Complexes of Graphs with Bounded Matching Size

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Abstract For positive integers k, n, we investigate the simplicial complex $\mathsf{NM}_k(n)$ of all graphs G on vertex set $[n]$ such that every matching in G has size less than k. This complex (along with other associated cell complexes) is found to be homotopy equivalent to a wedge of spheres. The number and dimension of the spheres in the wedge are determined, and (partially conjectural) links to other combinatorially defined complexes are described. In addition we study for positive integers r, s and k the simplicial complex $\mathsf{BNM}_k(r, s)$ of all bipartite graphs G on bipartition $[r] \cup [\bar{s}]$ such that there is no matching of size k in G, and obtain results similar to those obtained for $\mathsf{NM}_k(n)$.

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1 Introduction

A monotone graph property is a collection Δ of (simple, loopless) graphs on a fixed labelled vertex set V such that

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(C) if $G \in \Delta$ and H can be obtained from G by removing an edge then $H \in \Delta$, and
(P) if $G \in \Delta$ and H can be obtained from G by relabeling the vertices then $H \in \Delta$.

Condition (C) allows one to associate to each monotone graph property Δ an abstract simplicial complex (also called Δ) whose k-dimensional faces are (indexed by) those graphs in Δ which have $k + 1$ edges, and condition (P) then says that the natural action of the symmetric group S_V on V determines a simplicial action of S_V on Δ. Various classes of monotone graph properties arise naturally in problems from topology, algebra and combinatorics (see [1, 11, 15, 16] for references).

If $V = V_1 \cup V_2$ is a non-trivial partition of V then one can consider the following variant of condition (P):

(P') each $G \in \Delta$ is bipartite with fixed bipartition $V_1 \cup V_2$. If $G \in \Delta$ and H can be obtained from G by relabeling the vertices within V_1 and within V_2 then $H \in \Delta$.

Graph properties satisfying (C) and (P') admit a natural action of $S_{V_1} \times S_{V_2}$. See [1, 15]) for appearances of such properties in various settings.

Here we study the following two graph properties:

$\rightarrow$ Let n and k be positive integers. The complex $\mathsf{NM}_k(n)$ consists of all graphs on vertex set $[n] := \{1, \ldots, n\}$ that contain no matching of size k.
$\rightarrow$ Let r, s and k be positive integers. The complex $\mathsf{BNM}_k(r, s)$ consist of all bipartite graphs with bipartite vertex set $[r] \cup [\bar{s}] := \{1, \ldots, r, \bar{1}, \ldots, \bar{s}\}$ that do not contain a matching of size k.

These complexes turn out to have very nice topological structure. They are (mysteriously and partially conjecturally) related to that of some other combinatorially defined complexes, as will be described in Theorem 1.12 and Conjecture 1.13 below and the discussion directly preceding them. Moreover, in proving one of our main results, we introduce graphs called trees of triangles (see Sect. 2.2), which were approximately simultaneously found by Etingof, Henriques, Kamnitzer and Rains in their examination of the space of real points in the moduli space of algebraic curves of genus zero with a fixed number of marked points (see [4, Definition 4.12]). It remains to be seen whether there are any close relations between their work and ours.

Our proofs make use of the discrete Morse theory of Forman (see [5]). This theory has been applied several times in the study of monotone graph properties. We recommend [10] as an excellent survey on this topic. However, our results seem to be the first which make use of nonrudimentary results in graph theory. Namely, we use the Gallai–Edmonds structure theorem (see for example [13]).

Our main results are as follows.

Theorem 1.1 *For $n, k \in \mathbb{N}$, let $\Pi_{n-1}^1(k)$ be the set of all partitions τ of $[n-1]$ into $n - 2k + 1$ subsets $\tau_1, \ldots, \tau_{n-2k+1}$ of odd size. Then $\mathsf{NM}_k(n)$ has the homotopy type of a wedge of spheres of dimension $3k - 4$. The number of spheres in this wedge is*

$$\sum_{\tau \in \Pi_{n-1}^1(k)} \left(\prod_{i=1}^{n-2k+1} (\tau_i - 2)!! \right)^2.$$

A special case of this theorem is the following.

Corollary 1.2 *For $k \in \mathbb{N}$, let NPM_{2k} be the complex of all graphs on vertex set $[2k]$ which have no perfect matching. Then*

$$\mathsf{NPM}_{2k} \simeq \bigvee_{(2k-3)!!^2} S^{3k-4}.$$

Calculating the (reduced) Euler characteristic of $\mathsf{NM}_k(n)$ in two ways gives the following enumerative result.

Corollary 1.3 *For $n, k \in \mathbb{N}$, we have*

$$\sum_{G \in \mathsf{NM}_k(n)} (-1)^{|E(G)|} = (-1)^{k-1} \sum_{\tau \in \Pi_{n-1}^k} \left(\prod_{i=1}^{n-2k+1} (\tau_i - 2)!! \right)^2.$$

In particular,

$$\sum_{G \in \mathsf{NPM}_{2k}} (-1)^{|E(G)|} = (-1)^{k-1} (2k-3)!!^2.$$

Theorem 1.4 *For $r, s, k \in \mathbb{N}$, the homotopy type of $\mathsf{BNM}_k(r, s)$ is a wedge of spheres of dimension $2k - 3$. The number of spheres in this wedge is $\binom{r-1}{k-1}\binom{s-1}{k-1}$.*

Calculating the (reduced) Euler characteristic of $\mathsf{BNM}_k(r, s)$ in two ways gives the following enumerative result.

Corollary 1.5 *For $r, s, k \in \mathbb{N}$, we have*

$$\sum_{G \in \mathsf{BNM}_k(r,s)} (-1)^{|E(G)|} = \binom{r-1}{k-1}\binom{s-1}{k-1}.$$

In order to prove Theorem 1.1, we are forced to examine a certain quotient CW-complex. Let $n = 2m - 1$ be odd. A graph G on vertex set $[n]$ is called factor critical if for each vertex v of G, the graph obtained from G by removing v and all edges containing v has a perfect matching. The classification of bipartite factor critical graphs is very simple.

Remark 1.6 A factor critical bipartite graph consists of a single vertex.

The set NFC_n of not factor critical graphs on $[n]$ is a monotone graph property and therefore a subcomplex of the simplex $\Sigma(n)$ whose vertices are the $\binom{n}{2}$ edges

$\binom{[n]}{2} := \{\{i, j\} \mid 1 \leq i < j \leq n\}$ of the complete graph on vertex set $[n]$. The complex FC_n of factor critical graphs is the quotient space $\Sigma(n)/\mathsf{NFC}_n$. This space admits an obvious cell decomposition, as described in Sect. 2.3.

Theorem 1.7 *Let $n = 2m - 1 \in \mathbb{N}$. Then FC_n has the homotopy type of a wedge of $(n - 2)!!^2$ spheres of dimension $3m - 4$.*

We need the following topological fact.

Lemma 1.8 (see Example 0.14 [8]) *If Γ is a nonempty subcomplex of the simplex Σ then Σ/Γ is homotopy equivalent to the suspension of Γ.*

One can prove this by showing that the mapping cone of the identity embedding of Γ into Σ is the union of two contractible subspaces whose intersection is Γ.

Lemma 1.8 allows us to deduce the following result.

