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Pieri's formula for generalized Schur polynomials

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Abstract Young's lattice, the lattice of all Young diagrams, has the Robinson-Schensted-Knuth correspondence, the correspondence between certain matrices and pairs of semi-standard Young tableaux with the same shape. Fomin introduced generalized Schur operators to generalize the Robinson-Schensted-Knuth correspondence. In this sense, generalized Schur operators are generalizations of semi-standard Young tableaux. We define a generalization of Schur polynomials as expansion coefficients of generalized Schur operators. We show that the commutation relation of generalized Schur operators implies Pieri's formula for generalized Schur polynomials.

Keywords Pieri formula · Generarized Schur operators · Schur polynomials · Young diagrams · Planar binary trees · Differential posets · Dual graphs · Symmetric functions · Quasi-symmetric polynomials

1 Introduction

Young's lattice is a prototypical example of a differential poset which was first defined by Stanley [9, 10]. The Robinson correspondence is a correspondence between permutations and pairs of standard tableaux whose shapes are the same Young diagram. This correspondence was generalized for differential posets or dual graphs (generalizations of differential posets [3]) by Fomin [2, 4]. (See also [8].)

Young's lattice also has The Robinson-Schensted-Knuth correspondence, the correspondence between certain matrices and pairs of semi-standard tableaux. Fomin [5] introduced operators called generalized Schur operators, and generalized the Robinson-Schensted-Knuth correspondence to generalized Schur operators. We define

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a generalization of Schur polynomials as expansion coefficients of generalized Schur operators.

A complete symmetric polynomial is a Schur polynomial associated with a Young diagram consisting of only one row. Schur polynomials satisfy Pieri's formula, the formula describing the product of a complete symmetric polynomial and a Schur polynomial as a sum of Schur polynomials:

$$h_i(t_1,\dots,t_n)s_\lambda(t_1,\dots,t_n)=\sum_\mu s_\mu(t_1,\dots,t_n),$$

where the sum is over all μ's that are obtained from λ by adding i boxes, with no two in the same column, h_i is the i-th complete symmetric polynomial, and s_λ is the Schur polynomial associated with λ.

In this paper, we generalize Pieri's formula to generalized Schur polynomials.

Remark 1.1. Lam introduced a generalization of the Boson-Fermion correspondence [6]. In the paper, he also showed Pieri's and Cauchy's formulae for some families of symmetric functions in the context of Heisenberg algebras. Some important families of symmetric functions, e.g., Schur functions, Hall-Littlewood polynomials, Macdonald polynomials and so on, are examples of them. He proved Pieri's formula using essentially the same method as the one in this paper. Since the assumptions of generalized Schur operators are less than those of Heisenberg algebras, our polynomials are more general than his; e.g., some of our polynomials are not symmetric. An example of generalized Schur operators which provides non-symmetric polynomials is in Section 4.3. See also Remark 2.8 for the relation between [6] and this paper.

This paper is organized as follows: In Section 2.1, we recall generalized Schur operators, and define generalized Schur polynomials. We also define a generalization of complete symmetric polynomials, called weighted complete symmetric polynomials, in Section 2.2. In Section 3, we show Pieri's formula for these polynomials (Theorem 3.2). We also see that Theorem 3.2 becomes simple for special parameters, and that weighted complete symmetric polynomials are written as linear combinations of generalized Schur polynomials in a special case. Other examples are shown in Section 4.

2 Definition

We introduce two types of polynomials in this section. One is a generalization of Schur polynomials. The other is a generalization of complete symmetric polynomials.

2.1 Generalized schur polynomials

First we recall the generalized Schur operators defined by Fomin [5]. We define a generalization of Schur polynomials as expansion coefficients of generalized Schur operators.

Let K be a field of characteristic zero that contains all formal power series in variables $t, t', t_1, t_2, \dots$ Let V_i be finite-dimensional K-vector spaces for all $i \in \mathbb{Z}$. Fix a basis Y_i of each V_i so that $V_i = KY_i$. Let $Y = \bigcup_i Y_i$, $V = \bigoplus_i V_i$ and $\widehat{V} = \prod_i V_i$, i.e., V is the vector space consisting of all finite linear combinations of elements of Y and $\widehat{V}$ is the vector space consisting of all linear combinations of elements of Y. The *rank function* on V mapping $v \in V_i$ to i is denoted by ρ. We say that Y *has a minimum* $\varnothing$ if $Y_i = \emptyset$ for $i < 0$ and $Y_0 = \{\varnothing\}$.

For a sequence $\{A_i\}$ and a formal variable x, we write $A(x)$ for the generating function $\sum_{i\geq 0} A_i x^i$.

Definition 2.1. Let D_i and U_i be linear maps on V for nonnegative integers $i \in \mathbb{N}$. We call $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ *generalized Schur operators with* $\{a_m\}$ if the following conditions are satisfied:

- $\{a_m\}$ is a sequence of K.
- U_i satisfies $U_i(V_j) \subset V_{j+i}$ for all j.
- D_i satisfies $D_i(V_j) \subset V_{j-i}$ for all j.
- The equation $D(t')U(t) = a(tt')U(t)D(t')$ holds.

In general, $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ are not linear operators on V but linear operators from V to $\widehat{V}$.

Let $\langle , \rangle$ be the natural pairing, i.e., the bilinear form on $\widehat{V} \times V$ such that $\langle \sum_{\lambda\in Y} a_\lambda \lambda, \sum_{\mu\in Y} b_\mu \mu\rangle = \sum_{\lambda\in Y} a_\lambda b_\lambda$. For generalized Schur operators $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$, U_i^* and D_i^* denote the maps obtained from the adjoints of U_i and D_i with respect to $\langle , \rangle$ by restricting to V, respectively. For all i, U_i^* and D_i^* are linear maps on V satisfying $U_i^*(V_j) \subset V_{j-i}$ and $D_i^*(V_j) \subset V_{j+i}$. It follows by definition that

$$\langle v, U_i w\rangle = \langle w, U_i^* v\rangle, \quad \langle v, D_i w\rangle = \langle w, D_i^* v\rangle$$

for $v, w \in V$. We write $U^*(t)$ and $D^*(t)$ for $\sum U_i^* t^i$ and $\sum D_i^* t^i$. It follows by definition that

$$\langle U(t)\mu, \lambda\rangle = \langle U^*(t)\lambda, \mu\rangle, \quad \langle D(t)\mu, \lambda\rangle = \langle D^*(t)\lambda, \mu\rangle$$

for $\lambda, \mu \in Y$. The equation $D(t')U(t) = a(tt')U(t)D(t')$ implies the equation $U^*(t')D^*(t) = a(tt')D^*(t)U^*(t')$. Hence $U^*(t_1)\cdots U^*(t_n)$ and $D^*(t_n)\cdots D^*(t_1)$ are generalized Schur operators with $\{a_m\}$ when $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ are.

