$; i.e., $a_{i+1}$ is unbarred, and $\bar{a}_{i+1} > b_{i+1}$.

We proceed with an inductive argument on $i < j < k$. Suppose that $\begin{smallmatrix} a_j & a_{j+1} \\ b_j & b_{j+1} \end{smallmatrix}$ is a subtableau of $t$ such that $b_j$ and $b_{j+1}$ are barred, $a_j$ is unbarred, and $\bar{a}_j > b_j$. By reasoning identical to the above, we conclude that

$$\bar{a}_j > b_{j+1} \Rightarrow \bar{b}_{j+1} > a_j \Rightarrow \bar{b}_{j+1} > a_{j+1} \Rightarrow \bar{a}_{j+1} > b_{j+1}, \tag{10}$$

which once again means that $a_{j+1}$ is unbarred.

This inductively shows that $a_k$ is unbarred and $\bar{a}_k > b_k$, so we may append $\begin{smallmatrix} a_k \\ \bar{a}_k \end{smallmatrix} \cdots \begin{smallmatrix} a_k \\ \bar{a}_k \end{smallmatrix}$ to $t$ to get a tableau in $\mathcal{T}(s) \setminus B(s\varpi_2)$. By a symmetrical argument, we conclude that if $a_k$ is barred, then we may prepend $\begin{smallmatrix} \bar{b}_1 \\ b_1 \end{smallmatrix} \cdots \begin{smallmatrix} \bar{b}_1 \\ b_1 \end{smallmatrix}$ to $t$.

Now, suppose that $b_1$ is barred and $\bar{b}_1 > a_1$. This means that $a_1$ is unbarred and $\bar{a}_1 > b_1$, so the induction carried out in equation 10 applies. It follows that $a_k$ is unbarred and $\bar{a}_k > b_k$, so once again we may append $\begin{smallmatrix} a_k \\ \bar{a}_k \end{smallmatrix} \cdots \begin{smallmatrix} a_k \\ \bar{a}_k \end{smallmatrix}$ to $t$. Also, by a symmetrical argument, when $a_k$ is unbarred and $b_k > \bar{a}_k$, we may prepend $\begin{smallmatrix} \bar{b}_1 & \cdots & \bar{b}_1 \\ b_1 & & b_1 \end{smallmatrix}$ to $t$. Thus, when no subtableau of $t$ satisfy Criterion 4.4, either appending $\begin{smallmatrix} a_k & \cdots & a_k \\ \bar{a}_k & & \bar{a}_k \end{smallmatrix}$ or prepending $\begin{smallmatrix} \bar{b}_1 & \cdots & \bar{b}_1 \\ b_1 & & b_1 \end{smallmatrix}$ to $t$ will produce a tableau in $\mathcal{T}(s) \setminus B(s\varpi_2)$. □

**Lemma 4.7.** *Any tableau $t = \begin{smallmatrix} a_1 & \cdots & a_k \\ b_1 & & b_k \end{smallmatrix} \in \bigoplus_{k=0}^{s-1} B(k\varpi_2)$ has no more than two filling locations. If it has two, they are consecutive integers, and this choice has no effect on $F_{2,s}(t)$.*

**Proof:** Let $0 \le i_* \le k$ be minimal such that $i_*$ is a filling location of $t$. First assume that $0 < i_* < k$. This implies the existence of a subtableau $\begin{smallmatrix} a_{i_*} & a_{i_*+1} \\ b_{i_*} & b_{i_*+1} \end{smallmatrix}$ which satisfies Criterion 4.4.

Suppose that the first condition $b_{i_*} \le \bar{a}_{i_*} \le b_{i_*+1}$ of Criterion 4.4 is satisfied, and consider whether $i_* + 1$ can be a filling location. If $b_{i_*+1} \le \bar{a}_{i_*+1} \le b_{i_*+2}$, we have

$$b_{i_*+1} \le \bar{a}_{i_*+1} \le \bar{a}_{i_*} \le b_{i_*+1},$$

which implies that $\bar{a}_{i_*} = \bar{a}_{i_*+1} = b_{i_*+1}$, so that $t$ violates part 3 of Criterion 3.1. Similarly, if $a_{i_*+1} \le \bar{b}_{i_*+2} \le a_{i_*+2}$, then we have

$$\bar{a}_{i_*+1} \le \bar{a}_{i_*} \le b_{i_*+1} \le b_{i_*+2} \le \bar{a}_{i_*+1},$$

which also implies that $\bar{a}_{i_*} = \bar{a}_{i_*+1} = b_{i_*+1}$, once again violating part 3 of Criterion 3.1. We conclude that if $i_*$ is a filling location for which Criterion 4.4 is satisfied by $b_{i_*} \le \bar{a}_{i_*} \le b_{i_*+1}$, then $i_* + 1$ is not a filling location. Furthermore, this argument shows that $a_{i_*+1} > a_{i_*}$ or $b_{i_*+1} > \bar{a}_{i_*}$. By the partial ordering on our alphabet, it follows that $t$ has no other filling locations.

Now, suppose for the filling location $i_*$, Criterion 4.4 is satisfied by $a_{i_*} \le \bar{b}_{i_*+1} \le a_{i_*+1}$. The condition $a_{i_*+1} \le \bar{b}_{i_*+2} \le a_{i_*+2}$ for $i_* + 1$ to be a filling location implies that

$$\bar{b}_{i_*+2} \le \bar{b}_{i_*+1} \le a_{i_*+1} \le \bar{b}_{i_*+2},$$

which as above leads to a violation of part 3 of Criterion 3.1. However, $i_* + 1$ may be a filling location if Criterion 4.4 is satisfied by $b_{i_*+1} \le \bar{a}_{i_*+1} \le b_{i_*+2}$. Note that this inequality implies that $a_{i_*+1} \le \bar{b}_{i_*+1}$, which tells us that $a_{i_*+1} = \bar{b}_{i_*+1}$. Thus, choosing to insert $\begin{smallmatrix} \bar{b}_{i_*+1} & \cdots & \bar{b}_{i_*+1} \\ b_{i_*+1} & & b_{i_*+1} \end{smallmatrix}$ between columns $i_*$ and $i_* + 1$ or to insert $\begin{smallmatrix} a_{i_*+1} & \cdots & a_{i_*+1} \\ \bar{a}_{i_*+1} & & \bar{a}_{i_*+1} \end{smallmatrix}$ between columns $i_* + 1$ and $i_* + 2$ does not change $F_{2,s}(t)$. Since $i_* + 1$ is a filling location with Criterion 4.4 satisfied by $b_{i_*} \le \bar{a}_{i_*} \le b_{i_*+1}$, the preceding paragraph implies that there are no other filling locations in $t$.

Finally, suppose that $i_* = 0$ is a filling location for $t$; i.e., $b_1$ is barred, $a_1$ is unbarred, and $\bar{b}_1 \le a_1$. If 1 is a filling location, Criterion 4.4 is satisfied by $b_1 \le \bar{a}_1 \le b_2$; otherwise, part 3 of Criterion 3.1 is violated. Put together, this means that $\bar{a}_1 = b_1$, so prepending $\begin{smallmatrix} \bar{b}_1 & \cdots & \bar{b}_1 \\ b_1 & & b_1 \end{smallmatrix}$ to $t$ and inserting $\begin{smallmatrix} a_1 & \cdots & a_1 \\ \bar{a}_1 & & \bar{a}_1 \end{smallmatrix}$ between columns 1 and 2 results in the same tableau. As in the above cases, part 3 of Criterion 3.1 and the partial order on the alphabet prohibit any other filling locations. □

*Example 4.8.* Let $s = 4$. Then

$$t = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \bar{4} & \bar{2} & \bar{1} \\ \hline \end{array}, \quad F_{2,4}(t) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline \bar{4} & \bar{2} & \bar{2} & \bar{1} \\ \hline \end{array}.$$

While we could choose either column two or column three as the filling location, either choice results in the same tableau.

**Lemma 4.9.** *If a filling location of* $t = \begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix} \cdots \begin{smallmatrix} a_k \\ b_k \end{smallmatrix} \in \bigoplus_{i=0}^{s-1} B(i\varpi_2)$ *satisfies Criterion 4.4 with both inequalities, then* $F_{2,s}(t)$ *is independent of this choice.*

**Proof:** Suppose that $i_* \neq 0, k$ is a filling location for $t$ where both parts of Criterion 4.4 are satisfied. This means that the subtableau $\begin{smallmatrix} a_{i_*} & a_{i_*+1} \\ b_{i_*} & b_{i_*+1} \end{smallmatrix}$ satisfies both $\bar{a}_{i_*} \leq b_{i_*+1}$ and $a_{i_*} \leq \bar{b}_{i_*+1}$. The latter of these implies that $b_{i_*+1} \leq \bar{a}_{i_*}$, so we have $\bar{a}_{i_*} = b_{i_*+1}$ and $\bar{b}_{i_*+1} = a_{i_*}$. Thus, filling with either $\begin{smallmatrix} a_{i_*} \\ \bar{a}_{i_*} \end{smallmatrix} \cdots \begin{smallmatrix} a_{i_*} \\ \bar{a}_{i_*} \end{smallmatrix}$ or $\begin{smallmatrix} \bar{b}_{i_*+1} \\ b_{i_*+1} \end{smallmatrix} \cdots \begin{smallmatrix} \bar{b}_{i_*+1} \\ b_{i_*+1} \end{smallmatrix}$ between columns $i_*$ and $i_* + 1$ results in the same tableau $F_{2,s}(t)$. □

*Example 4.10.* To illustrate, for

$$t = \begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\ \hline \bar{4} & \bar{2} & \bar{1} \\ \hline \end{array} \quad \text{we have} \quad F_{2,s}(t) = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 3 \\ \hline \bar{4} & \bar{2} & \bar{2} & \bar{1} \\ \hline \end{array}.$$

By identifying $\mathcal{T}(s)$ with $\bigoplus_{i=0}^{s} B(i\varpi_2)$ via the maps $D_{2,s}$ and $F_{2,s}$, we have defined a $U_q(D_n)$-crystal with the decomposition (1), with vertices labeled by the $2 \times s$ tableaux of $\mathcal{T}(s)$. The action of the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ for $i \in \{1, \ldots, n\}$ on this crystal is defined in terms of the above bijection, given explicitly by

$$\begin{aligned} \tilde{e}_i(T) &= F_{2,s}(\tilde{e}_i(D_{2,s}(T))) \\ \tilde{f}_i(T) &= F_{2,s}(\tilde{f}_i(D_{2,s}(T))), \end{aligned} \tag{11}$$

for $T \in \mathcal{T}(s)$, where the $\tilde{e}_i$ and $\tilde{f}_i$ on the right are the standard Kashiwara operators on $U_q(D_n)$-crystals [17]. In section 6 we will discuss the action of $\tilde{e}_0$ and $\tilde{f}_0$ on $\mathcal{T}(s)$, which will make $\mathcal{T}(s)$ into an affine crystal called $\tilde{B}^{2,s}$.

Using the filling and dropping map we obtain a natural inclusion of $\mathcal{T}(s')$ into $\mathcal{T}(s)$ for $s' < s$.

*Definition 4.11.* For $s' < s$, the map $\Upsilon_{s'}^{s} : \mathcal{T}(s') \hookrightarrow \mathcal{T}(s)$ is defined by $\Upsilon_{s'}^{s} = F_{2,s} \circ D_{2,s'}$.

## 5. The branching component graph

The Dynkin diagram of $D_n^{(1)}$ has an automorphism interchanging the nodes 0 and 1, which induces a map $\sigma : B^{2,s} \to B^{2,s}$ on the crystals such that $\tilde{e}_0 = \sigma \tilde{e}_1 \sigma$ and $\tilde{f}_0 = \sigma \tilde{f}_1 \sigma$. With this in mind, suppose we have defined $\tilde{f}_0$ on $\mathcal{T}(s)$ to produce $\tilde{B}^{2,s}$, and consider the following operations on $\tilde{B}^{2,s}$: Let $K \subset I$, and denote by $B_K$ the graph which results from removing all $k$-colored edges from $\tilde{B}^{2,s}$ for $k \in K$. Then as directed graphs, we expect $B_{\{0\}}$ to be isomorphic to $B_{\{1\}}$; otherwise, $\tilde{B}^{2,s}$ and $B^{2,s}$ will not be isomorphic. We can gain some information about $\sigma$ by considering $B_{\{0,1\}}$. The combinatorial structure of $B_{\{0,1\}}$ is encoded in the branching component graph to be defined in this section.

The definition of $\sigma$ relies on several sets of data, which will be defined in sections 5 and 6. For all $k \geq 0$ there is a filtration of $B(k\varpi_2)$ by subgraphs isomorphic to $B(\ell\varpi_2)$ for

$\ell \leq k$; this relates any classical component of $\tilde{B}^{2,s}$ to the other classical components. Once this filtration is understood, we will see that the following data uniquely determine a vertex $b$ of $\tilde{B}^{2,s}$:

(1) its classical component $k$ in the direct sum $\bigoplus_{k=0}^{s} B(k\varpi_2)$;
(2) its position $\ell$ in the filtration $B(k\varpi_2) \supset \cdots \supset B(\ell\varpi_2) \supset \cdots \supset B(0)$;
(3) the number of 1-arrows in a path to $b$ from the highest weight vector of $B(k\varpi_2)$;
(4) the $D_{n-1}$-highest weight $\lambda$ of its connected component in $B_{\{0,1\}}$;
(5) its position $b = \tilde{f} v_\lambda = \tilde{f}_{i_1}^{m_1} \tilde{f}_{i_2}^{m_2} \cdots v_\lambda$ in the $D_{n-1}$-crystal $B(\lambda)$.

The involution $\sigma$ has a very simple description in terms of these data. In fact, $\sigma$ changes only items (1) and (3), leaving the other data fixed.

## 5.1. Definitions and preliminary discussion

The connected components of $B_{\{0,1\}}$ are $U_q(D_{n-1})$-crystals, indexed by partitions as described in this section. The decomposition of $\tilde{B}^{2,s}$ into $B_{\{0,1\}}$ produces a branching component graph for $\tilde{B}^{2,s}$, which we denote $\mathcal{BC}(\tilde{B}^{2,s})$. The vertices of this graph correspond to the connected $U_q(D_{n-1})$-crystals; a vertex $v_\lambda$ is labeled (non-uniquely) by the partition $\lambda$ indicating the classical highest weight of the corresponding $U_q(D_{n-1})$-crystal. The edges of $\mathcal{BC}(\tilde{B}^{2,s})$ are defined by placing an edge from $v_\lambda$ to $v_\mu$ if there is a tableau $b \in B(v_\lambda)$ such that $\tilde{f}_1(b) \in B(v_\mu)$, where $B(v_\lambda)$ denotes the set of tableaux contained in the $U_q(D_{n-1})$-crystal indexed by $v_\lambda$.