Corollary 1.9 *Let $n = 2m - 1 \in \mathbb{N}$. Then NFC_n has the homotopy type of a wedge of $(n - 2)!!^2$ spheres of dimension $3m - 5$.*

Proof Our claim can be confirmed by direct observation when $n < 5$, so assume that $n \geq 5$. We have $\mathsf{FC}_n = \Sigma(n)/\mathsf{NFC}_n$, and it follows from Theorem 1.7 and the following remark that NFC_n has the same homology as the given wedge of spheres. Since no graph on $[n]$ with three edges is factor critical, the complex NFC_n contains the entire 2-skeleton of $\Sigma(n)$ and is therefore simply connected. The corollary now follows from the uniqueness of Moore spaces (see for example [8, p. 368]). $\square$

By Remark 1.6 the only factor critical bipartite graphs are singletons. Thus for our purposes it is useful to replace the concept of factor critical graphs by a new concept in the bipartite setting. We say a bipartite graph with bipartition $X \cup Y$ is q-factor critical if $|X| = q$, $|Y| > q$ and for each $y \in Y$ the graph $G - y$ has a matching of size q. In the sequel we will mostly speak of factor critical graphs since the number q will be clear from the context. The set $\mathsf{NBFC}(q, s)$ of bipartite graphs on vertex set $[q] \cup [\overline{s}]$ which are not factor critical form a subcomplex of the simplex $\Sigma(q, s)$ whose vertices are the edges in the complete bipartite graph on bipartition $[q] \cup [\overline{s}]$. (Here $[\overline{s}] := \{\overline{1}, \ldots, \overline{s}\}$.) The complex $\mathsf{BFC}(q, s)$ of factor critical bipartite graphs on $[q] \cup [\overline{s}]$ is the quotient complex $\Sigma(q, s)/\mathsf{NBFC}(q, s)$. Once again, this complex admits an obvious cell decomposition (see Sect. 2.3). We will prove the following result, from which Theorem 1.4 will follow after some straightforward additional arguments.

Theorem 1.10 *For $1 \leq q < s$ the complex $\mathsf{BFC}(q, s)$ is homotopy equivalent to a wedge of $\binom{s-1}{q}$ spheres of dimension $2q - 1$.*

As in the non-bipartite case we deduce the following corollary.

Corollary 1.11 *For $1 \leq q < s$ the complex $\mathsf{NBFC}(q, s)$ is homotopy equivalent to a wedge of $\binom{s-1}{q}$ spheres of dimension $2q - 2$.*

All of the complexes NPM_n, FC_n and NFC_n have homology concentrated in a single dimension, and in each case the rank of the unique nontrivial homology group is $(n-3)!!^2$ if n is even and $(n-2)!!^2$ if n is odd. Moreover, in each case the action of S_n on the vertex set $[n]$ determines a simplicial (or cellular) action of S_n on the given complex. This action determines a representation of S_n on the nontrivial homology group. Representations of the groups S_n whose degrees are the ranks of these homology groups have arisen previously in work of Calderbank, Hanlon and Robinson ([3]), Hanlon and Wachs ([7]), and Hanlon ([6]). Fix $k \in \mathbb{N}$ and for each $n \in \mathbb{N}$ let $\Pi_n^{1,k}$ be the subposet of the partition lattice Π_n consisting of those nontrivial proper partitions in which each part has size equal to 1 mod k. In [2], Björner showed that the order complex $\Delta\Pi_n^{1,k}$ is Cohen-Macaulay and therefore has a unique nontrivial homology group, which occurs in the top dimension $t := \lfloor \frac{n-2}{k} \rfloor - 1$. In [7], it is shown that character realized by the representation of S_n on $\widetilde{H}_t(\Delta\Pi_n^{1,k})$ is equal to the character $\mathsf{lie}_n^{(k)}$ realized by the action of S_n on a particular subspace of a vector space with a $(k+1)$-ary operation, called a Lie-k-algebra. In [3], the character $\mathsf{lie}_n^{(2)}$ is determined. It is shown that if n is odd then this character has degree $(n-2)!!^2$. In [6], when $n \equiv 2$ mod k a poset $\mathcal{L}_n^{(k)}$ whose elements are trees with n leaves labelled bijectively with $[n]$ and nonleaves having degree equal to 2 mod k (but not equal to 2) is defined. It is shown that this poset is Cohen-Macaulay of dimension t and that the character for the representation of S_n on $\widetilde{H}_t(\Delta\mathcal{L}_n^{(k)})$ is

$$\mathsf{lie}_{n-1}^{(k)} \uparrow_{S_{n-1}}^{S_n} - \mathsf{lie}_n^{(k)}.$$

Here $\uparrow$ denotes induction of representations or characters, and below $\downarrow$ will represent restriction. Moreover, the dimension of $\widetilde{H}_t(\Delta\mathcal{L}_n^{(2)})$ is $(n-3)!!^2$. Let ε_n be the sign character of S_n. Wachs and the second author of this paper have proved the following result

Theorem 1.12 (Shareshian-Wachs) *Let $n = 2k \in \mathbb{N}$ be even. Then*

$$\widetilde{H}_{3k-4}(\mathsf{NPM}_n) \downarrow_{S_{n-1}}^{S_n} \cong_{S_{n-1}} \widetilde{H}_{3k-5}(\mathsf{NFC}_{n-1}) \cong_{S_{n-1}} \widetilde{H}_t(\Delta\Pi_{n-1}^{1,2}) \otimes \varepsilon_n.$$

We conjecture the following stronger result.

Conjecture 1.13 *Let $n = 2k$ be even. Then*

$$\widetilde{H}_{3k-4}(\mathsf{NPM}_n) \cong_{S_n} \widetilde{H}_t(\Delta\mathcal{L}_n^{(2)}) \otimes \varepsilon_n.$$

More generally, it would be interesting to know an answer to the following question.

Question 1.14 *What is the S_n-module structure of $\widetilde{H}_{3k-4}(\mathsf{NM}_k(n))$?*

The complex $\mathsf{BNM}_k(r, s)$ also has homology concentrated in a single dimension. This time the complex is invariant under the natural action of $S_r \times S_s$. Experiments in small cases and the rank of the homology groups given by Theorem 1.4 suggest the following conjecture.

Conjecture 1.15 *As an $(S_r \times S_s)$-module the reduced homology group $\widetilde{H}_{2k-3}(\mathsf{BNM}_k(r, s))$ is isomorphic to the tensor product $H(r, k) \otimes H(s, k)$, where $H(n, l)$ is the irreducible S_n-module corresponding to the hook Young diagram with l rows and $n - l + 1$ columns.*

One can consider the complex $\mathsf{NM}_k(n)$ as the complex of all n-partite graphs on n vertices that do not have a matching of size k. Let $\mathsf{t-NM_k}(\mathsf{s}_1, \ldots, \mathsf{s_t})$ be the complex of t-partite graphs on a t-partition with parts of size $\mathsf{s}_1, \ldots, \mathsf{s}_t$ with no matching of size k. Computational evidence indicates that in general this complex has homology concentrated in a single dimension.

Question 1.16 *What is the homotopy type of $\mathsf{t-NM_k}(\mathsf{s}_1, \ldots, \mathsf{s_t})$? Is it homotopic to a wedge of spheres, all of the same dimension?*

The remainder of this paper is organized as follows. In Sect. 2.1 we introduce our graph theoretic notation and discuss the Gallai–Edmonds structure theorem. In Sect. 2.2 we introduce certain graphs, called forests of triangles, which will play a prominent role in our arguments. Namely, these graphs will represent critical cells of discrete Morse functions on some of our complexes. Discrete Morse theory is discussed briefly in Sect. 2.3. In Sect. 3 we give the proofs of Theorems 1.7 and 1.10. Then in Sect. 4 we deduce Theorems 1.1 and 1.4 from these theorems.