Definition 2.2. Let $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ be generalized Schur operators with $\{a_m\}$. For $v \in V$ and $\mu \in Y$, $s^D_{v,\mu}(t_1, \ldots, t_n)$ and $s_U^{\mu,v}(t_1, \ldots, t_n)$ are respectively defined by

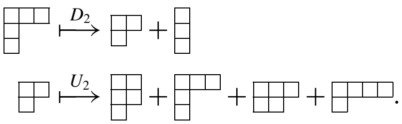
$$s^D_{v,\mu}(t_1, \ldots, t_n) = \langle D(t_1)\cdots D(t_n)v, \mu\rangle,$$
$$s_U^{\mu,v}(t_1, \ldots, t_n) = \langle U(t_n)\cdots U(t_1)v, \mu\rangle.$$

We call these polynomials $s^D_{v,\mu}(t_1,\ldots,t_n)$ and $s^{\mu,v}_U(t_1,\ldots,t_n)$ *generalized Schur polynomials*.

Remark 2.3. Generalized Schur polynomials $s^D_{v,\mu}(t_1,\ldots,t_n)$ are symmetric in the case when $D(t)D(t') = D(t')D(t)$, but not symmetric in general. Similarly, generalized Schur polynomials $s^{\mu,v}_U(t_1,\ldots,t_n)$ are symmetric if $U(t)U(t') = U(t')U(t)$.

If U_0 (resp. D_0) is the identity map on V, generalized Schur polynomials $s^D_{v,\mu}(t_1,\ldots,t_n)$ (resp. $s^{\mu,v}_U(t_1,\ldots,t_n)$) are quasi-symmetric. In [1], Bergeron, Mykytiuk, Sottile and van Willigenburg considered graded representations of the algebra of noncommutative symmetric functions on the $\mathbb{Z}$-free module whose basis is a graded poset, and gave a Hopf-morphism from a Hopf algebra generated by intervals of the poset to the Hopf algebra of quasi-symmetric functions.

Example 2.4. Our prototypical example is Young's lattice $\mathbb{Y}$ that consists of all Young diagrams. Let Y be Young's lattice $\mathbb{Y}$, V the K-vector space $K\mathbb{Y}$ whose basis is $\mathbb{Y}$, and ρ the ordinary rank function mapping a Young diagram λ to the number of boxes in λ. Young's lattice $\mathbb{Y}$ has a minimum $\varnothing$, the Young diagram with no boxes. We call a skew Young diagram μ/λ a horizontal strip if μ/λ has no two boxes in the same column. Define U_i by $U_i(\mu) = \sum_\lambda \lambda$, where the sum is over all λ's that are obtained from μ by adding a horizontal strip consisting of i boxes; and define D_i by $D_i(\lambda) = \sum_\mu \mu$, where the sum is over all μ's that are obtained from λ by removing a horizontal strip consisting of i boxes. For example,



(See also Fig. 1, the graph of D_1 (U_1) and D_2 (U_2).)

In this case, $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ are generalized Schur operators with $\{a_m = 1\}$. Both $s^D_{\lambda,\mu}(t_1,\ldots,t_n)$ and $s^{\lambda,\mu}_U(t_1,\ldots,t_n)$ are equal to the skew Schur

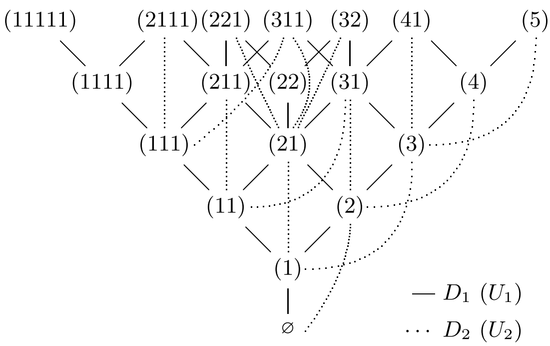


Fig. 1 Young's lattice

polynomial $s_{\lambda/\mu}(t_1, \ldots, t_n)$ for $\lambda, \mu \in \mathbb{Y}$. For example, since

$$\begin{aligned}
D(t_2)\yng(2,1) &= \yng(2,1) + t_2\yng(1,1) + t_2\yng(2) + t_2^2\yng(1)\\
D(t_1)D(t_2)\yng(2,1) &= \yng(2,1) + t_1\yng(1,1) + t_1\yng(2) + t_1^2\yng(1) + t_2\left(\yng(1,1) + t_1\yng(1)\right) + t_2\left(\yng(2) + t_1\yng(1) + t_1^2\varnothing\right)\\
&\quad + t_2^2(\yng(1) + t_1\varnothing),
\end{aligned}$$

$s^D_{(2,1),\varnothing}(t_1, t_2) = s_{(2,1)}(t_1, t_2) = t_1^2 t_2 + t_1 t_2^2.$

Example 2.5. Our second example is the polynomial ring $K[x]$ with a variable x. Let V be $K[x]$ and ρ the ordinary rank function mapping a monomial ax^n to its degree n. In this case, $\dim V_i = 1$ for all $i \geq 0$ and $\dim V_i = 0$ for $i < 0$. Hence its basis Y is identified with $\mathbb{N}$ and has a minimum c_0, a nonzero constant. Define D_i and U_i by $\frac{\partial^i}{i!}$ and $\frac{x^i}{i!}$, where ∂ is the partial differential operator in x. Then $D(t)$ and $U(t)$ are $\exp(t\partial)$ and $\exp(tx)$. Since $D(t)$ and $U(t)$ satisfy $D(t)U(t') = \exp(tt')U(t')D(t)$, $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ are generalized Schur operators with $\{a_m = \frac{1}{m!}\}$. In general, for differential posets, we can construct generalized Schur operators in a similar manner.

Since ∂ and x commute with t, the following equations hold:

$$\begin{aligned}
D(t_1)\cdots D(t_n) &= \exp(\partial t_1)\cdots\exp(\partial t_n) = \exp(\partial(t_1 + \cdots + t_n)),\\
U(t_n)\cdots U(t_1) &= \exp(xt_n)\cdots\exp(xt_1) = \exp(x(t_1 + \cdots + t_n)).
\end{aligned}$$

It follows from direct calculations that

$$\begin{aligned}
\exp(\partial(t_1 + \cdots + t_n))c_i x^i &= \sum_{j=0}^{i} \frac{(t_1 + \cdots + t_n)^j}{j!} \frac{i!}{(i-j)!} c_i x^{i-j}\\
&= \sum_{j=0}^{i} \frac{i!(t_1 + \cdots + t_n)^j c_i}{(i-j)!j!c_{i-j}} c_{i-j} x^{i-j},\\
\exp(x(t_1 + \cdots + t_n))c_i x^i &= \sum_j \frac{(t_1 + \cdots + t_n)^j x^j}{j!} c_i x^i\\
&= \sum_j \frac{(t_1 + \cdots + t_n)^j c_i}{j!c_{i+j}} c_{i+j} x^{i+j}.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
s^D_{c_{i+j}x^{i+j}, c_i x^i}(t_1, \ldots, t_n) &= \frac{(i+j)!}{i!j!} \frac{c_{i+j}}{c_i} (t_1 + \cdots + t_n)^j\\
s_U^{c_{i+j}x^{i+j}, c_i x^i}(t_1, \ldots, t_n) &= \frac{1}{j!} \frac{c_i}{c_{i+j}} (t_1 + \cdots + t_n)^j,
\end{aligned}$$

if we take $\{c_i x^i\}$ as the basis Y.