Note that it suffices to describe the decomposition of the component of $\tilde{B}^{2,s}$ with $U_q(D_n)$ highest weight $k\varpi_2$ into $U_q(D_{n-1})$-crystals for any $k \geq 0$, since

$$\mathcal{BC}\left(\bigoplus_{k=0}^{s} B(k\varpi_2)\right) = \bigoplus_{k=0}^{s} \mathcal{BC}(B(k\varpi_2)).$$

Denote the branching component subgraph with classical highest weight $k\varpi_2$ by $\mathcal{BC}(k\varpi_2)$. Since $\mathcal{BC}(k\varpi_2)$ is determined by the action of $\tilde{e}_i$ and $\tilde{f}_i$ on $B(k\varpi_2)$ for $i = 1, \ldots, n$, which is in turn defined by composing the classical Kashiwara operators with $D_{2,s}$ and $F_{2,s}$ (see equation (11)), it in fact suffices to determine the structure of $\mathcal{BC}(s\varpi_2) \subset \mathcal{BC}(\tilde{B}^{2,s})$.

The branching component graph $\mathcal{BC}(s\varpi_2)$ is characterized by the following proposition. We denote by $v_s$ the "highest weight" branching component vertex (that is to say the vertex $v$ such that the highest weight vector $u_s$ of $B(s\varpi_2)$ is in $B(v)$) of $\mathcal{BC}(s\varpi_2)$.

**Proposition 5.1.** *The graph distance from $v_s$ defines a rank function on $\mathcal{BC}(s\varpi_2)$. This graph has $2s+1$ ranks, and is symmetric as a non-directed graph over rank $s$. For $j \leq s$, the $j^{\text{th}}$ rank contains one of each partition $\lambda = (\lambda_1, \lambda_2) \subset (s, j)$ such that $|\lambda| = s - j + 2m$ for some $m \in \mathbb{Z}_{\geq 0}$. For all ranks $0 \leq j \leq 2s-1$, a vertex $v_\lambda$ with rank $j$ has an arrow to a vertex $v_\mu$ with rank $j+1$ if and only if $\lambda$ and $\mu$ are joined by an edge in Young's lattice.*

We begin by examining the first few ranks of $\mathcal{BC}(s\varpi_2)$ in detail, then show that this proposition is true in general in sections 5.2 and 5.3.

The highest weight branching component vertex $v_s$ is indexed by the one-part partition $(s)$. To see that this is true, simply observe that the highest weight tableau of $B(s\varpi_2)$ is

$\underbrace{\begin{smallmatrix}1 & & 1\\ 2 & \cdots & 2\end{smallmatrix}}_{s}$, and acting by $\tilde{f}_2, \ldots, \tilde{f}_n$ in the most general possible way will affect only the bottom row. When we map these bottom row subtableaux componentwise by $a \mapsto a-1$ and $\bar{a} \mapsto \overline{a-1}$ to tableaux of shape $(s)$, and apply the same map to the colors of the arrows, this is clearly isomorphic to the $U_q(D_{n-1})$-crystal with highest weight $s\varpi_1$.

Now, consider what can result from acting on a tableau $T = \begin{smallmatrix}a_1 & \cdots & a_s\\ b_1 & & b_s\end{smallmatrix}$ in $B(v_s)$ by $\tilde{f}_1$. Since $a_1 = \cdots = a_s = 1$, this will turn $a_s$ into a 2. There are two cases to consider: if $b_s = \bar{2}$, this results in a tableau with a configuration $\begin{smallmatrix}2\\ \bar{2}\end{smallmatrix}$ at the right end (note that $\tilde{f}_i$, $\tilde{e}_i$ for $i = 2, \ldots, n$ do not act on this subtableau); otherwise, it is a tableau with $a_1 = \cdots = a_{s-1} = 1$ where some element of $U_q(D_{n-1})$ can act on the rightmost column. In either case, we can act with $\tilde{e}_2, \ldots, \tilde{e}_n$ to find a $U_q(D_{n-1})$ highest weight vector $T' = \begin{smallmatrix}a'_1 & \cdots & a'_s\\ b'_1 & & b'_s\end{smallmatrix}$, where we have $b'_1 = \cdots = b'_{s-1} = 2$; in the first case, we have $b'_s = \bar{2}$, in the other, we have $b'_s = 3$. Remove those parts of these tableaux on which $\tilde{e}_i$ and $\tilde{f}_i$ for $i = 2, \ldots, n$ do not act; in both cases, we remove $a'_1, \ldots, a'_{s-1}$, and in the first case we also remove the $\begin{smallmatrix}2\\ \bar{2}\end{smallmatrix}$ at the end. We then have a skew tableau, which when rectified by Lecouvey $D$ equivalence (or, since there are no barred letters remaining, *jeu de taquin*), is either the tableau $2 \cdots 2$ of shape $(s-1)$, or the tableau of shape $(s, 1)$ with 2's in the first row and a 3 in the second. We conclude that there are two vertices of rank 1 in $\mathcal{BC}(s\varpi_2)$, corresponding to the partitions $(s-1)$ and $(s, 1)$.

Before we generalize this construction, we have a few technical remarks.

The number of 1-arrows in a minimal path in the crystal graph between the highest weight tableau and a tableau $T$ is the "$\alpha_1$-height" of $T$. Thus, the function

$$r_s(v) = d(v, v_s) = \min_{P(v,v_s)} \{\text{number of edges in } P(v, v_s)\}$$

where $P(v, v_s)$ is the set of all paths from $v$ to $v_s$ in $\mathcal{BC}(s\varpi_2)$, is a rank function on $\mathcal{BC}(s\varpi_2)$.

*Definition 5.2.* A null-configuration of size $k$ is

$$\underbrace{\begin{matrix}1 & & 1 & 2 & & 2\\ \bar{2} & \cdots & \bar{2} & \bar{1} & \cdots & \bar{1}\end{matrix}}_{k} \qquad \text{if } k \text{ is even,}$$

$$\underbrace{\begin{matrix}1 & & 1 & 2 & 2 & & 2\\ \bar{2} & \cdots & \bar{2} & \bar{2} & \bar{1} & \cdots & \bar{1}\end{matrix}}_{k} \qquad \text{if } k \text{ is odd,}$$

where the number of 1's equals the number of $\bar{1}$'s and the number of 2's equals the number of $\bar{2}$'s.

Null-configurations are named thus because $\tilde{e}_i$ and $\tilde{f}_i$ for $i = 2, \ldots, n$ send $T$ to 0, where $T$ is the $2 \times s$ tableau which is a null-configuration of size $s$. Therefore, $T$ is the basis vector for the trivial representation of $U_q(D_{n-1})$ in $\mathcal{BC}(s\varpi_2)$. Put another way, inserting a null-configuration into a tableau $T$ has no effect on $\varepsilon_i(T)$ or $\varphi_i(T)$ for $i = 2, \ldots, n$. This generalizes the phenomenon we observed in the case of $\begin{smallmatrix}2\\ \bar{2}\end{smallmatrix}$.

### 5.2. Content of rank $j$

We now characterize the partitions occuring in any rank $0 \le j \le s$ of the branching component graph. (Ranks greater than $s$ will be defined by the $*$-duality of the crystal as defined in section 3.3.) We defer the discussion of the edges of the branching component graph to section 5.3.

Let $T \in B(s\varpi_2)$. We wish to determine the vertex $v_\lambda$ of $\mathcal{BC}(s\varpi_2)$ for which $T \in B(v_\lambda)$, and also to determine $r_s(v_\lambda)$. As demonstrated for ranks 0 and 1 above, determine the parts of $T$ on which $\tilde{e}_i$ and $\tilde{f}_i$ for $i = 2, \dots, n$ do not act: this will be a null-configuration of size $r_2$ (possibly of size 0), $r_1$ many 1's in the first row before the null-configuration, and $r_3$ many $\bar{1}$'s in the second row after the null-configuration. We can extract from these data the pair

$$(t_1, t_2) = (r_1 + r_2, r_2 + r_3), \tag{12}$$

where $t_1, t_2 \le s$. By observing the number of times 1 appears in a sequence $i_1, \dots, i_p$ such that the highest weight vector of $B(s\varpi_2)$ is $u_s = \tilde{e}_{i_1} \dots \tilde{e}_{i_p} T$, it is easily seen that $r_s(v_\lambda) = s - t_1 + t_2$.

Consider the set $\mathcal{J}$ of tableaux such that $s - t_1 + t_2 = j \le s$. We wish to determine the partitions $\lambda$ such that $T \in \mathcal{J}$ are in a $U_q(D_{n-1})$-crystal with highest weight specified by $\lambda$. First, note that $|\lambda| = 2s - t_1 - t_2$, since this is precisely the number of boxes where $\tilde{e}_i$ and $\tilde{f}_i$ for $i = 2, \dots, n$ act non-trivially. It follows that $|\lambda| = s + j - 2t_2$, so $|\lambda| \equiv s + j \pmod 2$, and since $t_2$ ranges from 0 to $j$, we have $s - j \le |\lambda| \le s + j$. Based on the definition of $\tilde{e}_i$ and $\tilde{f}_i$ given in section 3.2, it is clear that other than the $t_1 + t_2$ boxes with 1's, $\bar{1}$'s, and the null-configuration, a $U_q(D_{n-1})$-highest weight tableau must have only 2's and 3's. We may remove the irrelevant $t_1 + t_2$ boxes from $T$ resulting in a skew tableau $T^\#$. All the letters in $T^\#$ are unbarred, so the Lecouvey relations applied to $w_{T^\#}$ yield the column word of the rectification of $T^\#$ (we call this rectified tableau the completely reduced form of $T$), whose shape has no more than two parts. Let $\mathcal{I} \subset \mathcal{J}$ be the set of $U_q(D_{n-1})$-highest weight tableaux with specified values for $t_1$ and $t_2$. Then $\mathcal{I}$ includes tableaux where the number of 2's ranges from $s - t_2$ up to $\min(2s - t_1 - t_2, s)$, and the number of 3's ranges simultaneously from $s - t_1$ down to $\max(0, s - t_1 - t_2)$. The algorithm described above can therefore produce a tableau of any shape $\lambda$ with two parts such that $|\lambda| = 2s - t_1 - t_2$, $\lambda_1 \le s$, and $\lambda_2 \le s - t_1 = j - t_2$. By properties of the plactic monoid, no two $U_q(D_{n-1})$-highest weight tableaux in $\mathcal{J}$ correspond to the same partition.

To summarize: In rank $j \le s$ of $\mathcal{BC}(s\varpi_2)$, the vertices correspond exactly to partitions $\lambda = (\lambda_1, \lambda_2) \subset (s, j)$ such that $|\lambda| = s - j + 2m$ for some $m \in \mathbb{Z}_{\ge 0}$.

By the $*$-symmetry of $B(s\varpi_2)$ as described in section 3.3, it is clear that the $U_q(D_{n-1})$-crystals of rank $j$ are the same as the $U_q(D_{n-1})$-crystals of rank $2s - j$. This completely characterizes the vertices of $\mathcal{BC}(s\varpi_2)$ by rank, and leads us to the following remark.

*Remark 5.3.* If we consider the embedding $U_q(D_{n-1}) \hookrightarrow U_q(D_n)$ as implicitly described above, and think of the action of $e_1, f_1 \in U_q(D_n)$ as specifying a rank function on the embedded $U_q(D_{n-1})$-modules in a given $U_q(D_n)$-module with highest weight $s\varpi_2$, this provides a combinatorial proof that the ranks are multiplicity-free.

### 5.3. Edges of $\mathcal{BC}(s\varpi_2)$

We must now confirm that the pairs of vertices which have an arrow between them are precisely those $v_\lambda$ and $v_\mu$ such that $r_s(v_\lambda) = j$ and $r_s(v_\mu) = j + 1$ for some $0 \le j \le 2s - 1$, and for which $\lambda$ and $\mu$ are adjacent in Young's lattice, that is, $\mu$ is obtained from $\lambda$ by either adding or removing a box. To do this, we will construct tableaux in $B(v_\lambda)$ such that the shape of the completely reduced form of their image under $\tilde{f}_1$ is the result of adding a box to $\lambda$. The question of removing boxes from $\lambda$ then is simply a matter of appealing to the $*$-symmetry of the crystal graph as described in section 3.3.