2 Preliminaries

2.1 Graph theory

By a graph we always mean $G = (V(G), E(G))$ where $V = V(G)$ is a finite vertex set and $E = E(G) \subseteq \binom{V}{2}$ is the edge set of G. An edge $\{x, y\} \in E$ will often be denoted by xy. For $X \subseteq V$, the subgraph of G induced on X will be denoted by $G|_X$, so $G|_X = (X, E \cap \binom{X}{2})$. For $v \in V$, $G - v$ will denote $G|_{V-\{v\}}$. For distinct $v, w \in V$, $G + vw$ will denote the graph $(V, E \cup \{vw\})$ obtained by adding vw to $E(G)$ and $G - vw$ will denote the graph $(V, E \setminus \{vw\})$ obtained by removing vw from $E(G)$ (so $G + vw = G$ if $vw \in E$ and $G - vw = G$ if $vw \notin E$). A matching in G is a subset M of $E(G)$ such that each $v \in V$ is contained in at most one $e \in M$. We say a matching M covers $v \in V$ if some $e \in M$ contains v. A matching M is perfect if every $v \in V$ is covered by M. Thus we can reformulate the definition of being factor critical from Sect. 1 as follows. If $|V|$ is odd, we call G factor critical if for each $v \in V$, $G - v$ contains a perfect matching. A maximum matching in G is one which contains at least as many edges as every other matching in G. The size of each maximum matching of G will be denoted by $\nu(G)$. For $v \in V(G)$, the neighborhood $N_G(v)$ is the set of all $w \in V(G)$ such that $vw \in E(G)$.

For a graph $G = (V, E)$, set

- $D = D(G) := \{v \in V : \text{Some maximum matching of } G \text{ does not cover } v\}$,
- $A = A(G) := \{v \in V - D \mid N_G(v) \cap D \neq \emptyset\}$, and
- $C = C(G) := V \setminus (A \cup D)$.

Theorem 2.1 (Gallai–Edmonds structure theorem, see [13]) *For every graph G, the following conditions hold.*

(1) *Each connected component of $G|_D$ is factor critical.*
(2) *Each maximum matching of G consists of*

- *a perfect matching on $G|_C$,*
- *for each $a \in A$ an edge ad_a, such that $d_a \in D$ and for distinct $a, b \in A$, d_a and d_b are in distinct connected components of $G|_D$, and*
- *a matching of size $\frac{|V(X)|-1}{2}$ on each connected component X of $G|_D$.*

(3) *Let $\mathrm{con}(G)$ be the number of connected components of $G|_D$. The number of vertices of G not covered by a given maximum matching is $\mathrm{con}(G) - |A|$. In other words,*

$$\nu(G) = \frac{1}{2}\left(|V| - \mathrm{con}(G) + |A|\right).$$

2.2 Trees of triangles

Definition 2.2 Let $V = \{v_1, v_2, \ldots, v_n\} \subseteq \mathbb{N}$ with $v_i < v_{i+1}$ for all $i < n$. A connected graph G on vertex set V is a tree of triangles if either $|V| = 1$ or

- $v_1 v_2 \in E(G)$,
- there exists a unique $m \in \{3, \ldots, n\}$ such that $v_1 v_m \in E(G)$ and $v_2 v_m \in E(G)$, and
- the graph obtained from G by removing edges $v_1 v_2, v_1 v_m, v_2 v_m$ has three connected components, each of which is a tree of triangles.

A forest of triangles on V is a graph F on V such that each connected component of F is a tree of triangles on its vertex set.

Induction on n shows that if there is a tree of triangles with n vertices then n is odd. For odd $n \in \mathbb{N}$ we write $n!!$ for the product of all odd $j \in \mathbb{N}$ such that $j \leq n$. By convention, $(-1)!! = 1$.

Proposition 2.3 Let $V \subseteq \mathbb{N}$ with $|V| = 2k - 1$ for some $k \in \mathbb{N}$. Then the number $\mathsf{TT}(|V|)$ of trees of triangles on vertex set V is $(2k-3)!!^2$. Each tree of triangles on V is factor critical and has $3k - 3$ edges.

Proof We proceed by induction on k, the case $k = 1$ being trivial. Assume $k > 1$. We may assume that $V = [2k - 1]$. Note that

$$(2k-3)!! = \frac{(2k-2)!}{(k-1)!2^{k-1}}. \tag{1}$$

In each tree of triangles G on vertex set V, there is a unique $v \in \{3, \ldots, 2k-1\}$ such that $12, 1v, 2v \in E(G)$. We may choose v in $2k-3$ ways, and having chosen v, the removal of edges $12, 1v, 2v$ leaves a graph with three connected components D_1, D_2, D_v, such that $i \in D_i$ and the subgraph of G induced on each D_i is a tree of triangles. It now follows that $|E(G)| = 3k - 3$ as claimed, and that if we let $\Pi^0(v)$ be the set of all ordered partitions of $\{3, \ldots, 2k-1\} \setminus \{v\}$ into three parts of even size, then we have

$$\mathsf{TT}(|V|) = (2k-3) \sum_{(I,J,L) \in \Pi^0(v)} \mathsf{TT}(|I|+1)\mathsf{TT}(|J|+1)\mathsf{TT}(|L|+1)$$

$$= (2k-3)$$

$$\times \sum_{i+j+l=k-2} \binom{2k-4}{2i, 2j, 2l} \mathsf{TT}(2i+1)\mathsf{TT}(2j+1)\mathsf{TT}(2l+1)$$

$$= \frac{(2k-3)!}{2^{2k-4}} \sum_{i+j+l=k-2} \binom{2i}{i}\binom{2j}{j}\binom{2l}{l},$$

the last equality following from the inductive hypothesis and equation (1). Our formula for the number of trees of triangles now follows from the facts (easily verified using Taylor's theorem) that

$$(1-4x)^{-1/2} = \sum_{i \geq 0} \binom{2i}{i} x^i$$

and

$$(1-4x)^{-3/2} = \sum_{j \geq 0} \binom{2j}{j}(2j+1)x^j.$$

It remains to show that every tree of triangles G is factor critical. For any $x \in V$, take $i \in \{1, 2, v\}$ with $x \in D_i$ (where the D_i are as defined above). Then $D_i - x$ has a perfect matching by inductive hypothesis, as do both $D_j - j$ for $j \neq i$. A perfect matching on $G - x$ is obtained by taking these three perfect matchings and the edge jk, where $\{i, j, k\} = \{1, 2, v\}$. $\qquad \square$

2.3 Discrete Morse theory

Here we give a brief summary of some results from discrete Morse theory for regular cell complexes as developed by Forman (see [5]). We will only consider cell complexes which are obtained by starting with a simplicial complex Δ and taking its quotient by a (possibly empty) subcomplex Γ. A simplicial complex Δ is realized as a CW-complex with one k-cell for each of its k-dimensional faces and gluing maps determined by its face structure. Note that we do not distinguish between an (abstract) simplicial complex and its geometric realization.

Let Γ be any simplicial subcomplex of Δ. If Γ is empty then we define $\Delta / \Gamma = \Delta$. Otherwise, Δ / Γ is a cell complex with one point p_0 and one k-cell σ for each face

σ of Δ which is not contained in Γ. If σ, τ are two such faces, then any portion of the boundary $\partial\sigma$ of σ which was glued to τ in Δ remains glued to τ in the same manner, while any portion of $\partial\sigma$ which was glued in Δ to a face of Γ is now glued to p_0.