If $c_i = 1$ for all i, then $s^D_{x^{i+j},x^i}(t_1, \dots, t_n) = \frac{(i+j)!}{i!j!}(t_1 + \cdots + t_n)^j$, and $s_U^{x^{i+j},x^i}(t_1, \dots, t_n) = \frac{1}{j!}(t_1 + \cdots + t_n)^j$.

Lemma 2.6. *Generalized Schur polynomials satisfy the following equations*:

$$s^D_{\lambda,\mu}(t_1, \dots, t_n) = s_{D^*}^{\lambda,\mu}(t_1, \dots, t_n),$$
$$s_U^{\lambda,\mu}(t_1, \dots, t_n) = s^{U^*}_{\lambda,\mu}(t_1, \dots, t_n)$$

for $\lambda, \mu \in Y$. *Generalized Schur polynomials also satisfy the following equations*:

$$s^D_{v,\mu}(t_1, \dots, t_n) = \sum_{\nu \in Y} \langle v, \nu \rangle s_{D^*}^{\nu,\mu}(t_1, \dots, t_n),$$
$$s_{D^*}^{\mu,v}(t_1, \dots, t_n) = \sum_{\nu \in Y} \langle v, \nu \rangle s^D_{\mu,\nu}(t_1, \dots, t_n),$$
$$s^{U^*}_{v,\mu}(t_1, \dots, t_n) = \sum_{\nu \in Y} \langle v, \nu \rangle s_U^{\nu,\mu}(t_1, \dots, t_n),$$
$$s_U^{\mu,v}(t_1, \dots, t_n) = \sum_{\nu \in Y} \langle v, \nu \rangle s^{U^*}_{\mu,\nu}(t_1, \dots, t_n)$$

for $\mu \in Y$, $v \in V$.

Proof: It follows by definition that

$$\begin{aligned} s^D_{\lambda,\mu}(t_1, \dots, t_n) &= \langle D(t_1) \cdots D(t_n)\lambda, \mu \rangle \\ &= \langle D^*(t_n) \cdots D^*(t_1)\mu, \lambda \rangle = s_{D^*}^{\lambda,\mu}(t_1, \dots, t_n). \end{aligned}$$

Similarly, we have $s_U^{\lambda,\mu}(t_1, \dots, t_n) = s^{U^*}_{\lambda,\mu}(t_1, \dots, t_n)$. The other formulae follow from $v = \sum_{\nu \in Y} \langle \nu, v \rangle \nu$ for $v \in V$. □

Remark 2.7. Rewriting the generalized Cauchy identity [5, 1.4. Corollary] with our notation, we obtain a Cauchy identity for generalized Schur polynomials:

$$\sum_{\nu \in Y} s^D_{\nu,\mu}(t_1, \dots, t_n) s_U^{\nu,v}(t'_1, \dots, t'_n) = \prod_{i,j} a(t_i t'_j) \sum_{\kappa \in Y} s_U^{\mu,\kappa}(t'_1, \dots, t'_n) s^D_{v,\kappa}(t_1, \dots, t_n)$$

for $v \in V$, $\mu \in Y$.

Remark 2.8. In this remark, we construct operators B_l from generalized Schur operators $D(t_1) \cdots D(t_n)$ and $U(t_n) \cdots U(t_1)$. These operators B_l are closely related to the results of Lam [6]. Furthermore we can construct other generalized Schur operators $D(t_1) \cdots D(t_n)$ and $U'(t_n) \cdots U'(t_1)$ from B_l.

Let $D(t_1) \cdots D(t_n)$ and $U(t_n) \cdots U(t_1)$ be generalized Schur operators with $\{a_m\}$. For a partition $\lambda \vdash l$, we define z_λ by $z_\lambda = 1^{m_1(\lambda)} m_1(\lambda)! \cdot 2^{m_2(\lambda)} m_2(\lambda)! \cdots$, where

$m_i(\lambda) = |\{j|\lambda_j = i\}|$. Let $U_0 = D_0 = I$, where I is the identity map. For positive integers l, we inductively define b_l, B_l and B_{-l} by

$$b_l = a_l - \sum_\lambda \frac{b_\lambda}{z_\lambda},$$

$$B_l = D_l - \sum_\lambda \frac{B_\lambda}{z_\lambda},$$

$$B_{-l} = U_l - \sum_\lambda \frac{B_{-\lambda}}{z_\lambda},$$

where $b_\lambda = b_{\lambda_1} \cdot b_{\lambda_2} \cdots$, $B_\lambda = B_{\lambda_1} \cdot B_{\lambda_2} \cdots$, $B_{-\lambda} = B_{-\lambda_1} \cdot B_{-\lambda_2} \cdots$ and the sums are over all partitions λ of l such that $\lambda_1 < l$. Let $b_l \neq 0$ for any l. It follows from direct calculations that

$$[B_l, B_{-l}] = l \cdot b_l \cdot I,$$

$$[B_l, B_{-k}] = 0$$

for positive integers $l \neq k$. If U_i and D_i respectively commute with U_j and D_j for all i, j, then $\{B_l, B_{-l} | l \in \mathbb{Z}_{>0}\}$ generates the Heisenberg algebra. In this case, we can apply the results of Lam [6]. See also Remark 2.13 for the relation between his complete symmetric polynomials $h_i[b_m](t_1, \ldots, t_n)$ and our weighted complete symmetric polynomials $h_i^{\{a_m\}}(t_1, \ldots, t_n)$.

For a partition $\lambda \vdash l$, let $\mathrm{sgn}(\lambda)$ denote $(-1)^{\sum_i (\lambda_i - 1)}$, where the sum is over all i's such that $\lambda_i > 0$. Although U_i and D_i do not commute with U_j and D_j, we can define dual generalized Schur operators $D(t_1) \cdots D(t_n)$ and $U'(t_n) \cdots U'(t_1)$ with $\{a'_m\}$ by

$$a'_l = \sum_\lambda \frac{\mathrm{sgn}(\lambda) b_\lambda}{z_\lambda},$$

$$U'_{-l} = \sum_\lambda \frac{\mathrm{sgn}(\lambda) B_{-\lambda}}{z_\lambda},$$

where the sums are over all partitions λ of l. In this case, it follows from direct calculations that $a(t) \cdot a'(-t) = 1$.

2.2 **Weighted complete symmetric polynomials**

Next we introduce a generalization of complete symmetric polynomials. We define weighted symmetric polynomials inductively.

Definition 2.9. Let $\{a_m\}$ be a sequence of elements of K. We define the i-th weighted complete symmetric polynomial $h_i^{\{a_m\}}(t_1, \ldots, t_n)$ to be the coefficient of t^i in $a(t_1 t) \cdots a(t_n t)$.

By definition, for each i, the i-th weighted complete symmetric polynomial $h_i^{\{a_m\}}(t_1, \dots, t_n)$ is a homogeneous symmetric polynomial of degree i.