Our analysis breaks into two cases, where our tableau $T \in B(v_\lambda)$ may be of one of the following two forms:

(1)

$$T = \underbrace{\begin{matrix} 1 \\ b_1 \end{matrix} \cdots\cdots \begin{matrix} 1 \\ b_{r_1} \end{matrix}}_{r_1} \underbrace{\begin{matrix} 1 \\ \bar{2} \end{matrix} \cdots \begin{matrix} 1 \\ \bar{2} \end{matrix} \begin{pmatrix} 2 \\ \bar{2} \end{pmatrix} \begin{matrix} 2 \\ \bar{1} \end{matrix} \cdots \begin{matrix} 2 \\ \bar{1} \end{matrix}}_{r_2} \underbrace{\begin{matrix} a_{s-r_3+1} \\ \bar{1} \end{matrix} \cdots \begin{matrix} a_s \\ \bar{1} \end{matrix}}_{r_3},$$

(2)

$$T = \underbrace{\begin{matrix} 1 \\ b_1 \end{matrix} \cdots \begin{matrix} 1 \\ b_{r_1} \end{matrix}}_{r_1} \underbrace{\begin{matrix} a_{r_1+1} \\ b_{r_1+1} \end{matrix} \cdots \begin{matrix} a_{s-r_3} \\ b_{s-r_3} \end{matrix}}_{u=s-r_1-r_3} \underbrace{\begin{matrix} a_{s-r_3+1} \\ \bar{1} \end{matrix} \cdots \begin{matrix} a_s \\ \bar{1} \end{matrix}}_{r_3},$$

where in case (1), the block of length $r_2$ is a maximal null-configuration, and in case (2), $a_{r_1+1} \neq 1$ and $b_{s-r_3} \neq \bar{1}$ (we set $r_2 = 0$ here). We now determine for which partitions $\mu$ we can have $\tilde{f}_1(T) \in B(v_\mu)$. Recall from the previous subsection that for $T \in B(v_\lambda)$, we defined $T^\#$ to be the skew tableau which results from removing all 1's, $\bar{1}$'s, and null-configurations from $T$. Observe that

$$w_{T^\#} = \begin{cases} b_1 \cdots b_{r_1} a_{s-r_3+1} \cdots a_s & \text{for case (1)} \\ b_1 \cdots b_{r_1} b_{r_1+1} a_{r_1+1} \cdots b_{s-r_3} a_{s-r_3} a_{s-r_3+1} \cdots a_s & \text{for case (2).} \end{cases}$$

In either case, if $b_{r_1} = \bar{2}$, the size of the null-configuration in $\tilde{f}_1(T)$ is $r_2 + 1$, since in case (1) $\tilde{f}_1$ acts on the middle of the null-configuration, and in case (2) $\tilde{f}_1$ acts on $a_{r_1} = 1$. It follows that $w_{\tilde{f}_1(T)^\#}$ is simply $w_{T^\#}$ with the $\bar{2}$ contributed by $b_{r_1}$ removed. If $b_{r_1} \neq \bar{2}$, we see that $w_{\tilde{f}_1(T)^\#}$ is $w_{T^\#}$ with a 2 inserted from $a_{s-r_3}$ in case (1), and from $a_{r_1}$ in case (2). Since we are currently concerned with adding boxes to $\lambda$, let us assume that $b_{r_1} \neq \bar{2}$, and analyze how inserting a 2 as above affects the shape of the rectifications of these words.

Our augmented words are

$$w_{\tilde{f}_1(T)^\#} = \begin{cases} b_1 \cdots b_{r_1} 2\, a_{s-r_3+1} \cdots a_s & \text{for case (1)} \\ b_1 \cdots b_{r_1} 2\, b_{r_1+1} a_{r_1+1} \cdots b_{s-r_3} a_{s-r_3} a_{s-r_3+1} \cdots a_s & \text{for case (2).} \end{cases} \tag{13}$$

Recall that we have assumed that $b_{r_1} \neq \bar{2}$, which in turn implies that all letters $b_1, \ldots, b_{r_1}$ are strictly less than $\bar{2}$. Using relation (1) of Lecouvey type $D$ equivalence, we may therefore move the 2 from position $a_{r_1}$ to the second position in the word. This new word begins $b_1 2 b_2$, with $b_2 > 2$. Since we may view all the plactic operations on this word as sliding moves, the subword $b_2 \cdots a_s$ can be rectified to give a tableau with no more than two rows. Thus, all we have done is added one box to our shape.

We now show that this process can add a box to the top row of $\lambda$ unless $\lambda_1 = s$, and it can add a box to the bottom row unless $\lambda_2 = \lambda_1$. In the $U_q(D_{n-1})$-crystal $B(v_\lambda)$, we know that there is a $U_q(D_{n-1})$ highest weight tableau $T_\lambda$ of the form

$$T_\lambda = \begin{cases} \underbrace{\begin{matrix}1\\2\end{matrix}\cdots\begin{matrix}1\\2\end{matrix}}_{r_1-\lambda_2}\underbrace{\begin{matrix}1\\3\end{matrix}\cdots\begin{matrix}1\\3\end{matrix}}_{\lambda_2}\underbrace{\begin{matrix}1\\\bar{2}\end{matrix}\cdots\begin{matrix}1\\\bar{2}\end{matrix}\begin{pmatrix}2\\\bar{2}\end{pmatrix}\begin{matrix}2\\\bar{1}\end{matrix}\cdots\begin{matrix}2\\\bar{1}\end{matrix}}_{r_2}\underbrace{\begin{matrix}2\\\bar{1}\end{matrix}\cdots\begin{matrix}2\\\bar{1}\end{matrix}}_{r_3} & \text{for case (1)}\\ \underbrace{\begin{matrix}1\\2\end{matrix}\cdots\begin{matrix}1\\2\end{matrix}}_{r_1-(\lambda_2-u)}\underbrace{\begin{matrix}1\\3\end{matrix}\cdots\begin{matrix}1\\3\end{matrix}}_{\lambda_2-u}\underbrace{\begin{matrix}2\\3\end{matrix}\cdots\begin{matrix}2\\3\end{matrix}}_{u}\underbrace{\begin{matrix}2\\\bar{1}\end{matrix}\cdots\begin{matrix}2\\\bar{1}\end{matrix}}_{r_3} & \text{for case (2).}\end{cases}$$

Note that in case (1) we have $\lambda_2 \leq r_3$ and in case (2) we have $\lambda_2 - u \leq r_3$; otherwise, acting by $\tilde{e}_2$ can turn another 3 into a 2.

These tableaux yield the words

$$w_{T_\lambda^\#} = \begin{cases} \underbrace{2\cdots 2}_{r_1-\lambda_2}\underbrace{3\cdots 3}_{\lambda_2}\underbrace{2\cdots 2}_{r_3} & \text{for case (1)}\\ \underbrace{2\cdots 2}_{r_1-(\lambda_2-u)}\ \underbrace{3\cdots 3}_{\lambda_2-u}\underbrace{32\cdots 32}_{2u}\underbrace{2\cdots 2}_{r_3} & \text{for case (2).}\end{cases}$$

The completely reduced form of these tableaux is a two-row tableau with $r_1 + r_3 - \lambda_2$ 2's in the top row and $\lambda_2$ 3's in the bottom row, or $2u + r_1 + r_3 - \lambda_2$ 2's in the top row and $\lambda_2$ 3's in the bottom row, respectively. It is easy to see that by adding a 2 to $w_{T^\#}$ as in (13), we simply add a box containing a 2 to the top row of the completely reduced form of $T_\lambda$. Note that this procedure fails precisely when $T_\lambda$ can have no 2's added to it, in which case there are $s$ 2's in $T_\lambda$, and thus $\lambda_1 = s$.

Now suppose that $\lambda_1 - \lambda_2 > 0$, so that adding a box to the second row will produce a legal diagram. Consider $\tilde{T}_\lambda = \tilde{f}_2^{\lambda_1-\lambda_2}(T_\lambda)$ (note that $\lambda_1$ is the number of 2's in $T_\lambda$). This tableau is in $B(v_\lambda)$, so its completely reduced form has shape $\lambda$, and we see that

$$w_{\tilde{T}_\lambda^\#} = \begin{cases} \underbrace{3\cdots 3}_{|\lambda|-r_3}\underbrace{2\cdots 2}_{\lambda_2}\underbrace{3\cdots 3}_{r_3-\lambda_2} & \text{for case (1)}\\ \underbrace{3\cdots 3}_{|\lambda|-r_3-2u}\ \underbrace{32\cdots 32}_{2u}\underbrace{2\cdots 2}_{\lambda_2-u}\ \underbrace{3\cdots 3}_{r_3-\lambda_2+u} & \text{for case (2).}\end{cases}$$

The rectified tableau has $\lambda_2$ 2's followed by $\lambda_1 - \lambda_2$ 3's in the top row, and $\lambda_2$ 3's in the bottom row. From this description, we see that adding a 2 to $w_{\tilde{T}_\lambda^\#}$ as in (13) affects the completely reduced tableau by preventing one of the 3's from the bottom row from being slid up to the top row; i.e., $\lambda_2$ is increased by 1. Since we add only one box at a time and the only shape in rank 0 is $(s, 0)$, we know that the number of boxes in the second row can never exceed the rank.

We now invoke the $*$-duality of the crystal graph to deal with how boxes can be removed from $\lambda$. If $v_\lambda \in \mathcal{BC}(s\varpi_2)$ has rank $p$, there is a unique vertex $v'_\lambda$, called the complementary vertex of $v_\lambda$, with rank $2s - p$ for which the corresponding $U_q(D_{n-1})$-crystal is $B(\lambda)$. This involution agrees with the $*$-crystal involution of section 3.3. We wish to show that there is an arrow from $v_\lambda$ to $v_\mu$, where $\lambda/\mu$ is a single box and $r_s(v_\mu) = r_s(v_\lambda) + 1$. Recall that by definition, this is the case when for some $T \in B(v_\lambda)$ we have $\tilde{f}_1(T) \in B(v_\mu)$. Observe that $r_s(v'_\lambda) = r_s(v'_\mu) + 1$, and $\lambda$ is the result of adding a box to $\mu$; therefore, there is an arrow from

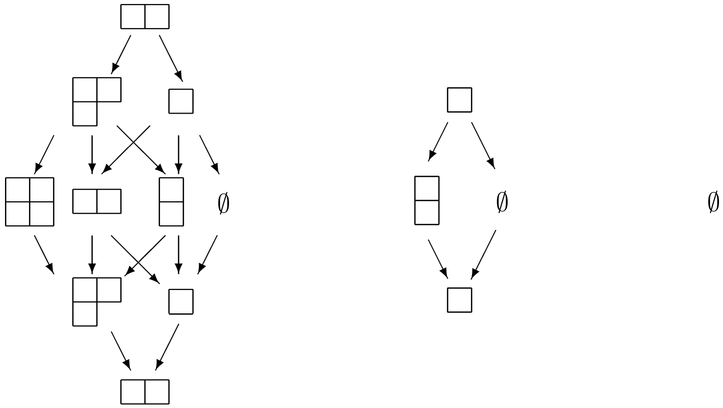



**Fig. 1** Branching component graph $\mathcal{BC}(\tilde{B}^{2,2})$

$v'_\mu$ to $v'_\lambda$. It follows that we can find some $T \in B(v'_\mu)$ such that $\tilde{f}_1(T) \in B(v'_\lambda)$. In turn, we have $T^* \in B(v_\mu)$ and $(\tilde{f}_1(T))^* = \tilde{e}_1(T^*) \in B(v_\lambda)$. Since we know that $\tilde{f}_1(\tilde{e}_1(T^*)) = T^* \in B(v_\mu)$, we have shown that there is an arrow from $v_\lambda$ to $v_\mu$.

The arguments of sections 5.2 and 5.3 prove Proposition 5.1.

5.4. Construction of $\mathcal{BC}(\tilde{B}^{2,s})$

Observe that $\mathcal{BC}(\tilde{B}^{2,s}) = \bigcup_{i=0}^{s} \mathcal{BC}(i\varpi_2)$. Let $v_\lambda \in \mathcal{BC}(i\varpi_2) \subset \mathcal{BC}(\tilde{B}^{2,s})$. Define $R(v_\lambda) = r_i(v_\lambda) + s - i$. This defines a rank on all of $\mathcal{BC}(\tilde{B}^{2,s})$. Note that $\mathcal{BC}(i\varpi_2) \subset \mathcal{BC}((i+1)\varpi_2)$, and this inclusion is compatible with $R$. Also note that if $R(v_\lambda) = p$, then $v'_\lambda$, the complementary vertex to $v_\lambda$, is now defined to be the vertex of rank $2s - p$ with the same shape and in the same component as $v_\lambda$.

To illustrate, $\mathcal{BC}(\tilde{B}^{2,2})$ is given in Fig. 1, with rank 0 in the first line, rank 1 in the second, etc.

## 6. Affine Kashiwara operators

Since we know that $B_{\{0\}}$ and $B_{\{1\}}$ are isomorphic as directed graphs, it is clear that we can put 0-colored edges in the branching component graph in such a way that interchanging the 1-edges and the 0-edges and applying some shape-preserving bijection $\breve{\sigma}$ to the vertices of the branching component graph will produce an isomorphic colored directed graph. Such a bijection can be naturally extended to $\sigma : \tilde{B}^{2,s} \to \tilde{B}^{2,s}$ as follows. Let $b \in B(v_\lambda) \subset \tilde{B}^{2,s}$ for some branching component vertex $v_\lambda$, and let $u_\lambda$ denote the $U_q(D_{n-1})$-highest weight vector of $B(v_\lambda)$. Then for some finite sequence $i_1, \ldots, i_k$ of integers in $\{2, \ldots, n\}$, we know that $\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\lambda = b$. Let $v_\lambda^\dagger = \breve{\sigma}(v_\lambda)$, and let $u_\lambda^\dagger$ be the highest weight vector of $B(v_\lambda^\dagger)$. We may define $\sigma(b) = \tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\lambda^\dagger$. This involution of $\tilde{B}^{2,s}$ allows us to define the affine structure of the crystal by the following equations:

$$\tilde{f}_0 = \sigma \tilde{f}_1 \sigma \qquad \text{and} \qquad \tilde{e}_0 = \sigma \tilde{e}_1 \sigma. \tag{14}$$

*Definition 6.1.* The affine crystal $\tilde{B}^{2,s}$ is given by the set $\mathcal{T}(s)$ as defined in section 4 with $\tilde{e}_i, \tilde{f}_i$ for $1 \le i \le n$ as in (11) and $\tilde{e}_0, \tilde{f}_0$ as in (14).

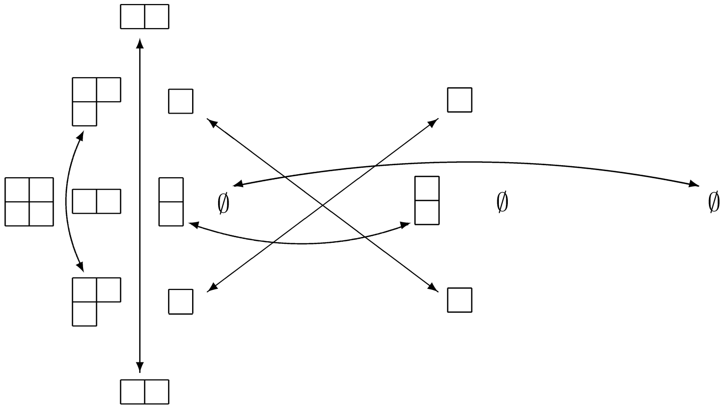



**Fig. 2** Definition of $\breve{\sigma}$ on $\mathcal{BC}(\tilde{B}^{2,2})$

Using $\breve{\sigma}$ as defined in section 6.1, it will be shown in section 7 that the resulting $U_q'(D_n^{(1)})$-crystal $\tilde{B}^{2,s}$ is perfect.