In order to formulate discrete Morse theory we need to introduce some concepts from the theory of directed graphs. A directed graph $\mathcal{D} = (V, \mathcal{A})$ on vertex set V is determined by its set of arcs (i.e. directed edges) $\mathcal{A} \subseteq V \times V$. If $W \subseteq V$ then we denote by $\mathcal{D}|_W = (W, \mathcal{A} \cap W \times W)$ the digraph induced by $\mathcal{D}$ on W. Let $\mathcal{D} = (V, \mathcal{A})$ be any directed graph (digraph) which has no directed cycle, in particular $\mathcal{D}$ has no loops. A Morse matching on $\mathcal{D}$ is a subset $\mathcal{M}$ of $\mathcal{A}$ such that

(M1) each vertex of $\mathcal{D}$ is the head or tail of at most one arc in $\mathcal{M}$, and
(M2) the digraph $\mathcal{D}_{\mathcal{M}}$ obtained from $\mathcal{D}$ by reversing the direction of each arc in $\mathcal{M}$ has no directed cycle.

A critical cell of a Morse matching $\mathcal{M}$ on $\mathcal{D}$ is a vertex of $\mathcal{D}$ which is not the head or tail of an arc in $\mathcal{M}$.

If P is a partially ordered set (poset), $\mathcal{D}(\mathsf{P}) = (V(\mathsf{P}), \mathcal{A}(\mathsf{P}))$ is the digraph obtained by directing each edge in the Hasse diagram of P downwards, that is, the arcs in $\mathcal{A}(\mathsf{P})$ are pairs (x, y) where x covers y in P. If Δ is a simplicial complex, let $\mathcal{D} = \mathcal{D}(\Delta)$ be the directed graph with one vertex for each face of Δ (including the empty face) and an arc (σ, τ) (directed from σ to τ) whenever τ is a codimension one face of σ. That is, $\mathcal{D}(\Delta) = \mathcal{D}(\mathsf{P}\Delta)$, where $\mathsf{P}\Delta$ is the poset of faces of Δ. The following proposition is a version of Forman's main result from [5] formulated for quotient complexes. A direct proof of the proposition can be found in [12].

Proposition 2.4 *Let Δ be a simplicial complex and let Γ be a subcomplex of Δ. Let $\mathcal{M}$ be a Morse matching on $\mathcal{D}(\Delta)$ such that every face of Γ is a critical cell of $\mathcal{M}$. If Γ is the empty complex, assume that the empty face is not a critical cell of $\mathcal{M}$. Then the quotient complex Δ / Γ has the homotopy type of a CW-complex with one vertex p along with one k-cell for each k-dimensional critical cell of $\mathcal{M}$ that does not lie in Γ. In particular, if the critical cells of $\mathcal{M}$ all have the same dimension k then the given complex has the homotopy type of a wedge of k-dimensional spheres, one sphere for each critical cell.*

The previous theorem is not stated explicitly in the work of Forman, but it is an immediate consequence of Forman's theory and the fact that the incidence of two adjacent cells in Δ / Γ none of which is the distinguished cell p_0 is regular.

The next two elementary results are useful in confirming that a set $\mathcal{M}$ of arcs is a Morse matching. For the formulation we adopt the following standard notation. A subset $S \subseteq \mathsf{P}$ of a poset P is called convex if $x \leq y \leq z$ and $x, z \in S$ implies $y \in S$.

Lemma 2.5 (Cluster Lemma—[9, Lemma 2]) *Let $\mathsf{P}_1, \dots, \mathsf{P}_r$ be pairwise disjoint, order convex subposets of P. For each $i \in [r]$, let $\mathcal{M}^i$ be a Morse matching on $D(\mathsf{P}_i)$. Define a relation on the P_i by $\mathsf{P}_i \leq_c \mathsf{P}_j$ if there exist $x \in \mathsf{P}_i$ and $y \in \mathsf{P}_j$ such that $x \leq y$. Assume that the P_i satisfy the condition*

($\mathcal{P}$) *The relation $\leq_c$ defines a partial order on the P_i's.*

Then

$$\mathcal{M} := \bigcup_{i=1}^{r} \mathcal{M}^i$$

is a Morse matching on $\mathcal{D}(\mathsf{P})$.

Lemma 2.6 (Cycle Lemma—[14, Proposition 3.1]) *Let* P *be an order convex sub-poset of the face poset of a simplicial complex* Σ *and assume that* $\mathcal{M} \subseteq \mathcal{A}(\mathsf{P})$ *satisfies condition* (M1). *Then every directed cycle in* $\mathcal{D}_{\mathcal{M}}(\mathsf{P})$ *is of the form* $\sigma_1, \tau_1, \sigma_2, \tau_2, \ldots, \sigma_{r-1}, \tau_{r-1}, \sigma_r = \sigma_1$, *where*

(1) $r \geq 3$,
(2) *for each* $i \in [r-1]$, *there is some* $x_i \in \tau_i$ *such that* $\tau_i = \sigma_i \cup \{x_i\}$ *and* $(\tau_i, \sigma_i) \in \mathcal{M}$,
(3) *for each* $i \in [r-1]$, *there is some* $y_i \in \tau_i$ *such that* $\sigma_{i+1} = \tau_i \setminus \{y_i\}$, *and*
(4) *the multisets* $\{x_i \mid i \in [r]\}$ *and* $\{y_i \mid i \in [r]\}$ *are equal.*

3 Proofs of Theorems 1.7, 1.10

3.1 The complex of factor critical graphs

Here we prove Theorem 1.7, which follows immediately from Proposition 2.3, Proposition 2.4 and the following result. Recall that $\Sigma(n)$ is the simplex on vertex set $\binom{[n]}{2}$.

Lemma 3.1 *Let* $n \in \mathbb{N}$ *be odd. Then there exists a Morse matching* $\mathcal{M}$ *in* $\mathcal{D}(\Sigma(n))$ *whose critical cells are the trees of triangles and the not-factor-critical graphs on vertex set* $[n]$.

As mentioned in the introduction, the use of the Gallai–Edmonds structure theorem as a fundamental ingredient in our proof of Lemma 3.1 distinguishes this proof from those of previous results which use discrete Morse theory in examining monotone graph properties. The rest of the proof is similar in form and spirit to many of the previous proofs, see [10]. We will not give as many details as were given in these proofs. In particular, at several junctures we will leave it to the reader to confirm that we have used Cluster Lemma 2.5 appropriately or that one can use Cycle Lemma 2.6 to show that a given matching is actually a Morse matching.

Proof We prove Lemma 3.1 by induction on n, the case $n = 1$ being trivial. So, assume $n > 1$. We will construct our Morse matching $\mathcal{M}$ in several steps.

Step 0: We begin with $\mathcal{M}$ empty and then add to $\mathcal{M}$ the set $\mathcal{M}^0$ of all arcs $(G + 12, G)$ where G is factor critical but $12 \notin E(G)$. Using Lemma 2.6, it is easy to confirm that these arcs form a matching and that their reversal leaves a digraph without directed cycles. Certainly every graph which is covered by an arc in $\mathcal{M}^0$ is factor critical. The factor critical graphs which are not covered by any arc in $\mathcal{M}^0$ are

those factor critical graphs G such that $12 \in E(G)$ and $G - 12$ is not factor critical. Let $\mathcal{C}^0$ be the set of all such graphs.