Remark 2.10. For a sequence $\{a_m\}$ of elements of K, the i-th weighted complete symmetric polynomial $h_i^{\{a_m\}}(t_1, \dots, t_n)$ coincides with the polynomial defined by

$$h_i^{\{a_m\}}(t_1, \dots, t_n) = \begin{cases} a_i t_1^i & (\text{for } n = 1), \\ \sum_{j=0}^{i} h_j^{\{a_m\}}(t_1, \dots, t_{n-1}) h_{i-j}^{\{a_m\}}(t_n) & (\text{for } n > 1). \end{cases} \tag{1}$$

Example 2.11. When a_m equals 1 for each m, $a(t) = \sum_i t^i = \frac{1}{1-t}$. In this case, $h_j^{\{1,1,\dots\}}(t_1, \dots, t_n)$ equals the complete symmetric polynomial $h_j(t_1, \dots, t_n)$.

Example 2.12. When a_m equals $\frac{1}{m!}$ for each m, $\sum_j h_j^{\{\frac{1}{m!}\}}(t) = \exp(t) = a(t)$ and $h_j^{\{\frac{1}{m!}\}}(t_1, \dots, t_n) = \frac{1}{j!}(t_1 + \dots + t_n)^j$.

Remark 2.13. In this remark, we compare the complete symmetric polynomials $h_i[b_m](t_1, \dots, t_n)$ of Lam [6] and our weighted complete symmetric polynomials $h_i^{\{a_m\}}(t_1, \dots, t_n)$. Let $\{b_m\}$ be a sequence of elements of K. The polynomials $h_i[b_m](t_1, \dots, t_n)$ of Lam are defined by

$$h_i[b_m](t_1, \dots, t_n) = \sum_{\lambda \vdash i} \frac{b_\lambda p_\lambda(t_1, \dots, t_n)}{z_\lambda},$$

where $b_\lambda = b_{\lambda_1} \cdot b_{\lambda_2} \cdots$, $p_\lambda(t_1, \dots, t_n) = p_{\lambda_1}(t_1, \dots, t_n) \cdot p_{\lambda_2}(t_1, \dots, t_n) \cdots$ and $p_i(t_1, \dots, t_n) = t_1^i + \dots + t_n^i$. These polynomials satisfy the equation

$$h_i[b_m](t_1, \dots, t_n) = \sum_{j=0}^{i} h_j[b_m](t_1, \dots, t_{n-1}) h_{i-j}[b_m](t_n).$$

Let $a_i = \sum_{\lambda \vdash i} \frac{b_\lambda}{z_\lambda}$. Then it follows $h_i[b_m](t_1) = a_i t^i$. Hence

$$h_i[b_m](t_1, \dots, t_n) = h_i^{\{a_m\}}(t_1, \dots, t_n).$$

3 Main results

In this section, we show some properties of generalized Schur polynomials and weighted complete symmetric polynomials.

Throughout this section, let $D(t_1) \cdots D(t_n)$ and $U(t_n) \cdots U(t_1)$ be generalized Schur operators with $\{a_m\}$.

3.1 **Main theorem**

In Proposition 3.1, we describe the commuting relation of U_i and $D(t_1)\cdots D(t_n)$, proved in Section 3.3. This relation implies Pieri's formula for our polynomials (Theorem 3.2), the main result in this paper. It also follows from this relation that the weighted complete symmetric polynomials are written as linear combinations of generalized Schur polynomials when Y has a minimum (Proposition 3.5).

First we describe the commuting relation of U_i and $D(t_1)\cdots D(t_n)$. We prove it in Section 3.3.

Proposition 3.1. *The equations*

$$D(t_1)\cdots D(t_n)U_i = \sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\ldots,t_n)U_j D(t_1)\cdots D(t_n), \tag{2}$$

$$D_i U(t_n)\cdots U(t_1) = \sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\ldots,t_n)U(t_n)\cdots U(t_1)D_j, \tag{3}$$

$$U_i^* D^*(t_n)\cdots D^*(t_1) = \sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\ldots,t_n)D^*(t_n)\cdots D^*(t_1)U_j^*, \tag{4}$$

$$U^*(t_1)\cdots U^*(t_n)D_i^* = \sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\ldots,t_n)D_j^* U^*(t_1)\cdots U^*(t_n). \tag{5}$$

hold for all i.

These equations imply the following main theorem.

Theorem 3.2 (Pieri's formula). *For each $\mu \in Y_k$ and each $v \in V$, generalized Schur polynomials satisfy*

$$s^D_{U_i v,\mu}(t_1,\ldots,t_n) = \sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\ldots,t_n) \sum_{\nu\in Y_{k-j}} \langle U_j v,\mu\rangle s^D_{v,\nu}(t_1,\ldots,t_n).$$

Proof: It follows from Proposition 3.1 that

$$\begin{aligned}\langle D(t_1)\cdots D(t_n)U_i v,\mu\rangle &= \left\langle \sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\ldots,t_n)U_j D(t_1)\cdots D(t_n)v,\mu\right\rangle \\ &= \sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\ldots,t_n)\langle U_j D(t_1)\cdots D(t_n)v,\mu\rangle\end{aligned}$$

for $v \in V$ and $\mu \in Y$. This says

$$s^D_{U_i v,\mu}(t_1,\dots,t_n) = \sum_{j=0}^{i} h^{\{a_m\}}_{i-j}(t_1,\dots,t_n) \sum_{\nu \in Y_{k-j}} \langle U_j \nu, \mu\rangle s^D_{v,\nu}(t_1,\dots,t_n).$$

□

This formula becomes simple in the case when $v \in Y$.

Corollary 3.3. *For each $\lambda, \mu \in Y$, generalized Schur polynomials satisfy*

$$s^D_{U_i\lambda,\mu}(t_1,\dots,t_n) = \sum_{j=0}^{i} h^{\{a_m\}}_{i-j}(t_1,\dots,t_n) \cdot s^{\lambda,U_j^*\mu}_{D^*}(t_1,\dots,t_n).$$

Proof: It follows from Theorem 3.2 that

$$s^D_{U_i\lambda,\mu}(t_1,\dots,t_n) = \sum_{j=0}^{i} h^{\{a_m\}}_{i-j}(t_1,\dots,t_n) \sum_{\nu\in Y} \langle U_j \nu, \mu\rangle s^D_{\lambda,\nu}(t_1,\dots,t_n).$$

Lemma 2.6 implies

$$\begin{aligned}\sum_{\nu\in Y}\langle U_j\nu,\mu\rangle s^D_{\lambda,\nu}(t_1,\dots,t_n) &= \sum_{\nu\in Y}\langle \nu, U_j^*\mu\rangle s^D_{\lambda,\nu}(t_1,\dots,t_n)\\ &= s^{\lambda,U_j^*\mu}_{D^*}(t_1,\dots,t_n).\end{aligned}$$

Hence

$$s^D_{U_i\lambda,\mu}(t_1,\dots,t_n) = \sum_{j=0}^{i} h^{\{a_m\}}_{i-j}(t_1,\dots,t_n) \cdot s^{\lambda,U_j^*\mu}_{D^*}(t_1,\dots,t_n).$$

□

If Y has a minimum $\varnothing$, Theorem 3.2 implies the following corollary.