## 6.1. Construction of $\breve{\sigma}$

We will define $\breve{\sigma}(v_\lambda)$ for $R(v_\lambda) \leq s$, and observe that $\breve{\sigma}(v_\lambda') = \breve{\sigma}(v_\lambda)'$, where $v'$ denotes the complementary vertex of $v$. Let $v_\lambda \in \mathcal{BC}(k\varpi_2)$, $R(v_\lambda) = p$, and $\ell$ be minimal such that $\breve{\iota}_k^s(v_\lambda) \in \breve{\iota}_\ell^s(\mathcal{BC}(\ell\varpi_2))$, where $\breve{\iota}_i^j$ is the embedding of $\mathcal{BC}(i\varpi_2)$ in $\mathcal{BC}(j\varpi_2)$ for $i < j$. Then by the inclusion $\mathcal{BC}(i\varpi_2) \subset \mathcal{BC}((i+1)\varpi_2)$ for $i = 0, \ldots, s-1$, there are $s - \ell + 1$ vertices of the same shape as $v_\lambda$ of rank $p$ in $\mathcal{BC}(\tilde{B}^{2,s})$, one in each $\mathcal{BC}(j\varpi_2)$ for $j = \ell, \ldots, s$. We define $\breve{\sigma}(v_\lambda)$ to be the vertex of the same shape as $v_\lambda$ of rank $2s - p$ in the component $\mathcal{BC}((s + \ell - k)\varpi_2)$.

The action of $\breve{\sigma}$ on $\mathcal{BC}(\tilde{B}^{2,2})$ is given in Fig. 2.

## 6.2. Combinatorial construction of $\sigma$

We can also give a direct combinatorial description of $\sigma(T)$ for any $T \in \tilde{B}^{2,s}$. As an auxilliary construction (which will also be useful in its own right later on), we combinatorially describe $\iota_i^j : B(i\varpi_2) \hookrightarrow B(j\varpi_2)$, the unique crystal embedding that agrees with $\breve{\iota}_i^j : \mathcal{BC}(i\varpi_2) \hookrightarrow \mathcal{BC}(j\varpi_2)$.

*Remark 6.2.* It will often be useful to identify $B(k\varpi_2)$ with its image in $\tilde{B}^{2,s}$. We will use the notation $T \in B(k\varpi_2) \subset \tilde{B}^{2,s}$ to indicate this identification.

Let $i \in \{0, \ldots, s-1\}$, so $\iota_i^{i+1}$ denotes the embedding of $B(i\varpi_2)$ in $B((i+1)\varpi_2)$. Let $T \in B(i\varpi_2) \subset \tilde{B}^{2,s}$. This embedding can be combinatorially understood through the following observations:

*Remark 6.3.*

- $\varphi_k(T) = \varphi_k(\iota_i^{i+1}(T))$ and $\varepsilon_k(T) = \varepsilon_k(\iota_i^{i+1}(T))$ for $k = 2, \ldots, n$;
- $D_{2,s}(\iota_i^{i+1}(T))$ has one more column than $D_{2,s}(T)$ (recall $D_{2,s}$ from section 4);

- Let $v(T)$ be the branching component vertex containing $T$. Then $R(v(T)) = R(v(\iota_i^{i+1}(T)))$, so the rank of $v(\iota_i^{i+1}(T))$ in $\mathcal{BC}((i+1)\varpi_2)$ is one greater than the rank of $v(T)$ in $\mathcal{BC}(i\varpi_2)$.

In other words, we know that $T$ has a maximal $a$-configuration of size $s-i$ (section 4), and has completely reduced form $T^{\#}$ (section 5). Furthermore, let $r_{1T}$ be the number of 1's in the first row of $D_{2,s}(T)$ to the left of a null-configuration, and similarly define $r_{2T}, r_{3T}, t_{1T}$, and $t_{2T}$ as in (12). Then the rank of $v(T)$ in $\mathcal{BC}(i\varpi_2)$ is $i - t_{1T} + t_{2T}$. We wish to construct a tableau $S$ with an $a$-configuration of size $s-i-1$ such that $S^{\#} = T^{\#}$ and $(i+1) - t_{1S} + t_{2S} = (i - t_{1T} + t_{2T}) + 1$; i.e., $t_{1S} - t_{2S} = t_{1T} - t_{2T}$. Based on properties of the height 2 type $D$ sliding algorithm of section 3.4, these conditions can only be satisfied when $t_{jS} = t_{jT} + 1$ for $j = 1, 2$.

We can calculate $\iota_i^{i+1}(T)$ by the following algorithm:

*Algorithm 6.4.*

(1) Remove the $a$-configuration of size $s-i$ from $T$ and slide it to get a $2 \times i$ tableau.
(2) Remove the 1's, $\bar{1}$'s and the null-configuration from the result to get a skew tableau of shape $(i, i - t_{2T})/(t_{1T})$.
(3) Using the type $D$ sliding algorithm, produce a skew tableau of shape $((i+1), (i+1) - (t_{2T}+1))/(t_{1T}+1)$.
(4) Fill this tableau with 1's, $\bar{1}$'s, and a null-configuration so that the result is a $2 \times (i+1)$ tableau.
(5) Use the height 2 fill map $F_{2,s}$ (section 4) to insert $s-i-1$ columns into the tableau.

This produces the unique tableau satisfying the three properties of Remark 6.3.

*Example 6.5.* Let

$$T = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 2 & \bar{3} & \bar{2} \\ \hline 2 & 2 & 3 & \bar{2} & \bar{2} & \bar{2} & \bar{1} \\ \hline \end{array} \in B(5\varpi_2) \subset \tilde{B}^{2,7}.$$

Running through the steps of our algorithm (using relation (2) of section 3.4 for step (3)) gives us

(1)
$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & \bar{3} & \bar{2} \\ \hline 2 & 2 & 3 & \bar{2} & \bar{1} \\ \hline \end{array}$$

(2)
$$\begin{array}{ccccc} \cline{3-5} & & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{\bar{3}} & \multicolumn{1}{c|}{\bar{2}} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{\bar{2}} & \\ \cline{1-4} \end{array}$$

(3)
$$\begin{array}{cccccc} \cline{4-6} & & & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{\bar{3}} & \multicolumn{1}{c|}{\bar{2}} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{\bar{3}} & & \\ \cline{1-4} \end{array}$$

(4)
$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 & \bar{3} & \bar{2} \\ \hline 2 & 2 & 3 & \bar{3} & \bar{1} & \bar{1} \\ \hline \end{array}$$

$$\iota_5^6(T) = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 & 3 & \bar{3} & \bar{2} \\ \hline 2 & 2 & 3 & \bar{3} & \bar{3} & \bar{1} & \bar{1} \\ \hline \end{array} \in B(6\varpi_2) \subset \tilde{B}^{2,7}. \tag{5}$$

Composing these maps gives us the following algorithm for calculating $\iota_i^j(T)$, where $T \in B(i\varpi_2)$ and $s \geq j > i$.

*Algorithm 6.6.*

(1) Remove the $a$-configuration of size $s - i$ from $T$ and slide it to get a $2 \times i$ tableau.
(2) Remove the 1's, $\bar{1}$'s and the null-configuration from the result to get a skew tableau of shape $(i, i - t_{2T})/(t_{1T})$.
(3) Using the type $D$ sliding algorithm, produce a skew tableau of shape $((j), (j) - (t_{2T} + (j - i)))/(t_{1T} + (j - i))$.
(4) Fill this tableau with 1's, $\bar{1}$'s, and a null-configuration so that the result is a $2 \times j$ tableau.
(5) Use the height 2 fill map $F_{2,s}$ (section 4) to insert $s - j$ columns into the tableau.

We can also define a map $\iota_i^j : B(i\varpi_2) \to B(j\varpi_2) \cup \{0\}$ for $j < i$ by

$$\iota_i^j(T) = \begin{cases} (\iota_j^i)^{-1}(T) & \text{if } T \in \iota_j^i(B(j\varpi_2)), \\ 0 & \text{otherwise .} \end{cases}$$

Reversing the above algorithm makes this map explicit. Lastly, we define $\iota_i^i$ to be the identity map on $B(i\varpi_2)$, so $\iota_i^j$ is defined for all $i, j \in \{0, \ldots, s\}$.

We have already observed that by the $*$-duality of $B(k\varpi_2) \subset \tilde{B}^{2,s}$, each vertex $v_\lambda \in B(k\varpi_2)$ has a complementary vertex $v'_\lambda \in B(k\varpi_2)$ such that $R(v_\lambda) + R(v'_\lambda) = 2s$. We define the involution $*_{\mathcal{BC}}$ on $\tilde{B}^{2,s}$ as follows: Let $T \in B(v_\lambda)$ such that $T = \tilde{f}_{i_1} \cdots \tilde{f}_{i_m} u_\lambda$, where $u_\lambda$ is the $U_q(D_{n-1})$-highest weight tableau of $B(v_\lambda)$. Then $T^{*_{\mathcal{BC}}} = \tilde{f}_{i_1} \cdots \tilde{f}_{i_m} u'_\lambda$, where $u'_\lambda$ is the $U_q(D_{n-1})$-highest weight tableau of $B(v'_\lambda)$. Alternatively, this map is the composition of $*$ with the "local $*$" map, which applies only to the tableaux in $B(v_\lambda)$ viewed as a $U_q(D_{n-1})$-crystal.

We now define $\sigma(T)$ combinatorially. Suppose $T \in B(k\varpi_2) \subset \tilde{B}^{2,s}$, and let $\ell$ be minimal such that $\iota_k^s(T) \in \iota_\ell^s(B(\ell\varpi_2))$. Then

$$\sigma(T) = \iota_k^{s+\ell-k}(T^{*_{\mathcal{BC}}}) = (\iota_k^{s+\ell-k}(T))^{*_{\mathcal{BC}}}, \tag{15}$$

where it was used that $\iota_i^j$ commutes with $*_{\mathcal{BC}}$.

## 6.3. Properties of $\tilde{f}_0$ and $\tilde{e}_0$

This combinatorial approach immediately gives us useful information about this crystal, such as the following lemma.

**Lemma 6.7.** *For $k = 0, 1, \ldots, s$, let $u_k$ denote the highest weight vector of the classical component $B(k\varpi_2) \subset \tilde{B}^{2,s}$. Then*

$$\tilde{f}_0(u_k) = \begin{cases} u_{k+1} & \text{if } k < s, \\ 0 & \text{if } k = s. \end{cases}$$

**Proof:** Observe that

$$u_k = \underbrace{\begin{matrix}1 & \cdots & 1\\ 2 & & 2\end{matrix}}_{k}\underbrace{\begin{matrix}1 & \cdots & 1\\ \bar{1} & & \bar{1}\end{matrix}}_{s-k};$$

We wish to calculate $\tilde{f}_0(u_k) = \sigma \tilde{f}_1 \sigma(u_k)$.

Note that $u_k \notin \iota^k_{k-1}(B((k-1)\varpi_2))$, so $\ell = k$ in the combinatorial definition of $\sigma$ above. It follows that $\sigma(u_k) = \iota^s_k(u_k^{*_{BC}})$, which is

$$\iota^s_k(u_k^{*_{BC}}) = \iota^s_k\left(\underbrace{\begin{matrix}1 & \cdots & 1\\ \bar{1} & & \bar{1}\end{matrix}}_{s-k}\underbrace{\begin{matrix}2 & \cdots & 2\\ \bar{1} & & \bar{1}\end{matrix}}_{k}\right) = \boxed{\emptyset_{s-k}}\underbrace{\begin{matrix}2 & \cdots & 2\\ \bar{1} & & \bar{1}\end{matrix}}_{k},$$

where $\boxed{\emptyset_i}$ denotes a null-configuration of size $i$ (see Definition 5.2). If $k = s$, $\tilde{f}_1$ kills this tableau, as claimed in the second case of the lemma. Otherwise, acting by $\tilde{f}_1$ will decrease the size of the null-configuration by 1 and add another $\begin{smallmatrix}2\\ \bar{1}\end{smallmatrix}$ to the columns on the right. It follows that $\iota^k_s$ kills this tableau, but $\iota^{k+1}_s$ does not, so now $\ell = k+1$ in the combinatorial definition of $\sigma$. Thus,

$$\sigma \tilde{f}_1 \sigma(u_k) = \iota^{k+1}_s\left(\underbrace{\begin{matrix}1 & \cdots & 1\\ 2 & & 2\end{matrix}}_{k+1}\boxed{\emptyset_{s-k-1}}\right) = \underbrace{\begin{matrix}1 & \cdots & 1\\ 2 & & 2\end{matrix}}_{k+1}\underbrace{\begin{matrix}1 & \cdots & 1\\ \bar{1} & & \bar{1}\end{matrix}}_{s-k-1} = u_{k+1}.$$

□

**Corollary 6.8.** *Let $u_k$ be as above for $k > 0$. Then*

$$\tilde{e}_0(u_k) = u_{k-1}.$$

A similar combinatorial analysis can be carried out on lowest weight tableaux to show that $\tilde{f}_0(u_k^*) = u_{k-1}^*$ and $\tilde{e}_0(u_k^*) = u_{k+1}^*$ for appropriate values of $k$. Since $u_0 = u_0^*$, this gives us the following corollary:

**Corollary 6.9.** *For highest weight vectors $u_k$ and lowest weight vectors $u_k^*$, we have*

$$\varphi_0(u_k) = \varepsilon_0(u_k^*) = s - k \qquad \text{and} \qquad \varphi_0(u_k^*) = \varepsilon_0(u_k) = s + k$$

## 7. Perfectness of $\tilde{B}^{2,s}$

### 7.1. Overview

To show that $\tilde{B}^{2,s}$ is perfect, it must be shown that all criteria of Definition 2.1 are satisfied with $\ell = s$. We have taken part 3 of Definition 2.1 as part of our hypothesis for Theorem 1.1, so we do not attempt to prove this here.

Part 2 of Definition 2.1 is satisfied by simply noting that $\lambda = \varpi_2 = s\Lambda_2 - 2s\Lambda_0$ is a weight in $P_{\rm cl}$ such that $B_\lambda = \{u_s\}$ contains only one tableau and all other tableaux in $\tilde{B}^{2,s}$ have "lower" weights.