In the following we collect properties of graphs in $\mathcal{C}^0$. Let $G \in \mathcal{C}^0$ (so G is factor critical but $G - 12$ is not). Set

$$G' := G - 12.$$

We consider the Gallai–Edmonds decomposition of G'. Since G is factor critical, we see that $G - 1 = G' - 1$ and $G - 2 = G' - 2$ both have perfect matchings. This gives

$$\nu(G') = \frac{n-1}{2} \tag{2}$$

and

$$1, 2 \in D(G'). \tag{3}$$

Claim 1 The vertices 1 and 2 lie in different connected components of $G'|_{D(G')}$.

◁ *Proof of Claim 1* Assume for contradiction that $1, 2$ lie in the same component $X = (V(X), E(X))$ of $G'|_{D(G')}$. Since G' is not factor critical it follows that there is a vertex v such that $G' - v$ does not have a perfect matching. Since $\nu(G') = \frac{n-1}{2}$, we see that v is not in $D(G')$. Consequently, $v \in A(G') \cup C(G')$. For any such v, there is a perfect matching K in $G - v$, and since $v \notin D(G')$, this matching K contains 12. By Theorem 2.1(1), there are oddly many elements of $V(X) \setminus \{1, 2\}$ and since all of them are contained in an edge from K it follows that there exist $x \in V(X)$ and $a \in A(G')$ such that $xa \in K$. By Theorem 2.1(3) and $\nu(G') = \frac{n-1}{2}$ it follows that $\mathrm{con}(G') = |A(G')| + 1$. But now there are $\mathrm{con}(G') - 1$ components of $G'|_{D(G')}$ other than X, each of which have odd size and $|A(G')| - 1 < \mathrm{con}(G') - 1$ elements of $A(G') \setminus \{a\}$ remaining to pair with elements of $D(G') \setminus V(X)$ in K. Thus by Theorem 2.1(2) K cannot be a perfect matching, giving the desired contradiction. ▷

Claim 2 $A(G') \neq \emptyset$.

◁ *Proof of Claim 2* By (2) and Theorem 2.1(3) it follows that $\mathrm{con}(G') = 1 + |A(G')|$. From Claim 1 we know that $\mathrm{con}(G') \geq 2$ which implies the assertion. ▷

Step 1: The set of graphs in $\mathcal{C}^0$ forms an ideal in the poset of factor critical graphs, so we can apply Cluster Lemma 2.5 after describing a Morse matching on the subgraph $\mathcal{D}^0$ of $\mathcal{D}(\Sigma(n))$ induced on $\mathcal{C}^0$.

For all $G \in \mathcal{C}^0$ such that $|A(G')| > 1$, let $a(G'), b(G')$ be the two smallest elements of $A(G')$. Define the set $\mathcal{M}^1$ of arcs in $\mathcal{D}^0$ by

$$\mathcal{M}^1 := \{(G + a(G')b(G'), G) \mid G \in \mathcal{C}^0, a(G')b(G') \notin E(G)\}.$$

We will see that $\mathcal{M}^1$ is a Morse matching on $\mathcal{D}^0$. First note that for any graph H and any distinct $a, b \in A(H)$, the graphs $H + ab$ and $H - ab$ have the same Gallai–Edmonds decomposition, since (by Theorem 2.1(2)) no maximum matching in H

uses an edge with both endpoints in $A(H)$. It follows that no vertex of $\mathcal{D}^0$ is a head or tail of more than one arc in $\mathcal{M}^1$, and it remains to show that $\mathcal{D}^0_{\mathcal{M}^1}$ has no directed cycle. Assume for contradiction that such a directed cycle exists. By Cycle Lemma 2.6, this cycle has vertices

$$G_1, H_1, \ldots, G_{r-1}, H_{r-1}, G_r = G_1,$$

where

- $r \geq 3$,
- $H_i = G_i - xy$ for some $xy \in E(G_i)$, and
- $G_{i+1} = H_i + a(H_i')b(H_i')$.

As noted above, H_i' and G_{i+1}' have the same Gallai–Edmonds decomposition for all i, in particular $D(G_{i+1}') = D(H_i')$. Moreover, since all the G_i and H_i lie in $\mathcal{C}^0$, we have

$$\nu(H_i') = \nu(G_i') = \frac{n-1}{2}$$

for all i. Thus any maximum matching of H_i' is a maximum matching of G_i'. It follows that $D(H_i') \subseteq D(G_i')$ for all i. Since $G_r = G_1$, we must have

$$D(H_i') = D(G_i') = D(G_1')$$

for all i.

Claim 3 $A(H_i') = A(G_i')$ for all i.

Claim 4 From Claim 3 the desired contradiction follows.

◁ *Proof of Claim 4:* Indeed, from the validity of Claim 3 it follows that $a := a(H_1') = a(G_1')$ and $b := b(H_1') = b(G_1')$. If $H_1' = G_1' - ab$ then $G_2' = G_1'$, a contradiction. If $ab \in E(H_1')$ then there is no arc (K, H_1) in $\mathcal{M}^1$, again a contradiction. Otherwise, we have $G_2' = H_1' + ab$. We may not have $H_2' = G_2' - ab = H_1'$ (a contradiction), so $ab \in H_2'$. Since H_1' and G_2' have the same Gallai–Edmonds decomposition we get $A(G_2') = A(H_1') = A(G_2') = A(H_2')$ by Claim 3. Thus there is no arc (K, H_2) in $\mathcal{M}^1$, a contradiction. ▷

◁ *Proof of Claim 3:* Since $\nu(H_i') = \nu(G_i')$ it follows from Theorem 2.1(3) that $\mathrm{con}(H_i') - |A(H_i')| = \mathrm{con}(G_i') - |A(G_i')|$. Since $D(H_i') = D(G_i')$ and $H_i' = G_i' - xy$, we either have $\mathrm{con}(H_i') = \mathrm{con}(G_i')$ or $\mathrm{con}(H_i') = \mathrm{con}(G_i') + 1$. In the first case $|A(G_i')| = |A(H_i')|$. In the second case xy must be an edge whose removal disconnects a connected component of $G'|_{D(G')}$ and $|A(G_i')| = |A(H_i')| + 1$. But again by $D(G_i') = D(H_i')$ this implies that xy is an edge which connects an element of $D(G_i')$ with an element of $A(G_i')$, a contradiction. Thus we may assume $|A(G_i')| = |A(H_i')|$ and it is sufficient to show that $A(H_i') \subseteq A(G_i')$. This follows since $D(H_i') = D(G_i')$ and $G_i' = H_i' + xy$. ▷

This completes the proof that $\mathcal{M}^1$ is a Morse matching on $\mathcal{D}^0$. As noted above, Cluster Lemma 2.5 guarantees that $\mathcal{M}^0 \cup \mathcal{M}^1$ is a Morse matching.

Step 2: Let $\mathcal{C}^1 \subseteq \mathcal{D}^0$ be the set of critical cells of $\mathcal{M}^0 \cup \mathcal{M}^1$.

Claim 5 $G \in \mathcal{C}^1$ if and only if

(a) G is factor critical,
(b) $|A(G')| = 1$, and
(c) $G'|_{D(G')}$ has exactly two connected components $D_1(G'), D_2(G')$ for which $i \in D_i(G')$, $i = 1, 2$ holds.