Corollary 3.4. *Let Y have a minimum $\varnothing$. For all $v \in V$, the following equations hold*:

$$s^D_{U_i v,\varnothing}(t_1,\dots,t_n) = u_0 \cdot h^{\{a_m\}}_{i}(t_1,\dots,t_n) \cdot s^D_{v,\varnothing}(t_1,\dots,t_n),$$

where u_0 is the element of K that satisfies $U_0\varnothing = u_0\varnothing$.

In the case when Y has a minimum $\varnothing$, weighted complete symmetric polynomials are written as linear combinations of generalized Schur polynomials.

Proposition 3.5. *Let Y have a minimum $\varnothing$. The following equations hold for all $i \geq 0$*:

$$s^D_{U_i\varnothing,\varnothing}(t_1,\dots,t_n) = d_0^n u_0 \cdot h^{\{a_m\}}_{i}(t_1,\dots,t_n),$$

where d_0, u_0 are the elements of K that satisfy $D_0\varnothing = d_0\varnothing$ and $U_0\varnothing = u_0\varnothing$.

Proof: By definition, $s^D_{\varnothing,\varnothing}(t_1,\dots,t_n)$ is d_0^n. Hence it follows from Corollary 3.4 that

$$s^D_{U_i\varnothing,\varnothing}(t_1,\dots,t_n) = u_0 h_i^{\{a_m\}}(t_1,\dots,t_n)d_0^n. \qquad \square$$

Example 3.6. In the prototypical example $\mathbb{Y}$ (Example 2.4), for $\lambda \in \mathbb{Y}$, $U_i\lambda$ is the sum of all Young diagrams obtained from λ by adding a horizontal strip consisting of i boxes. Hence $s^D_{U_i\lambda,\varnothing}(t_1,\dots,t_n)$ equals $\sum_\nu s_\nu$, where the sum is over all ν's that are obtained from λ by adding a horizontal strip consisting of i boxes. On the other hand, u_0 is 1, and $h_i^{\{1,1,1,\dots\}}(t_1,\dots,t_n)$ is the i-th complete symmetric polynomial $h_i(t_1,\dots,t_n)$ (Example 2.11). Thus Corollary 3.4 is nothing but the classical Pieri's formula. Theorem 3.2 is Pieri's formula for skew Schur polynomials; for a skew Young diagram λ/μ and $i \in \mathbb{N}$,

$$\sum_\kappa s_{\kappa/\mu}(t_1,\dots,t_n) = \sum_{j=0}^{i}\sum_\nu h_{i-j}(t_1,\dots,t_n)s_{\lambda/\nu}(t_1,\dots,t_n),$$

where the first sum is over all κ's that are obtained from λ by adding a horizontal strip consisting of i boxes; the last sum is over all ν's that are obtained from μ by removing a horizontal strip consisting of j boxes.

In this example, Proposition 3.5 says that the Schur polynomial $s_{(i)}$ corresponding to Young diagram with only one row equals the complete symmetric polynomial h_i.

Example 3.7. In the second example $\mathbb{N}$ (Example 2.5), Proposition 3.5 says that the constant term of $\exp(\partial(t_1+\cdots+t_n))\cdot\frac{x^i}{i!}$ equals $\frac{(t_1+\cdots+t_n)^i}{i!}$.

3.2 Some variations of Pieri's formula

In this section, we show some variations of Pieri's formula for generalized Schur polynomials, i.e., we show Pieri's formula not only for $s^D_{\lambda,\mu}(t_1,\dots,t_n)$ but also for $s_U^{\lambda,\mu}(t_1,\dots,t_n)$, $s_{D^*}^{\lambda,\mu}(t_1,\dots,t_n)$ and $s^{U^*}_{\lambda,\mu}(t_1,\dots,t_n)$.

Theorem 3.8 (Pieri's formula). *For each $\mu \in Y_k$ and each $v \in V$, generalized Schur polynomials satisfy the following equations*:

$$\sum_{\kappa\in Y}\langle D_i\kappa,\mu\rangle s_U^{\kappa,v}(t_1,\dots,t_n) = \sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\dots,t_n)s_U^{\mu,D_jv}(t_1,\dots,t_n),$$

$$s^{U^*}_{D_i^*v,\mu}(t_1,\dots,t_n) = \sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\dots,t_n)\sum_{\nu\in Y_{k-j}}\langle D_j^*v,\mu\rangle s^{U^*}_{v,\nu}(t_1,\dots,t_n),$$

$$\sum_{\kappa\in Y}\langle U_i^*\kappa,\mu\rangle s_{D^*}^{\kappa,v}(t_1,\dots,t_n) = \sum_{j=0}^{i} h_{i-j}^{\{a_m\}}(t_1,\dots,t_n)s_{D^*}^{\mu,U_j^*v}(t_1,\dots,t_n).$$

Proof: Applying Theorem 3.2 to $U^*(t_1)\cdots U^*(t_n)$ and $D^*(t_n)\cdots D^*(t_1)$, we obtain

$$s^{U^*}_{D^*_i v,\mu}(t_1,\dots,t_n)=\sum_{j=0}^{i} h^{\{a_m\}}_{i-j}(t_1,\dots,t_n)\sum_{\nu\in Y_{k-j}}\langle D^*_j\nu,\mu\rangle s^{U^*}_{v,\nu}(t_1,\dots,t_n).$$

It follows from Proposition 3.1 that

$$\langle D_i U(t_n)\cdots U(t_1)v,\mu\rangle=\left\langle\sum_{j=0}^{i} h^{\{a_m\}}_{i-j}(t_1,\dots,t_n)U(t_n)\cdots U(t_1)D_j v,\mu\right\rangle$$

for $v\in V$ and $\mu\in Y$. This equation says

$$\sum_{\kappa\in Y}\langle D_i\kappa,\mu\rangle s^{\kappa,v}_{U}(t_1,\dots,t_n)=\sum_{j=0}^{i} h^{\{a_m\}}_{i-j}(t_1,\dots,t_n)s^{\mu,D_j v}_{U}(t_1,\dots,t_n).$$

For generalized Schur operators $U^*(t_1)\cdots U^*(t_n)$ and $D^*(t_n)\cdots D^*(t_1)$, this equation says

$$\sum_{\kappa\in Y}\langle U^*_i\kappa,\mu\rangle s^{\kappa,v}_{D^*}(t_1,\dots,t_n)=\sum_{j=0}^{i} h^{\{a_m\}}_{i-j}(t_1,\dots,t_n)s^{\mu,U^*_j v}_{D^*}(t_1,\dots,t_n).$$

□

Corollary 3.9. *For all $v\in V$, the following equations hold*:

$$s^{U^*}_{D^*_i v,\varnothing}(t_1,\dots,t_n)=d_0\cdot h^{\{a_m\}}_{i}(t_1,\dots,t_n)\cdot s^{U^*}_{v,\varnothing}(t_1,\dots,t_n),$$

where d_0 is the element of K that satisfies $D_0\varnothing=d_0\varnothing$.