In section 7.2, we show that $\tilde{B}^{2,s} \otimes \tilde{B}^{2,s}$ is connected proving part 1 of Definition 2.1. Parts 4 and 5 of Definition 2.1 will be dealt with simultaneously in sections 7.4 and 7.5 by examining the levels of tableaux combinatorially. We will see that the level of a generic tableau is at least $s$ and the tableaux of level $s$ are in bijection with the level $s$ weights. In section 7.6 we show that $\tilde{B}^{2,s}$ is the unique affine crystal satsifying the properties of Conjecture 3.4 thereby proving Theorem 1.1.

## 7.2. Connectedness of $\tilde{B}^{2,s}$

**Lemma 7.1 (Part 1 of Definition 2.1).** *The crystal $\tilde{B}^{2,s} \otimes \tilde{B}^{2,s}$ is connected.*

**Proof:** (This proof is very similar to that in [18, Proposition 5.1].) For $k = 0, 1, \ldots, s$, let $u_k$ denote the highest weight vector of the classical component $B(k\varpi_2) \subset \tilde{B}^{2,s}$, as in Lemma 6.7. We will show that an arbitrary vertex $b \otimes b' \in \tilde{B}^{2,s} \otimes \tilde{B}^{2,s}$ is connected to $u_0 \otimes u_0$.

We know that for some $j \in \{0, \ldots, s\}$, we have $b' \in B(j\varpi_2)$. Then for some pair of sequences $i_1, i_2, \ldots, i_p$ (with entries in $\{1, \ldots, n\}$) and $m_1, m_2, \ldots, m_p$ (with entries in $\mathbb{Z}_{>0}$) and some $b^1 \in \tilde{B}^{2,s}$, we have $\tilde{e}_{i_1}^{m_1}\tilde{e}_{i_2}^{m_2}\cdots\tilde{e}_{i_p}^{m_p}(b \otimes b') = b^1 \otimes u_j$.

By Corollary 6.9, $\varphi_0(u_j) = s - j$, so if $\varepsilon_0(b^1) \le s - j$, Lemma 6.7 tells us that $\tilde{e}_0^j(b^1 \otimes u_j) = b^1 \otimes u_0$. If $\varepsilon_0(b^1) = r > s - j$, then $\tilde{e}_0^{r-s+j}(b^1 \otimes u_j) = b^2 \otimes u_j$, where $\varepsilon_0(b^2) = r - (r - s + j) = s - j$, so $\tilde{e}_0^j(b^2 \otimes u_j) = b^2 \otimes u_0$. In either case, our arbitrary $b \otimes b'$ is connected to an element of the form $b'' \otimes u_0$.

Let $j'$ be such that $b'' \in B(j'\varpi_2)$. Since $u_0$ is the unique element of $B(0)$, the crystal for the trivial representation of $U_q(D_n)$, we know that $B(j'\varpi_2) \otimes B(0) \simeq B(j'\varpi_2)$. Therefore, $b'' \otimes u_0$ is connected to $u_{j'} \otimes u_0$. Finally, we note that $\varphi_0(u_0) = s < s + j' = \varepsilon_0(u_{j'})$ for $j' \neq 0$, so $\tilde{e}_0^{j'}(u_{j'} \otimes u_0) = \tilde{e}_0^{j'}(u_{j'}) \otimes u_0 = u_0 \otimes u_0$, completing the proof. □

## 7.3. Preliminary observations

We first make a few observations.

**Proposition 7.2.** *Let $T \in B(k\varpi_2) \subset \tilde{B}^{2,s}$, and set $T_m = \iota_k^m(T)$ for $m = s, s-1, \ldots, \ell$, where $\ell$ is minimal such that $\iota_k^\ell(T) \neq 0$. If $\ell \neq s$, we have for $s > m \ge \ell$*

$$\varepsilon_1(T_{m+1}) = \varepsilon_1(T_m) + 1 \text{ and } \varepsilon_0(T_{m+1}) = \varepsilon_0(T_m) - 1,$$

$$\varphi_1(T_{m+1}) = \varphi_1(T_m) + 1 \text{ and } \varphi_0(T_{m+1}) = \varphi_0(T_m) - 1.$$

**Proof:** Let $\ell \le m \le s - 1$, so $\iota_m^{m+1}$ is defined. We first consider the difference between the reduced 1-signatures of $D_{2,s}(T_m)$ and $D_{2,s}(T_{m+1}) = D_{2,s}(\iota_m^{m+1}(T_m))$, since the action of $\tilde{e}_1$ on these tableaux is defined by the action of the classical $\tilde{e}_1$ on their image under $D_{2,s}$. Let $-^M+^P$ be the reduced 1-signature of $D_{2,s}(T_m)$. As in section 5.3, let $r_1$ denote the number of 1's in $D_{2,s}(T_m)$, $r_3$ the number of $\bar{1}$'s, $r_2$ the size of the null-configuration, and $t_1 = r_1 + r_2$, $t_2 = r_2 + r_3$. Then there is a contribution $-^{r_2}+^{r_2}$ to the 1-signature from the null-configuration, and the remaining $-$'s and $+$'s come from 1's with a letter greater than 2 below them and $\bar{1}$'s with a letter less than $\bar{2}$ above them, respectively.

We now have two cases, as in section 5.3. If $t_1 + t_2 \geq s$, $\iota_m^{m+1}$ simply increases the size of the null-configuration in $D_{2,s}(T_m)$ by 1. It follows that the reduced 1-signature of $D_{2,s}(T_{m+1})$ is $-^{M+1}+^{P+1}$, as we wished to show. On the other hand, if $t_1 + t_2 < s$, after step (2) of Algorithm 6.4 for $\iota_m^{m+1}$ we have a tableau of shape $(m, m - r_3)/(r_1)$. In step (3), we slide this into shape $(m+1, m+1-(r_3+1))/(r_1+1)$. We claim that the rightmost "uncovered" letter in the second row of this tableau is greater than 2 and the leftmost "unsupported" letter in the first row is less than $\bar{2}$. As observed in the preceding paragraph, this implies that after refilling the empty spaces as in step (4) of Algorithm 6.4 the reduced 1-signature of our tableau is $-^{M+1}+^{P+1}$ in this case as well.

Let us first consider the leftmost "unsupported" letter. After step (2), our tableau is of the form

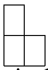
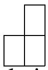
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | $a_{r_1+1}$ | $\cdots$ | $a_{m-r_3}$ | $a_{m-r_3+1}$ | $\cdots$ | $a_s$ |
| $b_1$ | $\cdots$ | $b_{r_1}$ | $b_{r_1+1}$ | $\cdots$ | $b_{m-r_3}$ | | | |

and its column word is unchanged by the slide

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | $a_{r_1+1}$ | $\cdots$ | | $a_{m-r_3}$ | $a_{m-r_3+1}$ | $\cdots$ | $a_s$ |
| $b_1$ | $\cdots$ | $b_{r_1}$ | $b_{r_1+1}$ | $\cdots$ | $b_{m-r_3}$ | | | | |

,

so we have $a_{m-r_3} < b_{m-r_3} \leq \bar{2}$.

The second row of this tableau has $m - r_3$ boxes just as it did before sliding, so the boxes in the bottom row will never be moved. It follows that this sliding procedure only changes L-shaped subtableaux into ⅃-shapes (i.e.,  into ) and never involves any Γ- or ꓶ-shapes. According to the Lecouvey $D$-equivalence relations from section 3.4, such moves can only be made when the letters in the bottom row are strictly greater than 2. Specifically, in relations (3) and (4), the letter which is "uncovered" is either $n$ or $\bar{n}$, while in relations (1) and (2) only the second case of each relation applies. This proves our claim, and thus the first half of the proposition.

Since $\tilde{e}_0 = \sigma \circ \tilde{e}_1 \circ \sigma$, we can derive the statements about $\varepsilon_0$ and $\varphi_0$ from the corresponding statements about $\varepsilon_1$ and $\varphi_1$. More precisely, $\varepsilon_0(T) = \varepsilon_1(\sigma(T))$ and $\varphi_0(T) = \varphi_1(\sigma(T))$ and by (15) we have

$$\sigma(T_m) = \left(\iota_m^{s+\ell-m} \circ \iota_k^m(T)\right)^{*_{BC}} = \iota_k^{s+\ell-m}(T^{*_{BC}}).$$

Hence

$$\begin{aligned}\varepsilon_0(T_{m+1}) &= \varepsilon_1(\sigma(T_{m+1})) = \varepsilon_1(\iota_k^{s+\ell-m-1}(T^{*_{BC}})) \\ &= \varepsilon_1(\iota_k^{s+\ell-m}(T^{*_{BC}})) - 1 = \varepsilon_1(\sigma(T_m)) - 1 = \varepsilon_0(T_m) - 1.\end{aligned}$$

A similar computation can be carried out for $\varphi_0$. □

**Corollary 7.3.** *Given the above hypotheses, we have*

$$\langle h_0 + h_1, \varepsilon(T_s)\rangle = \langle h_0 + h_1, \varepsilon(T_{s-1})\rangle = \cdots = \langle h_0 + h_1, \varepsilon(T_\ell)\rangle \neq 0.$$

The following observation is an immediate consequence of Remark 6.3:

**Corollary 7.4.** *For $i = 2, \dots, n$,*

$$\langle h_i, \varepsilon(T_s)\rangle = \langle h_i, \varepsilon(T_{s-1})\rangle = \cdots = \langle h_i, \varepsilon(T_\ell)\rangle.$$

**Lemma 7.5.** *The map $\Upsilon^s_{s-1} : \mathcal{T}(s-1) \hookrightarrow \mathcal{T}(s)$ viewed as sending the set underlying $\tilde{B}^{2,s-1}$ into the set underlying $\tilde{B}^{2,s}$ increases the level of a tableau by exactly* 1.

**Proof:** This map is defined by sending each summand $B(k\varpi_2) \subset \tilde{B}^{2,s-1}$ to $B(k\varpi_2) \subset \tilde{B}^{2,s}$ for $k = 0, \dots, s-1$, so $\varphi_i(\Upsilon^s_{s-1}(T)) = \varphi_i(T)$ for $i = 1, \dots, n$. To calculate the change in $\varphi_0(T)$, we must consider the difference between $\varphi_1(\sigma_{s-1}(T))$ and $\varphi_1(\sigma_s(\Upsilon^s_{s-1}(T)))$. By our descriptions of maps on crystals, we have $\sigma_s(\Upsilon^s_{s-1}(T)) = \Upsilon^s_{s-1}(\iota^{j+1}_j(\sigma_{s-1}(T)))$, where $j$ is determined by $\sigma_{s-1}(T) \in B(j\varpi_2) \subset \tilde{B}^{2,s-1}$. By Proposition 7.2, $\varphi_1(\iota^{j+1}_j(\sigma_{s-1}(T))) = \varphi_1(\sigma_{s-1}(T)) + 1$. □

## 7.4. Surjectivity

Given a weight $\lambda \in (P^+_{\rm cl})_s$, we construct a tableau $T_\lambda \in \tilde{B}^{2,s}$ such that $\varepsilon(T_\lambda) = \varphi(T_\lambda) = \lambda$. This amounts to constructing $T_\lambda$ so that its reduced $i$-signature is $-^{\varepsilon_i(T_\lambda)}+^{\varepsilon_i(T_\lambda)}$. Note that such a tableau is invariant under the $*$-involution, so its symmetry allows us to define it beginning with the middle, and proceeding outwards.

For $i = 0, \dots, n$, let $k_i = \langle h_i, \lambda\rangle$. We first construct a tableau $T_{\lambda'}$ corresponding to the weight $\lambda' = \sum_{i=2}^n k_i \Lambda_i$. We begin with the middle $k_{n-1} + k_n$ columns of $T_{\lambda'}$. If $k_{n-1} + k_n$ is even and $k_n \geq k_{n-1}$, these columns of $T_{\lambda'}$ are

$$\underbrace{\begin{array}{ccc} n-2 & & n-2 \\ n-1 & \cdots & n-1 \end{array}}_{k_{n-1}} \underbrace{\begin{array}{ccc} n-1 & & n-1 \\ n & \cdots & n \end{array}}_{(k_n-k_{n-1})/2} \underbrace{\begin{array}{ccc} \bar{n} & & \bar{n} \\ \overline{n-1} & \cdots & \overline{n-1} \end{array}}_{(k_n-k_{n-1})/2} \underbrace{\begin{array}{ccc} \overline{n-1} & & \overline{n-1} \\ \overline{n-2} & \cdots & \overline{n-2} \end{array}}_{k_{n-1}}$$

If $k_{n-1} + k_n$ is odd and $k_n \geq k_{n-1}$, we have

$$\underbrace{\begin{array}{ccc} n-2 & & n-2 \\ n-1 & \cdots & n-1 \end{array}}_{k_{n-1}} \underbrace{\begin{array}{ccc} n-1 & & n-1 \\ n & \cdots & n \end{array}}_{(k_n-k_{n-1}-1)/2} \begin{array}{c} \bar{n} \\ n \end{array} \underbrace{\begin{array}{ccc} \bar{n} & & \bar{n} \\ \overline{n-1} & \cdots & \overline{n-1} \end{array}}_{(k_n-k_{n-1}-1)/2} \underbrace{\begin{array}{ccc} \overline{n-1} & & \overline{n-1} \\ \overline{n-2} & \cdots & \overline{n-2} \end{array}}_{k_{n-1}}$$

In either case, if $k_n < k_{n-1}$, interchange $n$ and $\bar{n}$, and $k_n$ and $k_{n-1}$ in the above configurations.

Next we put a configuration of the form

$$\underbrace{\begin{array}{ccc} 1 & & 1 \\ 2 & \cdots & 2 \end{array}}_{k_2} \underbrace{\begin{array}{ccc} 2 & & 2 \\ 3 & \cdots & 3 \end{array}}_{k_3} \quad \cdots \quad \underbrace{\begin{array}{ccc} n-3 & & n-3 \\ n-2 & \cdots & n-2 \end{array}}_{k_{n-2}}$$

on the left, and a configuration of the form

$$\underbrace{\begin{array}{c}\overline{n-2}\\ \overline{n-3}\end{array}\cdots\begin{array}{c}\overline{n-2}\\ \overline{n-3}\end{array}}_{k_{n-2}}\underbrace{\begin{array}{c}\overline{n-3}\\ \overline{n-4}\end{array}\cdots\begin{array}{c}\overline{n-3}\\ \overline{n-4}\end{array}}_{k_{n-3}}\quad\cdots\quad\underbrace{\begin{array}{c}\bar{2}\\ \bar{1}\end{array}\cdots\begin{array}{c}\bar{2}\\ \bar{1}\end{array}}_{k_2}$$

on the right. Denote the set of tableaux constructed by the procedure up to this point by $\mathcal{M}(s')$.