◁ *Proof of Claim 5:* Since all critical cells of $\mathcal{C}^0$ are factor critical the same holds for $\mathcal{C}^1 \subseteq \mathcal{C}^0$. From Claim 2 we know that $|A(G')| \geq 1$ for all $G \in \mathcal{C}^1 \subseteq \mathcal{C}^0$. Recall that the Gallai–Edmonds decomposition is the same for G' and $G' \pm a(G')b(G')$. So exactly those $G \in \mathcal{C}^0$ with $|A(G')| \geq 2$ are covered by an edge from $\mathcal{M}^1$. Thus (b) holds for all $G \in \mathcal{C}^1$. By Claim 1 the vertices 1 and 2 lie in different connected components $D_1(G')$ and $D_2(G')$ of $G'|_{D(G')}$. By (2) and Theorem 2.1(3) it follows that $\mathrm{con}(G) = 1 + |A(G')|$. From (b) we know that $|A(G')| = 1$ and hence $\mathrm{con}(G') = 2$. Thus $D_1(G')$ and $D_2(G')$ are the only connected components of $G'|_{D(G')}$.

We have shown that every element of $G \in \mathcal{C}^1$ satisfies (a)-(c). Conversely it is easily checked that no graph satisfying (a)-(c) is covered by an edge in $\mathcal{M}^0 \cup \mathcal{M}^1$. ▷

For each $a \in [n] - \{1, 2\}$ and each ordered partition (X, Y, Z) of $[n] \setminus \{1, 2, a\}$ into three possibly empty subsets we define the subset $\mathcal{C}^1[a, (X, Y, Z)] \subseteq \mathcal{C}^1$ by $G \in \mathcal{C}^1[a, (X, Y, Z)]$ if and only if

- $G \in \mathcal{C}^1$,
- $A(G') = \{a\}$,
- $V(D_1(G')) = X \cup \{1\}$,
- $V(D_2(G')) = Y \cup \{2\}$, and
- $C(G') = Z$.

It follows from Claim 5 that the $\mathcal{C}^1[(a, X, Y, Z)]$ actually partition $\mathcal{C}^1$. The Cluster Lemma 2.5 applies to this partition, and we will define a Morse matching $\mathcal{M}[a, (X, Y, Z)]$ on each $\mathcal{C}^1[a, (X, Y, Z)]$.

Fix $a, (X, Y, Z)$ and let $\mathcal{C} = \mathcal{C}^1[a, (X, Y, Z)]$. We will show that there is a Morse matching $\mathcal{M}[a, (X, Y, Z)]$ on the subgraph $\mathcal{D}|_{\mathcal{C}}$ of $\mathcal{D}^0$ induced on $\mathcal{C}$ whose critical cells are exactly those trees of triangles G such that

- $12, 1a, 2a \in E(G)$ and
- the connected components of $G - \{12, 1a, 2a\}$ are $X \cup \{1\}$, $Y \cup \{2\}$ and $Z \cup \{a\}$.

Once this is done, the proof of our lemma is completed by applying Cluster Lemma 2.5 to $\mathcal{M}^0$, $\mathcal{M}^1$ and all of the $\mathcal{M}[a, (X, Y, Z)]$. Set

$$\mathcal{M}(1) := \{(G + 1a, G) : G \in \mathcal{C}, 1a \notin E(G)\}.$$

It is straightforward to show that $\mathcal{M}(1)$ is a Morse matching on $\mathcal{D}|_{\mathcal{C}}$ whose critical cells are those $G \in \mathcal{C}$ such that $N_G(a) \cap X = \emptyset$. Let $\mathcal{C}(1)$ be the set of all such G, and

define

$$\mathcal{M}(2) := \{(G + 2a, G) : G \in \mathcal{C}(1), 2a \notin E(G)\}.$$

It is again straightforward to show (using Cluster Lemma 2.5) that $\mathcal{M}(1) \cup \mathcal{M}(2)$ is a Morse matching on $\mathcal{D}|_{\mathcal{C}}$ whose critical cells are those $G \in \mathcal{C}(1)$ such that $N_G(a) \cap Y = \emptyset$. Let $\mathcal{C}(2)$ be the set of all such critical cells.

We claim now that if $G \in \mathcal{C}(2)$ then $G|_{Z \cup \{a\}}$ is factor critical. In other words, $\mathcal{C}(2)$ consists of all graphs $G \in \mathcal{C}$ such that

- $12, 1a, 2a \in E(G)$,
- $G - \{12, 1a, 2a\}$ has three connected components $D_1 = X \cup \{1\}$, $D_2 = Y \cup \{2\}$, $D_a = Z \cup \{a\}$, and
- the subgraph of G induced on each of these three components is factor critical.

If this claim holds then we can use the inductive hypothesis (and Cluster Lemma 2.5 one more time) to produce the desired Morse matching on $\mathcal{C}$ and our lemma follows.

So, let $G \in \mathcal{C}(2)$. Note that since $G|_Z = G|_{C(G')}$ contains a perfect matching by definition, it remains to show that if $z \in Z$ then the subgraph of G induced on $(Z \setminus \{z\}) \cup \{a\}$ contains a perfect matching. We know that $G - z$ contains a perfect matching but that $G' - z$ contains no perfect matching. Therefore, any perfect matching K in $G - z$ includes the edge 12. Since both $|X|$ and $|Y|$ are even and the connected components of $G'|_{D(G')}$ are $D_1(G')$ and $D_2(G')$, K cannot contain any edge av with $v \in D(G')$. Thus K consists of 12, a perfect matching on X, a perfect matching on Y and the desired perfect matching on $(Z \setminus \{z\}) \cup \{a\}$ and we are done. □

3.2 The complex of q-factor critical bipartite graphs

The following lemma immediately implies Theorem 1.10.

Lemma 3.2 *For $0 \leq q < s$, there is a Morse matching $\mathcal{M}$ on the digraph $D(\Sigma(q, s))$ whose critical cells are the elements of $\mathsf{NBFC}(q, s)$ along with $\binom{s-1}{q}$ cells of dimension $2q - 1$ from $\mathsf{BFC}(q, s)$.*

Proof We proceed by induction on q, the case $q = 0$ being trivial. Assume $q > 0$. As in the proof of Lemma 3.1 we construct our Morse function in several steps.

Step 0: We begin with $\mathcal{M}$ being empty and then add to $\mathcal{M}$ the set $\mathcal{M}^0$ of all arcs $(G + 1\bar{1}, G)$ where G is q-factor critical and $1\bar{1} \notin E(G)$. Then $\mathcal{M}^0$ is a matching, the digraph $D_{\mathcal{M}^0}$ contains no directed cycle and the set $\mathcal{C}^0$ of graphs not covered by $\mathcal{M}^0$ consists of $\mathsf{NBFC}(q, s)$ along with the elements $G \in \mathsf{BFC}(q, s)$ such that $1\bar{1} \in E(G)$ and $G - 1\bar{1} \notin \mathsf{BFC}(q, s)$.

For $G \in \mathcal{C}^0 \cap \mathsf{BFC}(q, s)$, set $G' := G - 1\bar{1}$ and consider the Gallai–Edmonds decompositions of the vertex set of G' into A', C' and D'. We prove the following claims:

Claim 1 $\bar{1} \in D'$ and $\nu(G') = q$.

◁ *Proof* Since $G \in \mathsf{BFC}(q, s)$, we have $\nu(G - \overline{1}) = q$. The assertion follows from $G - \overline{1} = G' - \overline{1}$. ▷

Claim 2 $D' \subseteq [\overline{s}]$ and therefore $A' \subseteq [q]$.

◁ *Proof* Since by Claim 1 we have $\nu(G') = q$ and G' is bipartite, we must have $D' \subseteq [\overline{s}]$. The second part of the claim then follows again from the fact that G' is bipartite. ▷

Claim 3 $1 \in C'$.