Proof: We obtain this proposition from Theorem 3.4 by applying to generalized Schur operators $U^*(t_1,\dots,t_n)$ and $D^*(t_1,\dots,t_n)$. □

Proposition 3.10. *Let Y have a minimum $\varnothing$. Then*

$$s^{U^*}_{D^*_i\varnothing,\varnothing}(t_1,\dots,t_n)=u_0^n d_0\cdot h^{\{a_m\}}_{i}(t_1,\dots,t_n),$$

where u_0 and d_0 are the elements of K that satisfy $D_0\varnothing=d_0\varnothing$ and $U_0\varnothing=u_0\varnothing$.

Proof: We obtain this proposition by applying Theorem 3.5 to generalized Schur operators $U^*(t_1)\cdots U^*(t_n)$ and $D^*(t_n)\cdots D^*(t_1)$. □

3.3 **Proof of Proposition 3.1**

In this section, we prove Proposition 3.1.

First, we prove the Eq. (2). The other equations follow from the Eq. (2).

Proof: Since $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ are generalized Schur operators with $\{a_m\}$, the equations $D(t)U_i = \sum_{j=0}^{i} a_j t^j U_{i-j} D(t)$ hold for all integers i. Hence $D(t_1)\cdots D(t_n)U_i$ is written as a K-linear combination of $U_j D(t_1)\cdots D(t_n)$. We write $H_{i,j}(t_1,\ldots,t_n)$ for the coefficient of $U_j D(t_1)\cdots D(t_n)$ in $D(t_1)\cdots D(t_n)U_i$.

It follows from the equation $D(t)U_i = \sum_{j=0}^{i} a_j t^j U_{i-j} D(t)$ that

$$H_{i,i-j}(t_1) = a_j t_1^j \tag{6}$$

for $0 \le j \le i$.

We apply the relation (6) to $D(t_n)$ and U_i to have

$$D(t_1)\cdots D(t_{n-1})D(t_n)U_i = \sum_{j=0}^{i} a_{i-j} t_n^{i-j} D(t_1)\cdots D(t_{n-1})U_j D(t_n).$$

Since $D(t_1)\cdots D(t_{n-1})U_i = \sum_j H_{i,j}(t_1,\ldots,t_{n-1})U_j D(t_1)\cdots D(t_{n-1})$ by the definition of $H_{i,j}(t_1,\ldots,t_{n-1})$, we have the equation

$$D(t_1)\cdots D(t_{n-1})D(t_n)U_i = \sum_{k=0}^{i}\sum_{j=k}^{i} a_{i-j} t_n^{i-j} H_{j,k}(t_1,\ldots,t_{n-1})U_k D(t_1)\cdots D(t_n).$$

Since $D(t_1)\cdots D(t_n)U_i$ equals $\sum_{k=0}^{i} H_{i,k}(t_1,\ldots,t_n)U_k D(t_1)\cdots D(t_n)$ by definition, the equation

$$\begin{aligned}&\sum_{k=0}^{i}\sum_{j=k}^{i} a_{i-j} t_n^{i-j} H_{j,k}(t_1,\ldots,t_{n-1})U_k D(t_1)\cdots D(t_n)\\&\quad= \sum_{k=0}^{i} H_{i,k}(t_1,\ldots,t_n)U_k D(t_1)\cdots D(t_n)\end{aligned}$$

holds. Hence the equation

$$\sum_{j=k}^{i} a_{i-j} t_n^{i-j} H_{j,k}(t_1,\ldots,t_{n-1}) = H_{i,k}(t_1,\ldots,t_n) \tag{7}$$

holds.

We claim that $H_{i+k,k}(t_1,\ldots,t_n)$ does not depend on k. It follows from this relation (7) that

$$\begin{aligned}H_{k+l,k}(t_1,\ldots,t_n) &= \sum_{j=k}^{k+l} a_{k+l-j} t_n^{k+l-j} H_{j,k}(t_1,\ldots,t_{n-1})\\&= \sum_{j'=0}^{l} a_{k+l-(j'+k)} t_n^{k+l-(j'+k)} H_{j'+k,k}(t_1,\ldots,t_{n-1})\\&= \sum_{j'=0}^{l} a_{l-j'} t_n^{l-j'} H_{j'+k,k}(t_1,\ldots,t_{n-1}).\end{aligned}$$

Since the monomials $a_{l-j'}t_n^{l-j'}$ do not depend on k, the equations

$$H_{(i-k)+k,k}(t_1,\dots,t_n) = H_{(i-k)+k',k'}(t_1,\dots,t_n)$$

hold if the equations $H_{k+j,k}(t_1,\dots,t_{n-1}) = H_{k'+j,k'}(t_1,\dots,t_{n-1})$ hold for all k, k' and $j \le i-k$. In fact, since $H_{i+k,k}(t_1)$ equals $a_i t_1^i$, $H_{i+k,k}(t_1)$ does not depend on k. Hence it follows inductively that $H_{i+k,k}(t_1,\dots,t_n)$ does not depend on k, either. Hence we may write $\tilde{H}_{i-j}(t_1,\dots,t_n)$ for $H_{i,j}(t_1,\dots,t_n)$.

It follows from the Eqs. (6) and (7) that

$$\begin{cases} \tilde{H}_i(t_1) = a_i t_1^i & \text{(for } n=1\text{)},\\ \tilde{H}_i(t_1,\dots,t_n) = \displaystyle\sum_{k=0}^{i} \tilde{H}_{i-k}(t_1,\dots,t_{n-1})\tilde{H}_k(t_n) & \text{(for } n>1\text{)}. \end{cases}$$

Since $\tilde{H}_i(t_1,\dots,t_n)$ equals the i-th weighted complete symmetric polynomial $h_i^{\{a_m\}}(t_1,\dots,t_n)$, we have the Eq. (2).

We obtain the Eq. (4) from the Eq. (2) by applying $*$.

Since $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ are generalized Schur operators with $\{a_m\}$, $U^*(t_1)\cdots U^*(t_n)$ and $D^*(t_n)\cdots D^*(t_1)$ are also generalized Schur operators with $\{a_m\}$. Applying the Eq. (4) Proposition 3.1 to $U^*(t_1)\cdots U^*(t_n)$ and $D^*(t_n)\cdots D^*(t_1)$, we obtain the Eqs. (3) and (5), respectively.

Hence Proposition 3.1 follows. □

4 More examples

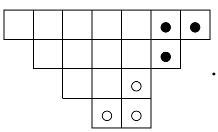
In this section, we consider some examples of generalized Schur operators.