Observe that the reduced 1-signature of $T_{\lambda'}$ is empty, so $\langle h_1, \varphi(T_{\lambda'})\rangle = 0$. Furthermore, since $\lambda'$ has the same number of 1's as $\bar{1}$'s, it is fixed by $\sigma$, so $\langle h_0, \varphi(T_{\lambda'})\rangle = 0$ as well. Thus Proposition 7.2 implies that $T_{\lambda'} \in \tilde{B}^{2,s}_{\min} \cap B(s'\varpi_2) \setminus \iota^{s'}_{s'-1}(B((s'-1)\varpi_2))$ as a subset of $\tilde{B}^{2,s'}$. Recall the embedding $\Upsilon^{s}_{s'} : \tilde{B}^{2,s'} \hookrightarrow \tilde{B}^{2,s}$ from Definition 4.11. Since $s'$ is the minimal $m$ for which $\iota^{m}_{s'}(T_{\lambda'}) \neq 0$, Lemma 7.5 and its proof tell us that $\varepsilon_0(F_{2,s}(T_{\lambda'})) = s - s'$, where the fill map $F_{2,s}$ inserts an $a$-configuration to increase the width of $T_{\lambda'}$ to $s$. By the same proposition the desired tableau is $T_\lambda = \iota^{s'+k_1}_{s'} \circ F_{2,s}(T_{\lambda'})$. We denote by $\mathcal{M}_{\min}(s)$ the set of tableaux constructed by this procedure.

### 7.5. Injectivity

In this subsection we show that the tableaux in $\mathcal{M}_{\min}(s)$ are all the minimal tableaux in $\tilde{B}^{2,s}$.

We first introduce some useful notation. Observe that any tableau $T$ can be written as $T = T_1T_2T_3T_4T_5$, where the block $T_i$ has width $k_i$, and all letters in $T_1$ (resp. $T_5$) are unbarred or $\bar{n}$ in the second row (resp. barred or $n$ in the first row), all columns in $T_2$ (resp. $T_4$) are of the form $\frac{a}{b}$ where $a < b \leq n-1$ (resp. $b < a \leq n-1$), and all columns in $T_3$ are of the form $\frac{a}{\bar{a}}$ for some $a$. Note that for a tableau in $B(s\varpi_2) \subset \tilde{B}^{2,s}$ we have $0 \leq k_3 \leq 1$. Also note that $T_2$ and $T_4$ do not contain any $n$'s or $\bar{n}$'s.

**Theorem 7.6.** *We have* $\mathcal{M}_{\min}(s) = \tilde{B}^{2,s}_{\min}$.

**Proof:** We prove the theorem by induction on $s$. For the base case, we checked explicitly that the statement of the theorem is true for $s = 0, 1, 2$.

By our induction hypothesis, $\mathcal{M}_{\min}(s-1) = \tilde{B}^{2,s-1}_{\min}$. By Lemma 7.5, $\Upsilon^{s}_{s-1}$ increases the level of a tableau by 1, and therefore $\Upsilon^{s}_{s-1}(\tilde{B}^{2,s-1}_{\min}) = (\tilde{B}^{2,s} \setminus B(s\varpi_2))_{\min}$. By Corollaries 7.3 and 7.4, $\iota^{s}_{s-1}$ does not change the level of a tableau, so it suffices to show that

$$\mathcal{M}(s) = \tilde{B}^{2,s}_{\min} \cap (B(s\varpi_2) \setminus \iota^{s}_{s-1}(B((s-1)\varpi_2))). \tag{16}$$

By Lemmas 7.8 and 7.9 below, if $T$ is a minimal tableau not in the image of $\iota^{s}_{s-1}$, it has $k_2 = k_4 = 0$, and if $k_3 = 1$, then $T_3 = \frac{n}{\bar{n}}$ or $T_3 = \frac{\bar{n}}{n}$. By Lemma 7.10, equation (16) follows. □

Here we state the lemmas used in the proof of Theorem 7.6. The proofs are given in Appendices A–D and together with Theorem 7.6 all rely on induction on $s$. The base cases $s = 0, 1, 2$ have been checked explicitly.

**Lemma 7.7.** *For all* $t \in \tilde{B}^{2,s}$ *we have* $\langle c, \varepsilon(t)\rangle \geq s$ *and* $\langle c, \varphi(t)\rangle \geq s$.

**Lemma 7.8.** *If $T \in (\tilde{B}^{2,s} \cap B(s\varpi_2))_{\min}$, we have $k_1 + k_2 = k_4 + k_5 = \lfloor s/2 \rfloor$ with $k_1, k_2, k_4$ and $k_5$ as defined in the beginning of this subsection.*

**Lemma 7.9.** *Suppose $T \in (\tilde{B}^{2,s} \cap B(s\varpi_2))_{\min}$ has both an unbarred letter and a barred letter in a single column $\frac{a}{b}$ other than $\frac{n}{\bar{n}}$, $\frac{\bar{n}}{n}$, $\frac{n-1}{\bar{n}}$, or $\frac{n}{n-1}$. Then $T \in \iota^s_{s-1}(B((s-1)\varpi_2)$.*

**Lemma 7.10.** *Let $T \in (\tilde{B}^{2,s} \cap B(s\varpi_2))_{\min}$ such that $T$ does not contain any column $\frac{a}{b}$ for $1 \le a, b \le n$ except possibly $\frac{n-1}{\bar{n}}$, $\frac{n}{\bar{n}}$, or $\frac{n}{n-1}$. Then $T \in \mathcal{M}(s)$.*

### 7.6. Uniqueness

Theorem 1.1 follows as a corollary from the next proposition.

**Proposition 7.11.** *$\tilde{B}^{2,s}$ is the unique affine finite-dimensional crystal structure satisfying the properties of Conjecture 3.4.*

For the proof of Proposition 7.11 we must show that our choice of $\sigma$ is the only choice satisfying the properties of Conjecture 3.4. Recall from the beginning of section 6 the relationship between $\sigma$ and $\breve{\sigma}$. Let $T \in \tilde{B}^{2,s}$. We know that

$$\mathrm{wt}(T) = \sum_{i=0}^{n} k_i \Lambda_i \Leftrightarrow \mathrm{wt}(\sigma(T)) = k_1\Lambda_0 + k_0\Lambda_1 + \sum_{i=2}^{n} k_i \Lambda_i. \tag{17}$$

Once $\breve{\sigma}$ is determined, the given definition of $\sigma$ (sending $D_{n-1}$ highest weight vectors to $D_{n-1}$ highest weight vectors, etc.) is the only involution of the set of tableaux in $\tilde{B}^{2,s}$ satisfying (17) and agreeing with $\breve{\sigma}$.

As we observed in section 6.1, $\breve{\sigma}(v)$ and $v$ must be associated with the same partition, and if $v'$ is the complementary vertex of $v$, $\breve{\sigma}(v')$ is the complementary vertex of $\breve{\sigma}(v)$. We now prove a few lemmas that uniquely determine $\breve{\sigma}$.

Please note that in this section we often use the phrase "the tableau $b$ is in the branching component vertex $v$" to mean $b \in B(v)$.

**Lemma 7.12.** *Suppose $b \in \tilde{B}^{2,s}$ is in a branching component vertex of rank $p$ and $\tilde{f}_0(b) \neq 0$. Then the branching component vertex containing $\tilde{f}_0(b)$ has rank $p-1$.*

**Proof:** Recall from the weight structure of type $D$ algebras that $\alpha_0 = 2\Lambda_0 - \Lambda_2$ and $\mathrm{wt}(\tilde{f}_0(b)) = \mathrm{wt}(b) - \alpha_0$. Define $\mathrm{cw}(b) = \mathrm{wt}(b) - (\varphi_0(b) - \varepsilon_0(b))\Lambda_0$. The above implies that $\mathrm{cw}(\tilde{f}_0(b)) = \mathrm{cw}(b) + \Lambda_2 = \mathrm{cw}(b) + \varepsilon_1 + \varepsilon_2$. Similarly, $\mathrm{cw}(\tilde{f}_i(b)) = \mathrm{cw}(b) - \alpha_i$, so that by (9) only $\tilde{f}_1$ changes the $\varepsilon_1$ component by $-1$. Since $\tilde{f}_1$ increases the rank by one and $\tilde{f}_0$ changes the $\varepsilon_1$ component by $+1$, it follows that $\tilde{f}_0$ decreases the rank by one. □

Since $\tilde{f}_1$ increases the rank by one, $\tilde{f}_0$ decreases the rank by one and $\tilde{f}_0 = \sigma \tilde{f}_1 \sigma$, the following corollary holds.

**Corollary 7.13.** *Suppose $b$ is in a branching component vertex $v$ of rank $p$. Then $\breve{\sigma}(v)$, the branching component vertex containing $\sigma(b)$, has rank $2s - p$.*

Note that this determines $\check{\sigma}$ on $B(s\varpi_2) \setminus \iota^s_{s-1}(B((s-1)\varpi_2))$, and is in agreement with our definition of $\check{\sigma}$ restricted to that domain.

**Lemma 7.14.** *Let $v \in \mathcal{BC}(k\varpi_2)$ be a branching component vertex of rank $s$ associated with a rectangular partition, and let $\ell$ be minimal such that $v \in \check{\iota}^k_\ell(\mathcal{BC}(\ell\varpi_2))$. Then the hypothesis that $\tilde{B}^{2,s}$ is perfect of level $s$ can only be satisfied if $\check{\sigma}(v)$ is the vertex associated with the same shape as $v$ with rank $s$ in $\mathcal{BC}((s+\ell-k)\varpi_2)$.*

**Proof:** We have already shown that $\check{\sigma}(v)$ has the same shape as $v$ and has rank $s$, so it only remains to show that $\check{\sigma}(v) \in \mathcal{BC}((s+\ell-k)\varpi_2)$.

First, observe that $v$ must contain a minimal tableau as constructed in section 7.4, according to the following table.

| shape associated with $v$ | weight of tableau in $v$ |
|---|---|
| $(2m)$ | $m\Lambda_2 + (k-2m)\Lambda_1 + (s-k)\Lambda_0$ |
| $(m, m)$ | $m\Lambda_3 + (k-2m)\Lambda_1 + (s-k)\Lambda_0$ |

Let $T$ be the tableau constructed by this prescription, so that $\langle c, \mathrm{cw}(T)\rangle = k$. The criterion that $\langle c, \varphi(T)\rangle \geq s$ forces us to have $\varphi_0(T) = \varphi_1(\sigma(T)) \geq s-k$. We denote $T_i = \iota^i_k(T)$, and thus have $\varphi_1(T_i) = i - \ell$.

We show inductively that $\sigma(T_i) = T_{s+\ell-i}$ for $\ell \leq i \leq s$. As a base case, we see that $\langle c, \mathrm{cw}(T_\ell)\rangle = \ell$, so we must have $\varphi_1(\sigma(T_\ell)) \geq s-\ell$. The only $T_i$ for which this inequality holds is $T_s$, where we have $\varphi_1(T_s) = s-\ell$.

For the induction step, assume that $\sigma$ sends $T_\ell, T_{\ell+1}, \ldots, T_{k-1}$ to $T_s, \ldots, T_{s+\ell-k+1}$, respectively. By the above inequality this implies $\varphi_1(\sigma(T_k)) \geq s-k$, which specifies that $\sigma(T_k) = T_{s+\ell-k}$. □

*Definition 7.15.* Recall the definition of $\Upsilon^s_{s'}$ from Definition 4.11. We define $\check{\Upsilon}^s_{s'} : \mathcal{BC}(\tilde{B}^{2,s'}) \hookrightarrow \mathcal{BC}(\tilde{B}^{2,s})$ by $\check{\Upsilon}^s_{s'}(v) = v'$ if for some $T \in B(v)$, we have $\Upsilon^s_{s'}(T) \in B(v')$.

**Lemma 7.16.** *Let $v \in \mathcal{BC}(k\varpi_2)$ be a branching component vertex of rank $1 \leq p \leq s-1$ associated to the shape $(\lambda_1, \lambda_2)$. Suppose that for the branching component vertex $w \in \mathcal{BC}(k\varpi_2)$ of rank $p+1$ with shape $(\lambda_1 - 1, \lambda_2)$, $\tilde{B}^{2,s}$ has the correct energy function and is perfect only if $\check{\sigma}(w)$ is as described in section 6.1. Then $\tilde{B}^{2,s}$ has the correct energy function only if $\check{\sigma}(v)$ is as described in section 6.1.*

**Proof:** First, recall that the partitions associated to vertices of rank $p$ in $\mathcal{BC}(\tilde{B}^{2,s})$ are produced by adding or removing one box from the partitions associated to vertices of rank $p-1$. Since the vertex of rank 0 is associated with a rectangle of shape $(s)$, the lowest rank for which we can have a two-row rectangle is $s$. It follows that removing a box from the first row of $v$ results in a partition of rank $p+1$, so there is in fact a vertex $w$ as described in the statement of the lemma.

Let $\ell$ be minimal such that $v \in \check{\iota}^k_\ell(\mathcal{BC}(\ell\varpi_2))$. We may assume $v \notin \mathcal{BC}(\ell\varpi_2)$, and let $\check{\sigma}(v)$ be determined by the involutive property of $\check{\sigma}$ in the case $v \in \mathcal{BC}(\ell\varpi_2)$. Specifically, we will show that the vertex $v' \in \mathcal{BC}(s\varpi_2)$ with the same shape and rank as $v$ has the property that $\check{\sigma}(v')$ is the complementary vertex of $v$, and therefore $\check{\sigma}(v)$ is the complementary vertex of $v'$.

The top row of $w$ is one box shorter than the top row of $v$, so that $w \in \check{\iota}^k_{\ell-1}(\mathcal{BC}((\ell-1)\varpi_2))$, and $\ell-1$ is minimal with this property. By our hypothesis, $\check{\sigma}(w)$ is the vertex with shape $(\lambda_1-1,\lambda_2)$ of rank $2s-p-1$ in $\mathcal{BC}((s+(\ell-1)-k)\varpi_2)$.