◁ *Proof* By Claim 2 it suffices to show that $1 \notin A'$. Assume for contradiction that $1 \in A'$. For each $y \in D'$ we have $N_{G'}(y) \subseteq A'$ and the assumption $1 \in A'$ implies $N_G(y) \subseteq A'$. Since $\nu(G') = q$ by Claim 1 and $G' \notin \mathsf{BFC}(q, s)$, we see that $[\overline{s}] \neq D'$. Thus there is some $\overline{x} \in [\overline{s}] \cap C'$. Let $p = |[\overline{s}] \cap C'|$. Note that $p = |[q] \cap C'|$, since C' contains a perfect matching. From $\nu(G') = q$ we infer $|A'| = q - p$ and $|D'| = s - p$. Let K be a matching of size q in $G - \overline{x}$. (K exists since $G \in \mathsf{BFC}(q, s)$.) At most $s - q$ elements of D' are not covered by any edge of K. Since D' has $s - p$ elements, all in $[\overline{s}]$, there must be at least $q - p$ edges of K with one endpoint in D' and one endpoint in A'. Since $|A'| = q - p$, every element of A' is the endpoint of one of the edges just mentioned. This means that there is no edge in K with one endpoint in C' and one in A', and certainly there is no edge in K with one endpoint in C' and one in D'. Thus the remaining p edges in K have both endpoints in $C' \setminus \{\overline{x}\}$. But this is impossible since $|(C' \setminus \{\overline{x}\}) \cap [\overline{s}]| < p$. ▷

Step 1: Set $\mathcal{B}^0 := \{G \in \mathcal{C}^0 \cap \mathsf{BFC}(q, s) \mid A' \neq \emptyset\}$, carrying forward the definitions of A', C' and D' from **Step 0**. For $G \in \mathcal{B}^0$, let a (resp. $\overline{c}$) be the smallest elements of A' (resp. $C' \cap [\overline{s}]$). Note that by Claim 3 we have $a \neq 1$. Define $\mathcal{M}^1 := \{(G, G - a\overline{c}) \mid G \in \mathcal{B}^0, a\overline{c} \in E(G)\}$.

Claim 4 For $G \in \mathcal{B}^0$ such that $a\overline{c} \in E(G)$ the graphs G' and $G' - a\overline{c}$ have the same Gallai–Edmonds decomposition.

◁ *Proof* By Theorem 2.1(2) the edge $a\overline{c}$ is not included in any maximum matching of G'. ▷

It follows immediately from Claim 4 that $\mathcal{M}^1$ is a matching.

Claim 5 For each $G \in \mathcal{B}^0$, we have $G - a\overline{c} \in \mathcal{B}^0$.

◁ *Proof* By Claim 4, it suffices to show that $G - a\overline{c} \in \mathcal{C}^0 \cap \mathsf{BFC}(q, s)$. Since $G \in \mathcal{C}^0$, it suffices to show that $G - a\overline{c} \in \mathsf{BFC}(q, s)$. Let $\overline{x} \in [\overline{s}]$. If $\overline{x} \in D'$ then there is a matching of size q in $G' - \overline{x}$ which does not contain $a\overline{c}$ by Theorem 2.1. Say $\overline{x} \in C'$. There is a matching K of size q in $G - \overline{x}$, and every such matching contains the edge $1\overline{1}$. As above, let $p = |[\overline{s}] \cap C'| = |[q] \cap C'|$, so $|A'| = q - p$ and $|D'| = s - p$. Let k be the number of edges of K which contain an element y of A' and an element $\overline{z}$ of C'. Note that since $y \in [q]$, we must have $\overline{z} \in [\overline{s}]$.

The only edge in G between $C' \cap [q]$ and D' is $1\overline{1}$. Thus $p - 1 - (p - k - 1) = k$ elements of $C' \cap [q]$ are not covered by K. Hence $k = 0$ and in particular, $a\overline{c} \notin K$ as desired. $\triangleright$

Step 2: Let $\mathcal{C}^1$ be the set of critical points of $\mathcal{M}^0 \cup \mathcal{M}^1$. Then $\mathcal{C}^1$ consists of $\mathsf{NBFC}(q, s)$ and those $G \in \mathsf{BFC}(q, s)$ such that $G' \notin \mathsf{BFC}(q, s)$ and $A' = \emptyset$. By Theorem 2.1(3) it then follows that $|D'| = s - q$.

Note that $\overline{1} \in D'$. For each $X \subseteq [\overline{s}] - \{\overline{1}\}$ such that $|X| = s - q - 1$, set $\mathcal{C}^1[X] :=$ $\{G \in \mathcal{C}^1 \cap \mathsf{BFC}(q, s) \mid D' = X \cup \{\overline{1}\}\}$. Note that there are $\binom{s-1}{s-q-1} = \binom{s-1}{q}$ choices for X. We can then apply the Cluster Lemma 2.5 to the decomposition of $\mathcal{C}^1$ into $\mathsf{NBFC}(q, s)$ and the sets $\mathcal{C}^1[X]$.

Fix one such X. Let $G \in \mathcal{C}^1[X]$. Then the $s - q$ elements of D' are by Remark 1.6 isolated in G'.

Claim 6 $G|_{C'} - 1$ is $(q - 1)$-factor critical.

$\triangleleft$ *Proof* The claim follows from the fact that, if $\overline{x} \in C' \cap [\overline{s}]$ then any matching of size q in $G - \overline{x}$ consists of the edge $1\overline{1}$ along with $q - 1$ edges in $C' - \overline{x}$. $\triangleright$

Write $N_{C'}(1)$ for $N_G(1) \cap C'$

Claim 7 $N_{C'}(1) \neq \emptyset$.

$\triangleleft$ *Proof* Otherwise $\nu(G - \overline{1}) = q - 1$. $\triangleright$

Step 3: For fixed X and $G \in \mathcal{C}^1[X]$, let $\overline{c} = \overline{c}(G)$ be the smallest element of $C' \cap [\overline{s}]$. Define

$$\mathcal{M}^2[X] := \{(G + 1\overline{c}, G) : G \in \mathcal{C}^1[X], \overline{c}(G) \notin N_G(1)\}.$$

It is straightforward to show that $\mathcal{M}^2[X]$ is a Morse matching on $\mathcal{D}(\Sigma(q, s))|_{\mathcal{C}^1[X]}$. Let $\mathcal{C}^2[X]$ be the set of critical points of $\mathcal{M}^2[X]$.

Claim 8 $G \in \mathcal{C}^2[X]$ if and only if

- Every element of $D' = X \cup \{\overline{1}\}$ is isolated in G,
- $G|_{C'} - 1$ is $(q - 1)$-factor critical, and
- $N_{C'}(1) = \{\overline{c}(G)\}$.

$\triangleleft$ *Proof* We have already seen that the first two conditions are necessary. To show that the third condition is also necessary, it suffices to show that if G satisfies the first two conditions and $\overline{c}(G) \neq \overline{y} \in N_{C'}(1)$ then $G - 1\overline{c}(G) \in \mathcal{C}^1[X]$. Equivalently, we must show that $G - 1\overline{c}(G) \in \mathsf{BFC}(q, s)$ and that $D(G' - 1\overline{c}(G)) = D'$.

Let $\overline{z} \in [\overline{s}]$. If $\overline{z} \in D'$ then a matching of size q in $G' - 1\overline{c}(G) - \overline{z}$ is obtained by taking a matching of size $q - 1$ in $G|_{C' \setminus \{1, \overline{y}\}}$ along with $1\overline{y}$. Thus $\overline{z} \in D(G' - 1\overline{c})$.