4.1 Shifted shapes

This example is the same as [5, Example 2.1]. Let Y be the set of shifted shapes, i.e.,

$$Y = \left\{ \left\{ (i,j) \in \mathbb{N}^2 \middle| i \le j \le \lambda_i + i \right\} \middle| \lambda = (\lambda_1 > \lambda_2 > \cdots), \lambda_i \in \mathbb{N} \right\}.$$

For $\lambda \subset \nu \in Y$, let $cc_0(\lambda \setminus \nu)$ denote the number of connected components of $\lambda \setminus \nu$ that do not intersect with the main diagonal, and $cc(\lambda \setminus \nu)$ the number of connected components of $\lambda \setminus \nu$. For example, let $\lambda = (7,5,3,2)$ and $\mu = (5,4,2)$. In this case, $\lambda \setminus \nu$ is the set of boxes $\circ$ and $\bullet$ in



Since the component of the boxes $\circ$ intersects with the main diagonal at $(4,4)$, $cc_0(\lambda \setminus \nu) = 1$ and $cc(\lambda \setminus \nu) = 2$.

For $\lambda \in Y$, D_i are defined by

$$D_i\lambda = \sum_{\nu} 2^{cc_0(\lambda\backslash\nu)}\nu,$$

where the sum is over all ν's that are obtained from λ by removing i boxes, with no two box in the same diagonal.

For $\lambda \in Y$, U_i are defined by

$$U_i\lambda = \sum_{\mu} 2^{cc(\mu\backslash\lambda)}\mu,$$

where the sum is over all μ's that are obtained from λ by adding i-boxes, with no two box in the same diagonal.

In this case, since $D(t)$ and $U(t)$ satisfy

$$D(t')U(t) = \frac{1+tt'}{1-tt'}U(t)D(t'),$$

$D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ are generalized Schur operators with $\{1, 2, 2, 2, \ldots\}$. (See [5].) In this case, for $\lambda, \mu \in Y$, generalized Schur polynomials $s^D_{\lambda,\mu}$ and $s^{\lambda,\mu}_U$ are respectively the shifted skew Schur polynomials $Q_{\lambda/\mu}(t_1, \ldots, t_n)$ and $P_{\lambda/\mu}(t_1, \ldots, t_n)$.

In this case, Proposition 3.5 reads as

$$h_i^{\{1,2,2,2,\ldots\}}(t_1, \ldots, t_n) = \begin{cases} 2Q_{(i)}(t_1, \ldots, t_n) & i > 0 \\ Q_{\varnothing}(t_1, \ldots, t_n) & i = 0 \end{cases}.$$

It also follows from Proposition 3.10 that

$$h_i^{\{1,2,2,2,\ldots\}}(t_1, \ldots, t_n) = P_{(i)}(t_1, \ldots, t_n).$$

Furthermore, Corollary 3.4 reads as

$$\sum_{\mu} 2^{cc(\mu\backslash\lambda)} Q_\mu(t_1, \ldots, t_n) = h_i^{\{1,2,2,2,\ldots\}} Q_\lambda(t_1, \ldots, t_n),$$

where the sum is over all μ's that are obtained from λ by adding i boxes, with no two in the same diagonal.

4.2 **Young's lattice: dual identities**

This example is the same as [5, Example 2.4]. Let Y be Young's lattice $\mathbb{Y}$, and D_i the same ones in the prototypical example, (i.e., $D_i\lambda = \sum_\mu \mu$, where the sum is over all μ's that are obtained from λ by removing i boxes, with no two in the same column.) For $\lambda \in Y$, U_i' are defined by $U_i'\lambda = \sum_\mu \mu$, where the sum is over all μ's that are obtained from λ by adding i boxes, with no two in the same row. (In other words, D_i removes horizontal strips, while U'_i adds vertical strips.)

In this case, since $D(t)$ and $U'(t)$ satisfy

$$D(t)U'(t') = (1 + tt')U'(t')D(t),$$

$D(t_1)\cdots D(t_n)$ and $U'(t_n)\cdots U'(t_1)$ are generalized Schur operators with $\{1, 1, 0, 0, 0, \ldots\}$. (See [5].) In this case, for λ, $\mu \in Y$, generalized Schur polynomials $s_{U'}^{\lambda,\mu}$ equal $s_{\lambda'/\mu'}(t_1, \ldots, t_n)$, where λ' and μ' are the transposes of λ and μ, and $s_{\lambda'/\mu'}(t_1, \ldots, t_n)$ are skew Schur polynomials.

In the prototypical example (Example 3.6), Corollary 3.4 is the classical Pieri's formula, the formula describing the product of a complete symmetric polynomial and a Schur polynomial. In this example, Corollary 3.4 is the dual Pieri's formula, the formula describing the product of a elementary symmetric polynomial and a Schur polynomial.

In this case, Corollary 3.5 reads as

$$h_i^{\{1,1,0,0,0,\ldots\}}(t_1, \ldots, t_n) = s_{(1^i)}(t_1, \ldots, t_n) = e_i(t_1, \ldots, t_n),$$

where $e_i(t_1, \ldots, t_n)$ denotes the i-th elementally symmetric polynomials.

Furthermore, Corollary 3.4 reads as

$$\sum_{\mu} s_\mu(t_1, \ldots, t_n) = e_i(t_1, \ldots, t_n)s_\lambda(t_1, \ldots, t_n),$$

where the sum is over all μ's that are obtained from λ by adding a vertical strip consisting of i boxes.

For a skew Young diagram λ/μ and $i \in \mathbb{N}$, Theorem 3.2 reads as

$$\sum_{\kappa} s_{\kappa/\mu}(t_1, \ldots, t_n) = \sum_{j=0}^{i} \sum_{\nu} e_{i-j}(t_1, \ldots, t_n)s_{\lambda/\nu}(t_1, \ldots, t_n),$$

where the first sum is over all κ's that are obtained from λ by adding a vertical strip consisting i boxes; the last sum is over all ν's that are obtained from μ by removing a vertical strip consisting j boxes.

4.3 **Planar binary trees**

This example is the same as [7]. Let F be the monoid of words generated by the alphabet $\{1, 2\}$ and 0 denote the word of length 0. We give F the structure of a poset by $v \le vw$ for $v, w \in F$. We call an ideal of the poset F a *planar binary tree* or shortly a *tree*. An element of a tree is called a *node* of the tree. We write $\mathbb{T}$ for the set of trees and $\mathbb{T}_i$ for the set of trees with i nodes. We respectively call nodes $v2$ and $v1$ *right* and *left children* of v. A node without a child is called a *leaf*. For $T \in \mathbb{T}$ and $v \in F$, we define T_v to be $\{w \in T | v \le w\}$.