We now use induction on $s$. Suppose that the only choice of $\check{\sigma}$ for which $\tilde{B}^{2,s-1}$ is perfect and has an energy function is the choice of section 6.1. Part 3 of Conjecture 3.4 states that in $\tilde{B}^{2,s}$, the energy on the component $B((s-k)\varpi_2)$ is $-k$, and so the difference in energy between $B((s-k)\varpi_2)$ and $B((s-j)\varpi_2)$ is $j-k$. In order for this to be true for all $1 \le k, j \le s-1$, the action of $\tilde{f}_0$ and $\tilde{e}_0$ on $\tilde{B}^{2,s}$ must agree with the action on $\tilde{B}^{2,s-1}$. More precisely, if $v$ and $w$ are in different classical components of $\tilde{B}^{2,s-1}$ and $\tilde{f}_0(v)=w$ in $\tilde{B}^{2,s-1}$, then $\tilde{f}_0(\Upsilon^s_{s-1}(v)) = \Upsilon^s_{s-1}(w)$; this statement extends naturally to $\mathcal{BC}(\tilde{B}^{2,s-1})$ and $\mathcal{BC}(\tilde{B}^{2,s})$.

Let $v^\dagger$ denote the vertex with shape $(\lambda_1,\lambda_2)$ of rank $2s-p$ in $\mathcal{BC}((s+\ell-k)\varpi_2)$. Since we assumed $k \neq \ell$, we know that $v^\dagger \notin \mathcal{BC}(s\varpi_2)$, and therefore $v^\dagger$ has a preimage under $\check{\Upsilon}^s_{s-1}$. From our construction of $\check{\sigma}$ we know that in $\mathcal{BC}(\tilde{B}^{2,s-1})$, $(\check{\Upsilon}^s_{s-1})^{-1}(v^\dagger)$ has a 0 arrow to $(\check{\Upsilon}^s_{s-1})^{-1}(\check{\sigma}(w))$. Our induction argument tells us that in $\mathcal{BC}(\tilde{B}^{2,s})$, $v^\dagger$ has a 0 arrow to $\check{\sigma}(w)$. Since $v$ has a 1 arrow to $w$ and we must have $\tilde{f}_0 = \sigma\tilde{f}_1\sigma$, we conclude that in fact $\check{\sigma}(v) = v^\dagger$. □

**Lemma 7.17.** *Let $v \in \mathcal{BC}(k\varpi_2)$ be a branching component vertex of rank $s$ associated to the non-rectangular shape $(\lambda_1,\lambda_2)$, and suppose that for the branching component vertex $w \in \mathcal{BC}(k\varpi_2)$ of rank $s-1$ with shape $(\lambda_1,\lambda_2+1)$, $\tilde{B}^{2,s}$ has the correct energy function and is perfect only if $\check{\sigma}(w)$ is as described in section 6.1. Then $\tilde{B}^{2,s}$ has the correct energy function only if $\check{\sigma}(v)$ is as described in section 6.1.*

**Proof:** (This proof is very similar to the proof of Lemma 7.16.)

Let $\ell$ be minimal such that $v \in \check{\iota}^k_\ell(\mathcal{BC}(\ell\varpi_2))$, assuming $k \neq \ell$. Note that $\ell$ is also minimal for $w \in \check{\iota}^k_\ell(\mathcal{BC}(\ell\varpi_2))$, since the shapes for $v$ and $w$ have the same number of boxes in the first row. By our hypothesis, $\check{\sigma}(w)$ is the vertex with shape $(\lambda_1,\lambda_2+1)$ of rank $s+1$ in $\mathcal{BC}((s+\ell-k)\varpi_2)$. Let $v^\dagger$ be the vertex with shape $(\lambda_1,\lambda_2)$ of rank $s$ in $\mathcal{BC}((s+\ell-k)\varpi_2)$. From our construction of $\check{\sigma}$, we know that in $\mathcal{BC}(\tilde{B}^{2,s-1})$, $(\check{\Upsilon}^s_{s-1})^{-1}(w)$ has a 0 arrow to $(\check{\Upsilon}^s_{s-1})^{-1}(\check{\sigma}(v^\dagger))$. It follows from our induction argument that in $\mathcal{BC}(\tilde{B}^{2,s})$, $w$ has a 0 arrow to $\check{\sigma}(v^\dagger)$. Since $w$ has a 1 arrow to $v$ and we must have $\tilde{f}_0 = \sigma\tilde{f}_1\sigma$, we conclude that in fact $\check{\sigma}(v) = v^\dagger$. □

**Corollary 7.18.** *Corollary 7.13 and Lemmas 7.14, 7.16 and 7.17 determine $\check{\sigma}$ on $\mathcal{BC}(\tilde{B}^{2,s})$ uniquely.*

**Proof:** For any vertex $v$ associated with shape $(\lambda_1,\lambda_2)$ with rank $p \le s$, $\check{\sigma}(v)$ is fixed by the image under $\check{\sigma}$ of a vertex with shape $(\lambda_1-s+p,\lambda_2)$ and rank $s$ by Lemma 7.16. If $\lambda_1-s+p \neq \lambda_2$, Lemmas 7.16 and 7.17 may be used together to reduce determining $\check{\sigma}(v)$ to determining the action of $\check{\sigma}$ on a rectangular vertex of rank $s$, which is given by Lemma 7.14. □

## 8. Discussion

In this section we discuss some applications and open problems regarding the crystals $\tilde{B}^{2,s}$ introduced in this paper.

The major open question regarding $\tilde{B}^{2,s}$ is of course its existence, which was assumed throughout this paper. A possible method of proof is a generalization of the fusion construction of [10].

In [16], Kashiwara conjectures that for any quantum affine algebra, $B^{r,s}$ is isomorphic as a classical crystal to a Demazure crystal in an irreducible affine highest weight module of weight $s\max(1, 2/(\alpha_r, \alpha_r))\varpi_r - s\Lambda_0$, where $\varpi_r = \Lambda_r - a_r^\vee \Lambda_0$ (except for type $A_{2n}^{(2)}$). The $\tilde{f}_0$ edges that stay within the Demazure crystal should be among the $\tilde{f}_0$ edges of $B^{r,s}$. The combinatorial structure of $\tilde{B}^{2,s}$ as constructed in this paper might give a hint on how to make this correspondence more precise.

For a tensor product of affine finite crystals $B = B_L \otimes \cdots \otimes B_1$ and a dominant integral weight $\lambda$ define the set of classically restricted paths as

$$\mathcal{P}(B, \lambda) = \left\{b \in B \mid \mathrm{wt}(b) = \lambda, \tilde{e}_i(b) = 0 \text{ for all } i \in J\right\}$$

where $J = \{1, 2, \ldots, n\}$. The classically restricted one dimensional sum is defined to be

$$X(B, \lambda; q) = \sum_{b \in \mathcal{P}(B,\lambda)} q^{D_B(b)}.$$

In [5, Section 4] fermionic formulas $M(B, \lambda; q)$ are defined which are sums of products of $q$-binomial coefficients, and it is conjectured that $X(B, \lambda; q) = M(B, \lambda; q)$. This conjecture has been proven for type $A_n^{(1)}$ [19, 20, 21] and various other cases [1, 24, 25, 26, 27, 29, 30]. It is expected that the $X = M$ conjecture can also be proven in the case of tensor products of crystals $\tilde{B}^{2,s}$ as constructed in this paper by using the splitting map (see [30]) and the single column bijection of type $D_n^{(1)}$ (see [29]). Using these maps, one should obtain a statistic-preserving bijection between the set of tableaux $\mathcal{T}(s)$ defined in section 4 and a set of rigged configurations which naturally indexes the $q$-binomial coefficients in the fermionic formulas. The statistics that are preserved by this bijection (energy in the case of tableaux, co-charge in the case of rigged configurations) are the exponents of $q$ in the $X = M$ formula.

## Appendix A. Proof of Lemma 7.7

By induction hypothesis, $\tilde{B}^{2,s-1}_{\min} = \mathcal{M}_{\min}(s-1)$.

Observe that by Corollaries 7.3 and 7.4 $\iota_i^j$ is level preserving, and by Lemma 7.5, the map $\Upsilon_{s-1}^s$ increases the level of a tableau by one. Our induction hypothesis therefore allows us to assume that $t \in B(s\varpi_2) \setminus \iota_{s-1}^s(B((s-1)\varpi_2)$. Combinatorially, we may characterize such tableaux as being those which are legal in the classical sense and for which removing all 1's, $\bar{1}$'s, and null configurations produces a tableau which is Lecouvey $D$-equivalent to a tableau whose first row has width $s$. This characterization follows from the combinatorial description of $\iota_{s-1}^s$ in Algorithm 6.4.

We may further restrict our attention by the observation that if $T$ is minimal, then so is $T^*$. We may therefore assume $T$ to be in the top half (inclusive of the middle row) of the branching component graph. This means that $T$ has no more $\bar{1}$'s than 1's.

Our approach is to consider the tableau $T'$ that results from removing the leftmost column from $T$. We will show that if $T'$ is minimal, the level of $T$ exceeds the level of $T'$ by at least 2, and if $T'$ is not minimal, the level of $T$ is at least as great as the level of $T'$.

First consider the case when $T'$ is minimal. Since $T$ is assumed to be such that removing all 1's and $\bar{1}$'s produces a tableau which is Lecouvey $D$-equivalent to a tableau whose first row has width $s$, it is the case that removing all 1's and $\bar{1}$'s from $T'$ produces a tableau which is Lecouvey $D$-equivalent to a tableau whose first row has width $s-1$. The minimal tableaux of $\tilde{B}^{2,s-1}$ with this property are precisely $\mathcal{M}(s-1)$. By properties of $\mathcal{M}(s-1)$, we know that $\varphi_0(T')=0$, so $\varphi_0(T)\geq\varphi_0(T')$. Since our base case is $s=2$, we know that the first column of $T$ is $\substack{a\\b}$, where $a$ and $b$ are both unbarred. Observe that $\varphi(T)=\varphi(T')+2\Lambda_b+$ non-negative weight if $b\neq n-1$ or $\varphi(T)=\varphi(T')+\Lambda_{n-1}+\Lambda_n+$ non-negative weight if $b=n-1$. Hence, the level is increased by at least 2.

Now suppose $T'$ is not minimal. The level of the $i$-signatures (that is to say, the level of the sum of the weights $\varphi_i(T')$ which depend on $i$-signatures) cannot have a net decrease for $i=1,\ldots,n$, but there is now a possibility that $\varphi_0(T)<\varphi_0(T')$. We will show that when $\varphi_0(T)<\varphi_0(T')$, the level of the $i$-signatures goes up by at least $\varphi_0(T')-\varphi_0(T)$.

First, suppose $T$ has no 1's. Then by one of our hypotheses, it also has no $\bar{1}$'s, and is therefore fixed by $\sigma$: it follows that $\varphi_0(T)=\varphi_1(T)$, so we may assume the upper-left entry of $T$ to be 1.

We know that $\varphi_0(T)$ is equal to the number of $-$'s in the reduced 1-signature of $\sigma(T)$. Consider the following tableaux:

$$T'=\underbrace{\begin{matrix}1 & \cdots & 1\\ b_1 & & b_{m_1}\end{matrix}}_{m_1}\begin{matrix}a_{m_1+1} & \cdots & a_{s-m_2}\\ b_{m_1+1} & & b_{s-m_2}\end{matrix}\underbrace{\begin{matrix}a_{s-m_2+1} & \cdots & a_s\\ \bar{1} & & \bar{1}\end{matrix}}_{m_2}$$

$$T=\begin{matrix}1\\ b\end{matrix}\underbrace{\begin{matrix}1 & \cdots & 1\\ b_1 & & b_{m_1}\end{matrix}}_{m_1}\begin{matrix}a_{m_1+1} & \cdots & a_{s-m_2}\\ b_{m_1+1} & & b_{s-m_2}\end{matrix}\underbrace{\begin{matrix}a_{s-m_2+1} & \cdots & a_s\\ \bar{1} & & \bar{1}\end{matrix}}_{m_2}$$

$$\sigma(T')=\underbrace{\begin{matrix}1 & \cdots & 1 & 1\\ b_1 & & b_{m_2-1} & b'_{m_2}\end{matrix}}_{m_2}\begin{matrix}a'_{m_2+1} & \cdots & a'_{s-m_1}\\ b'_{m_2+1} & & b'_{s-m_1}\end{matrix}\underbrace{\begin{matrix}a'_{s-m_1+1} & \cdots & a'_s\\ \bar{1} & & \bar{1}\end{matrix}}_{m_1}$$

$$\sigma(T)=\underbrace{\begin{matrix}1 & 1 & \cdots & 1\\ b & b_1 & & b_{m_2-1}\end{matrix}}_{m_2}\begin{matrix}a''_{m_2} & \cdots & a''_{s-m_1-1}\\ b''_{m_2} & & b''_{s-m_1-1}\end{matrix}\underbrace{\begin{matrix}a''_{s-m_1} & \cdots & a''_s\\ \bar{1} & & \bar{1}\end{matrix}}_{m_1+1}.$$

Note that our assumption that $m_2\leq m_1+1$ ensures that the absence of primes on $b,b_1,\ldots,b_{m_2-1}$ is accurate.

Let us consider all possible ways for the number of $-$'s in the 1-signature to be smaller for $\sigma(T)$ than for $\sigma(T')$. The number of 1's is the same, so the only way this contribution could be decreased is by having more 2's in the first $m_2$ letters of the bottom row. This can only come about by having $b=2$, and only one $-$ may be removed in this way.

The other possibility is for the number of $-$'s contributed by $\bar{2}$'s to be decreased. The only Lecouvey relation which removes a $\bar{2}$ assumes the presence of a column $\substack{2\\ \bar{2}}$, which we disallow (null-configuration). To decrease this contribution therefore requires an additional $+$ in the 1-signature of $\sigma(T)$ compared to that of $\sigma(T')$, which will bracket one of the $-$'s from a $\bar{2}$. The additional $+$ may come from one of the additional $\bar{1}$'s, or from a 2 that is "pushed out" from under the 1's at the beginning in the case $b=2$. Note that this second possibility is mutually exclusive with having more 2's bracketing 1's at the beginning.