Say $\overline{z} \in C'$. A matching of size q in $G - 1\overline{c} - \overline{z}$ is obtained by taking a matching of size $q - 1$ in $G|_{C' \setminus \{1, \overline{z}\}}$ along with $1\overline{1}$. Moreover, we have

$$\nu(G' - 1\overline{c} - \overline{z}) \leq \nu(G' - \overline{z}) < q,$$

so $\overline{z} \notin D(G' - 1\overline{c})$.

To show that the three conditions are sufficient, it suffices to show that $G \in$ BFC(q, s) but $G' \notin$ BFC(q, s). If 1 has no neighbor in C' other than $\overline{c}(G)$ and $A' = \emptyset$ then $\nu(G' - \overline{c}) < q$, so $G' \notin$ BFC(q, s). On the other hand, if $G|_{C'} - 1$ is $(q - 1)$-factor critical then for each $\overline{z} \in [\overline{s}]$ one obtains a matching of size q in $G - \overline{z}$ by taking a matching of size $q - 1$ in $G|_{C' \setminus \{1, \overline{z}\}}$ along with $1\overline{1}$. ▷

We see now that $\mathcal{D}(\mathcal{C}^2[X])$ is isomorphic to $\mathcal{D}(\text{BFC}(q - 1, q))$. thus by our inductive hypothesis there is a matching $\mathcal{M}^3[X]$ on $\mathcal{D}(\mathcal{C}^2[X])$ whose unique critical cell is a graph G with $2q$ edges ($2q - 2$ of them having one endpoint in $[q] \setminus \{1\}$ and one in $[\overline{s}] \setminus (X \cup \{\overline{1}\})$) and the other two being $1\overline{1}$ and $1\overline{c}(G)$).

Using Cluster Lemma 2.5, we see that $\mathcal{M}^0 \cup \mathcal{M}^1 \cup \{M^2[X] \cup M^3[X] : X \in \binom{[\overline{s}] \setminus \{\overline{1}\}}{s - q - 1}\}$ is the desired Morse matching. □

4 Proofs of Theorems 1.1, 1.4

4.1 The complexes $\text{NM}_k(n)$

We now prove Theorem 1.1

Proof We proceed in two steps:
Step 0: We define a Morse matching $\mathcal{M}$ on $\text{NM}_k(n)$ such that the graphs G corresponding to the critical cells are exactly those which satisfy:

- $N_G(n) = \emptyset$.
- For each $1 \leq v < n$, $G + vn$ contains a matching of size k.

For $1 \leq v \leq n - 1$, define $\mathcal{M}(i)$ on $\text{NM}_k(n)$ recursively as follows.

- $\mathcal{M}(1) := \{(G, G - 1n) \mid G \in \text{NM}_k(n), 1n \in E(G)\}$.
- For $1 < v < n$, let $\mathcal{C}(v)$ consist of those $G \in \text{NM}_k(n)$ such that no arc in $\bigcup_{w < v} \mathcal{M}(w)$ has G as its head or tail. Then $\mathcal{M}(v) := \{(G, G - vn) \mid G \in \mathcal{C}(v), vn \in E(G)\}$.

Note that the presence or absence in a graph G of an edge wn ($w < v$) has no effect on the existence of a matching of size k in G which includes the edge vn. Thus it follows from Cluster Lemma 2.5 that $\mathcal{M} := \bigcup_{v=1}^{n-1} \mathcal{M}(v)$ is a Morse matching on $\text{NM}_k(n)$.

The critical cells of this Morse matching are those $G \in \text{NM}_k(n)$ such that $N_G(n) = \emptyset$ and, for $1 \leq v < n$, $G + vn$ contains a matching of size k. For each such G and for $1 \leq v < n$, we have $\nu(G - v) = k - 1$.

Step 1: Let G be a graph corresponding to a critical cell of the Morse matching $\mathcal{M}$ from **Step 0**. Let $G^- := G - n$. Since for $1 \leq v < n$, $G + vn$ contains a matching of size k we have $\nu(G^- - v) = k - 1$ and $D(G^-) = [n - 1] = V(G^-)$.

Thus by Theorem 2.1(3) each connected component of $G^- = G^-|_{D(G^-)}$ is factor critical. Since $\nu(G) = k - 1$ and each component X of G^- satisfies $\nu(G^-|_X) = \frac{|X| - 1}{2}$, we have

$$c := \mathrm{con}(G^-) = n - 2k + 1.$$

Now for each partition τ of $[n - 1]$ into c parts, let $\mathcal{D}^\tau$ be the subdigraph of $\mathcal{D}(\mathsf{NM}_k(n))$ induced on those critical cells G described above such that the connected components of G^- determine the partition τ. Using Lemma 3.1 and Cluster Lemma 2.5, we can construct a Morse matching on $\mathcal{D}^\tau$ whose critical cells are those forests of triangles whose connected components determine τ. Theorem 1.1 follows from a final application of Cluster Lemma 2.5. $\square$

4.2 The complexes $\mathsf{BNM}_k(r, s)$

We now prove Theorem 1.4. Analogous to the proof for non-bipartite graphs we have to proceed in two steps of which **Step 0** is only a slight modification of **Step 0** from the proof of Theorem 1.1.

Proof **Step 0:** We inductively define a Morse matching $\mathcal{M}$ on $\mathsf{BNM}_k(r, s)$.
    For $\bar{1} \leq \bar{v} \leq \bar{s}$, define $\mathcal{M}^{\bar{v}}$ on $\mathsf{BNM}_k(r, s)$ recursively as follows.

- $\mathcal{M}^{\bar{1}} := \{(G, G - r\bar{1}) \mid G \in \mathsf{BNM}_k(r, s), r\bar{1} \in E(G)\}$.
- For $\bar{1} < \bar{v} \leq \bar{s}$, let $\mathcal{C}^{\bar{v}}$ consist of those $G \in \mathsf{BNM}_k(r, s)$ which are critical cells of $\bigcup_{\bar{w} < \bar{v}} \mathcal{M}^{\bar{w}}$, and set

$$\mathcal{M}^{\bar{v}} := \{(G, G - r\bar{v}) : G \in \mathcal{C}^{\bar{v}}, r\bar{v} \in E(G)\}.$$

Set $\mathcal{M} := \bigcup_{\bar{v} = \bar{1}}^{\bar{s}} \mathcal{M}^{\bar{v}}$. With the same arguments as for non-bipartite graphs we see that $\mathcal{M}$ actually is a Morse matching on $\mathsf{BNM}_k(r, s)$.

The graphs G corresponding to the critical cells of $\mathcal{M}$ are exactly those which satisfy:

- $N_G(r) = \emptyset$.
- For each $\bar{1} \leq \bar{v} < \bar{s}$, $G + r\bar{v}$ contains a matching of size k.

Step 1: It follows from König's Theorem (see [13]), which says that the size of a maximal matching equals the size of a minimal vertex cover, that exactly $k - 1$ elements of $[r]$ are not isolated. So, we make one of $\binom{r-1}{k-1}$ choices for the set of nonisolated vertices in $[r]$ and then assume that $r = k$. In this case, remaining graphs are those for which $[r - 1]$ can be completely matched into every $(s - 1)$-subset of $[\bar{s}]$. Theorem 1.4 now follows from (the proof of) Theorem 1.10 and Cluster Lemma 2.5. $\square$

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