First we define up operators. We respectively call T' a tree obtained from T by adding some nodes *right-strictly* and *left-strictly* if $T \subset T'$ and each $w \in T' \setminus T$ has no right children and no left children. We define linear operators U_i and U_i' on $K\mathbb{T}$ by

$$U_i T = \sum_{T'} T',$$
$$U_i' T = \sum_{T''} T'',$$

where the first sum is over all T''s that are obtained from T by adding i nodes right-strictly, and the second sum is over all T'''s that are obtained from T by adding i nodes left-strictly. For example,

$$U_2\{0\} = \{0, 1, 11\} + \{0, 1, 2\} + \{0, 2, 21\},$$
$$U_2'\{0\} = \{0, 2, 22\} + \{0, 1, 2\} + \{0, 1, 12\}.$$

Next we define down operators. For $T \in \mathbb{T}$, let r_T be $\{w \in T | w2 \notin T.$ If $w = v1w'$ then $v2 \notin T.\}$, i.e., the set of nodes which have no child on its right and which belong between 0 and the rightmost leaf of T. The set r_T is a chain. Let $r_T = \{w_{T,1} < w_{T,2} < \cdots\}$. We define linear operators D_i on $K\mathbb{T}$ by

$$D_i T = \begin{cases} (\cdots((T \ominus w_{T,i}) \ominus w_{T,i-1})\cdots) \ominus w_{T,1} & i \le |r_T| \\ 0 & i > |r_T| \end{cases}$$

for $T \in \mathbb{T}$, where

$$T \ominus w = (T \setminus T_w) \cup \{wv | w1v \in T_w\}$$

for $w \in T$ such that $w2 \notin T$. Roughly speaking, $D_i T$ is the tree obtained from T by evacuating the i topmost nodes without a child on its right and belonging between 0 and the rightmost leaf of T. For example, let T be $\{0, 1, 11, 12, 121\}$. Since $w_{T,1} = 0$, $w_{T,2} = 12$ and

$$\{0, 1, 11, 12, 121\} \xrightarrow{\ominus 12} \{0, 1, 11, 12\} \xrightarrow{\ominus 0} \{0, 1, 2\},$$

we have $D_2 T = \{0, 1, 2\}$.

These operators $D(t)$, $U(t')$ and $U'(t')$ satisfy the following equations:

$$D(t)U(t') = \frac{1}{1 - tt'} U(t')D(t),$$
$$D(t)U'(t') = (1 + tt')U'(t')D(t).$$

(See [7] for a proof of the equations.) Hence the generalized Schur polynomials for these operators satisfy the same Pieri's formula as in the case of the classical Young's lattice and its dual construction.

In this case, generalized Schur polynomials are not symmetric in general. For example, since

$$\begin{aligned}
&D(t_1)D(t_2)\{0,1,12\}\\
&\quad = D(t_1)\big(\{0,1,12\}+t_2\{0,2\}+t_2^2\{0\}\big)\\
&\quad = \big(\{0,1,12\}+t_1\{0,2\}+t_1^2\{0\}\big)+t_2\big(\{0,2\}+t_1\{0\}\big)+t_2^2\big(\{0\}+t_1\emptyset\big),
\end{aligned}$$

$s^D_{\{0,1,12\},\emptyset}(t_1,t_2)=t_1t_2^2$ is not symmetric.

We define three kinds of labeling on trees to give generalized Schur polynomials $s_U^{T,\emptyset}(t_1,\dots,t_n)$, $s_{U'}^{T,\emptyset}(t_1,\dots,t_n)$ and $s^D_{T,\emptyset}(t_1,\dots,t_n)$ presentations as generating functions of them.

Definition 4.1. Let T be a tree and m a positive integer. We call a map $\varphi: T\to\{1,\dots,m\}$ a *right-strictly-increasing labeling* if

- $\varphi(w)\le\varphi(v)$ for $w\in T$ and $v\in T_{w1}$ and
- $\varphi(w)<\varphi(v)$ for $w\in T$ and $v\in T_{w2}$.

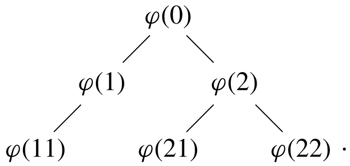
We call a map $\varphi: T\to\{1,\dots,m\}$ a *left-strictly-increasing labeling* if

- $\varphi(w)<\varphi(v)$ for $w\in T$ and $v\in T_{w1}$ and
- $\varphi(w)\le\varphi(v)$ for $w\in T$ and $v\in T_{w2}$.

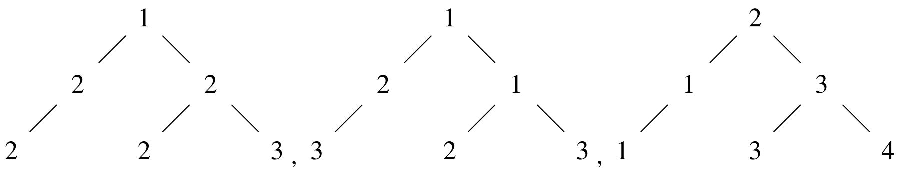
We call a map $\varphi: T\to\{1,\dots,m\}$ a *binary-searching labeling* if

- $\varphi(w)\ge\varphi(v)$ for $w\in T$ and $v\in T_{w1}$ and
- $\varphi(w)<\varphi(v)$ for $w\in T$ and $v\in T_{w2}$.

For example, let $T=\{0,1,2,11,21,22\}$. We write a labeling φ on T as the diagram



In this notation, the labelings



on T are a right-strictly-increasing labeling, a left-strictly-increasing labeling and a binary-searching labeling, respectively.

The inverse image $\varphi^{-1}(\{1,\dots,n+1\})$ of a right-strictly-increasing labeling φ is the tree obtained from the inverse image $\varphi^{-1}(\{1,\dots,n\})$ by adding some nodes right-strictly. Hence we identify right-strictly-increasing labelings with sequences $(\emptyset = T^0, T^1, \ldots, T^m)$ of $m+1$ trees such that T^{i+1} is obtained from T^i by adding some nodes right-strictly for each i.

Similarly, we identify left-strictly-increasing labelings with sequences $(\emptyset = T^0, T^1, \ldots, T^m)$ of $m+1$ trees such that T^{i+1} is obtained from T^i by adding some nodes left-strictly for each i.

For a binary-searching labeling $\varphi_m : T \to \{1, \ldots, m\}$, by the definition of binary-searching labeling, the inverse image $\varphi_m^{-1}(\{m\})$ equals $\{w_{T,1}, \ldots, w_{T,k}\}$ for some k. We can obtain a binary-searching labeling $\varphi_{m-1} : T \ominus \varphi_m^{-1}(\{m\}) \to \{1, \ldots, m-1\}$ from φ_m by evacuating k nodes $\varphi_m^{-1}(\{m\})$ together with their labels. Hence we identify binary-searching labelings with sequences $(\emptyset = T^0, T^1, \ldots, T^m)$ of $m+1$ trees such that $D_{k_i} T^i = T^{i-1}$ for some $k_1, k_2, \ldots, k_m$.

For a labeling φ from T to $\{1, \ldots, m\}$, we define $t^\varphi = \prod_{w \in T} t_{\varphi(w)}$. For a tree T, it follows that

$$s_U^{T,\emptyset}(t_1, \ldots, t_n) = \sum_\varphi t^\varphi,$$

$$s_{U'}^{T,\emptyset}(t_1, \ldots, t_n) = \sum_\phi t^\phi,$$

$$s_{T,\emptyset}^{D}(t_1, \ldots, t_n) = \sum_\psi t^\psi,$$

where the first sum is over all right-strictly-increasing labelings φ on T, the second sum is over all left-strictly-increasing labelings ϕ on T, and the last sum is over all binary-searching labelings ψ on T.

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