In any case, we see that $\varphi_0(T) - \varphi_0(T') \leq 2$, and that when this value is 2, the first column of $T$ is $\genfrac{}{}{0pt}{}{1}{2}$. This column adds no $+$'s to the $i$-signatures, but does provide a new $-$ in the 2-signature. Since $\Lambda_2$ is a level 2 weight, the level stays the same in this case.

If $\varphi_0(T) - \varphi_0(T') = 1$ and the first column of $T$ is $\genfrac{}{}{0pt}{}{1}{2}$, we in fact have a net increase in level. If $b \neq 2$, the $i$-signature levels go up by at least 1, so still the total level cannot decrease.

## Appendix B. Proof of Lemma 7.8

We first establish that $\langle c, \varphi(T)\rangle \geq 2k_1 + 2k_2 + k_3$, and thus by $*$-duality, $\langle c, \varepsilon(T)\rangle \geq k_3 + 2k_4 + 2k_5$ as well. Recall that $0 \leq k_3 \leq 1$.

First, observe that every letter in the bottom row of $T_1$ contributes:

- a $-$ to the reduced $a$-signature if $2 \leq a \leq n-2$ is in the bottom row;
- a $-$ to both the $(n-1)$-signature and the $n$-signature if $n-1$ is in the bottom row;
- a $-$ to the $n$-signature (resp. $(n-1)$-signature) if $n$ (resp. $\bar{n}$) is the bottom row.

Suppose $T_1$ has a column of the form $\genfrac{}{}{0pt}{}{a}{b}$ with $b \neq a+1$, or $b = \bar{n}$ and $a \neq n-1$. For the $-$ in the $a$-signature of $T$ contributed by this $a$ to be bracketed, we must have a column of the form $\genfrac{}{}{0pt}{}{a'}{a+1}$ to the left of this column in $T_1$, with $a' < a$. Applying this observation recursively, we see that to bracket as many $-$'s as possible we must eventually have a column of the form $\genfrac{}{}{0pt}{}{1}{c}$ for some $c \neq 2$. Note that in the case of columns of the form $\genfrac{}{}{0pt}{}{n-1}{n}$ (resp. $\genfrac{}{}{0pt}{}{n-1}{\bar{n}}$) the unbracketed $-$ in the $n$-signature (resp. $(n-1)$-signature) from $n-1$ cannot be bracketed, since $n$ and $\bar{n}$ may not appear in the same row.

Now, consider a column $\genfrac{}{}{0pt}{}{a}{\bar{b}}$ in $T_2$, so we have $a < b$, and thus also $\bar{a} > \bar{b}$. Recall that $T_2$ has no $n$'s or $\bar{n}$'s, so $b \leq n-1$. This column contributes $-$'s to the $a$-signature and the $(b-1)$-signature of $T$. In this case, these $-$'s may be bracketed. Due to the conditions that the rows and columns of $T$ are increasing, the $-$ from the $a$ can only be bracketed by an $a+1$ in the bottom row of $T_1$ and the $-$ from the $\bar{b}$ can only be bracketed by a $b$ in the bottom row of $T_1$. Furthermore, the letter above these must be strictly less than $a$ and $b-1$, respectively. By the reasoning in the previous paragraph, we see that to bracket every $-$ engendered by the column $\genfrac{}{}{0pt}{}{a}{\bar{b}}$ we must have two columns of the form $\genfrac{}{}{0pt}{}{1}{a'}$, with each $a' \neq 2$.

If $k_3 = 1$, $T$ has a column of the form $\genfrac{}{}{0pt}{}{a}{\bar{a}}$. We have two cases; $2 \leq a \leq n-1$, and $a = n$ (resp. $a = \bar{n}$). In the first case, we have a $-$ in the $(a-1)$-signature from the $\bar{a}$ in this column. Because of the prohibition against configurations of the form $\genfrac{}{}{0pt}{}{a}{} \genfrac{}{}{0pt}{}{a}{\bar{a}}$, this $-$ can only be bracketed by a $+$ from an $a$ in the bottom row of $T_1$. Therefore, this column engenders another column of the form $\genfrac{}{}{0pt}{}{1}{a'}$. In the case of $a = n$ (resp. $a = \bar{n}$), we have a $-$ in the $(n-1)$-signature (resp. $n$-signature) which cannot be bracketed.

To bracket the maximal number of $-$'s (i.e., to minimize $\langle c, \varphi(T)\rangle$) we see that unless $T_3 = \genfrac{}{}{0pt}{}{n}{\bar{n}}$ or $T_3 = \genfrac{}{}{0pt}{}{\bar{n}}{n}$, we must have

$$T_1T_2T_3 = \underbrace{\begin{matrix} 1 & \cdots & 1 \\ b_1 & & b_{2k_2+k_3} \end{matrix}}_{2k_2+k_3}\ \underbrace{\begin{matrix} a_{2k_2+k_3+1} & \cdots & a_{k_1} \\ b_{2k_2+k_3+1} & & b_{k_1} \end{matrix}}_{k_1-(2k_2+k_3)}\ \underbrace{\begin{matrix} a_{k_1+1} & \cdots & a_{k_1+k_2+k_3} \\ \bar{b}_{k_1+1} & & \bar{b}_{k_1+k_2+k_3} \end{matrix}}_{k_2+k_3}, \tag{18}$$

where each column in the first block contributes 3 to $\langle c, \varphi(T)\rangle$, each column in the second block contributes 2 to $\langle c, \varphi(T)\rangle$, and the third block contributes nothing. In the case $T_3 = \genfrac{}{}{0pt}{}{n}{\bar{n}}$ or $T_3 = \genfrac{}{}{0pt}{}{\bar{n}}{n}$, we have $k_2 = 0$, so we simply have $T_1T_2T_3 = T_1T_3$, where each column in $T_1$

increases $\langle c, \varphi(T)\rangle$ by at least 2 and $T_3$ increases $\langle c, \varphi(T)\rangle$ by 1. We therefore have in the first case $\langle c, \varphi(T)\rangle \geq 3(2k_2 + k_3) + 2(k_1 - 2k_2 - k_3) = 2k_1 + 2k_2 + k_3$, and in the second case $\langle c, \varphi(T)\rangle \geq 2k_1 + k_3 = 2k_1 + 2k_2 + k_3$ , as we wished to show.

Since by Lemma 7.7 elements in $\tilde{B}^{2,s}$ have level at least $s$, it follows that when $T \in (\tilde{B}^{2,s} \cap B(s\varpi_2))_{\min}$, we have $k_1 + k_2 \leq \lfloor s/2 \rfloor$, and by $*$-duality that $k_4 + k_5 \leq \lfloor s/2 \rfloor$. Furthermore, since $s = k_1 + k_2 + k_3 + k_4 + k_5$, it follows that $k_1 + k_2 = k_4 + k_5 = \lfloor \frac{s}{2} \rfloor$ and $k_3 = 0$ if $s$ is even and $k_3 = 1$ if $s$ is odd.

## Appendix C. Proof of Lemma 7.9

By using the reverse of Algorithm 6.4, it suffices to show the following:

(1) $T$ has a 1;
(2) $T$ has a $\bar{1}$;
(3) after removing all 1's and $\bar{1}$'s, applying the Lecouvey $D$ relations will reduce the width of $T$.

The proof of Lemma 7.8 shows that if $k_2 \neq 0$, or $k_3 = 1$ and $T_3 \neq \begin{smallmatrix} n \\ \bar{n} \end{smallmatrix}, \begin{smallmatrix} \bar{n} \\ n \end{smallmatrix}$, then $T$ has a 1. By $*$-duality, if $k_4 \neq 0$, or the same condition is placed on $k_3$ and $T_3$, then $T$ has a $\bar{1}$. We will show that if $k_2 + k_3 \neq 0$, then $k_3 + k_4 \neq 0$, which will prove statements (1) and (2) above.

If $k_3 = 1$ this statement is trivial, so we assume $k_3 = 0$. We show that the assumptions $k_2 \neq 0$ and $k_4 = 0$ lead to a contradiction. From the proof of Lemma 7.8, we know that for $T$ to be minimal, every $-$ from $T_5$ must be bracketed. Because of the increasing conditions on the rows and columns of $T$, the $-$'s from the bottom row of $T_5$ cannot be bracketed by $+$'s from $T_5$, so there must be at least $k_5$ $+$'s from $T_1T_2$. Inspection of (18) shows us that the first block contributes no $+$'s, the second block contributes $k_1 - 2k_2$ many $+$'s, and the third block contributes $2k_2$ many $+$'s. We thus have $k_1 \geq k_5$; but Lemma 7.8 tells us that $k_1 + k_2 = k_4 + k_5$, contradicting our assumption that $k_2 \neq 0$ and $k_4 = 0$.

For the proof of statement (3), we must show that every configuration $\begin{smallmatrix} & c \\ a & b \end{smallmatrix}$ in $T$ avoids the following patterns (recall the Lecouvey $D$ sliding algorithm from section 3.4): $\begin{smallmatrix} & n \\ x & \bar{n} \end{smallmatrix}$ and $\begin{smallmatrix} & \bar{n} \\ x & n \end{smallmatrix}$ with $x \leq n-1$; $\begin{smallmatrix} & \bar{n} \\ n-1 & \overline{n-1} \end{smallmatrix}$ ; $\begin{smallmatrix} & n \\ n-1 & \overline{n-1} \end{smallmatrix}$ ; and $c \geq a$, unless $c = a = \bar{b}$. If $T$ has any of these patterns, the top row will not slide over.

First, simply observe that the first four specified configurations exclude the possibility of having a column of the form $\begin{smallmatrix} a \\ \bar{b} \end{smallmatrix}$ other than $\begin{smallmatrix} n \\ \bar{n} \end{smallmatrix}$ or $\begin{smallmatrix} \bar{n} \\ n \end{smallmatrix}$. It therefore suffices to show that the presence of a column $\begin{smallmatrix} d \\ \bar{e} \end{smallmatrix}$, $2 \leq d, e \leq n-1$ implies that $T$ avoids $c \geq a$, unless $c = a = \bar{b}$. We break our analysis of this criterion into several special cases:

*Case 1:* $a$ and $b$ are barred, $c$ is unbarred: trivial.
*Case 2:* $a$ is unbarred, $b$ and $c$ are barred: This excludes the possibility of having $\begin{smallmatrix} d \\ \bar{e} \end{smallmatrix} \in T$.
*Case 3:* $a$ and $c$ are unbarred, $b$ is barred: We know the $-$ in the $c$-signature from $c$ must be bracketed; if it is by $b$, we have $b = \bar{c}$. As we saw in the proof of Lemma 7.8, we must have the $-$ in the $(c-1)$-signature from $\bar{c}$ bracketed by a $c$ in the bottom row. This forces $a \geq c$. If the $-$ in the $c$-signature from $c$ is bracketed by a $c+1$, it also must be in the bottom row, forcing $a > c$.
*Case 4:* $a, b, c$ all unbarred: Suppose $c \geq a$. Let $d$ be the leftmost unbarred letter weakly to the right of $c$ which does not have its $-$ bracketed by the letter immediately below it. (Such a letter exists, since we assume the occurence of $\begin{smallmatrix} d \\ \bar{e} \end{smallmatrix} \in T$, except when $d = e$; this case will

be treated below.) This letter $d$ must be bracketed by a $d+1$ in the bottom row, and it must be weakly to the left of $a$. But we have $d+1 > d \geq c \geq a \geq d+1$; contradiction.

If instead we have a $\frac{d}{\bar{d}}$ column, we must have the $-$ from the $\bar{d}$ bracketed by a $d$ in the bottom row weakly to the left of $a$. In this case $d \geq c \geq a \geq d$ so $c = d$, and we have a $\begin{smallmatrix} d & d \\ & \bar{d} \end{smallmatrix}$ configuration, contradicting our assumption that $T \in B(s\varpi_2)$.

*Case 5: $a, b, c$ all barred*: Similarly to case 4, suppose $c \geq a$ and let $d$ be the rightmost barred letter weakly to the left of $a$ which does not have its $+$ bracketed by the letter immediately above it. (If none exists, we have a $\frac{\bar{d}}{d}$ case, see below.) It must be bracketed by a $\overline{\bar{d}+1}$ in the top row to the right of $c$. We then have $d \leq a \leq c \leq \overline{\bar{d}+1}$; contradiction.

If we have a $\frac{\bar{d}}{d}$ column (note that $d$ is barred), the $+$ from the $\bar{d}$ must be bracketed by a $d$ in the top row to the right of $c$. This implies that $d \geq c \geq a \geq d$, so $a = d$ and we have a $\begin{smallmatrix} \bar{d} & \\ d & d \end{smallmatrix}$ configuration, again contradicting our assumption that $T \in B(s\varpi_2)$.

## Appendix D. Proof of Lemma 7.10

In the notation of section 7.5, we have $k_2 = k_4 = 0$, and if $k_3 = 1$, $T_3 = \frac{n}{\bar{n}}$ or $T_3 = \frac{\bar{n}}{n}$. Lemma 7.8 thus tells us that $k_1 = k_5$.

Next we show that a column $\frac{j}{i}$ must be of the form $\frac{i-1}{i}$ for $T$ to be in $\tilde{B}^{2,s}_{\min}$. For $i = 2$ we have $j = 1$ by columnstrictness. Now suppose that $\frac{j}{i}$ is the leftmost column such that $j < i - 1$. Then $j$ contributes a $\Lambda_j$ to $\varphi(T)$ and hence $\langle c, \varphi(T)\rangle \geq 2k_1 + k_3 + 1 = s + 1$, so that $T$ is not minimal. By a similar argument $\langle c, \varepsilon(T)\rangle > s$ unless all columns of the form $\frac{\bar{i}}{\bar{j}}$ must obey $j = i - 1$.

A column $\frac{\bar{i}}{i-1}$ for $i > 2$ (resp. $\frac{n}{n-1}$) contributes a $-$ to the $(i-2)$-signature (resp. $(n-2)$-signature) of $T$. This $-$ can only be compensated by a $+$ in the $(i-2)$-signature (resp. $(n-2)$-signature) from a column $\frac{i-1}{\bar{i}}$ (resp. $\frac{n-1}{\bar{n}}$). Hence for $T$ to be minimal the number of columns of the form $\frac{i-1}{\bar{i}}$ (resp. $\frac{n-1}{\bar{n}}$) needs to be the same as the number of columns of the form $\frac{\bar{i}}{i-1}$ (resp. $\frac{n}{n-1}$). This proves that $T \in \mathcal{M}(s)$.

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