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On the hyperbolic unitary geometry

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Abstract Hans Cuypers (Preprint) describes a characterisation of the geometry on singular points and hyperbolic lines of a finite unitary space—the hyperbolic unitary geometry—using information about the planes. In the present article we describe an alternative local characterisation based on Cuypers' work and on a local recognition of the graph of hyperbolic lines with perpendicularity as adjacency. This paper can be viewed as the unitary analogue of the second author's article (J. Comb. Theory Ser. A 105:97–110, 2004) on the hyperbolic symplectic geometry.

Keywords Hyperbolic unitary geometry · Root group geometry · Local recognition graphs · Centralisers of involutions

1 Introduction

The geometry on the points and hyperbolic lines of a non-degenerate finite unitary polar space (or, short, **hyperbolic unitary geometry**) is interesting for a number of reasons.

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One reason is the fact that it belongs to the family of (partially) linear geometries that are characterised by their planes (Cuypers [6]; restated in part as Theorem 4.6 of the present article). The most famous of such geometries is the projective space, which by a classical result is characterised as a linear geometry each of whose planes are projective (Veblen and Young [14], [15]). Another geometry characterised by its planes is the hyperbolic symplectic geometry (Cuypers [4], Hall [9]). It is closely related to the hyperbolic unitary geometry: while each plane of the former is dual affine (also called symplectic), a plane of the latter is either dual affine or linear. (The linear ones are, in fact, related to classical unitals, cf. [10], [11].)

A second reason why the hyperbolic unitary geometry is an interesting object to study is the 1-1 correspondence between the set of long root subgroups, resp. fundamental SL_2's of $SU_n(q^2)$ on one hand and the points, resp. hyperbolic lines of the corresponding unitary geometry on the other hand via the map that assigns the respective groups to their commutator in the module. This correspondence is well-known, see e.g. [13, Chapter 2]. Cuypers' article [5] underscores that root group geometries are highly interesting objects.

This paper can be viewed as a sister paper of [8] (where the root group geometry of $Sp_{2n}(\mathbb{F})$ is studied for arbitrary fields) and of [1] (where the authors study the line graph of a complex vector space endowed with an anisotropic unitary form). However, the situations covered by the sister papers [1], [8] of this paper are much more behaved and a lot easier to handle than the situation in this paper. The increased difficulty compared to [8] originates from the fact that we prove Theorem 1 for $n \geq 7$ instead of $n \geq 8$ (odd-dimensional non-degenerate symplectic forms do not exist), while the increased difficulty compared to [1] comes from the fact that subspaces of non-degenerate subspaces can be very far from being non-degenerate, whereas subspaces of anisotropic subspaces are anisotropic.

The first result of this paper focuses on the hyperbolic lines and their relative positions. More precisely, let U_n denote an n-dimensional vector space over $\mathbb{F}_{q^2}$ endowed with a non-degenerate hermitian form. The **hyperbolic line graph** $\mathbf{G}(U_n)$ is the graph on the hyperbolic lines, i.e., the non-degenerate two-dimensional subspaces of U_n, in which hyperbolic lines l and m are adjacent (in symbols $l \perp m$) if and only if l is perpendicular to m with respect to the unitary form. Equivalently, $l \perp m$ if and only if the corresponding fundamental SL_2's commute.

A graph Γ is **locally homogeneous** if and only if for any pair x, y of vertices of Γ, the induced subgraphs $\Gamma(x)$ and $\Gamma(y)$ on the set of neighbours of x, resp. y are isomorphic. Such a locally homogeneous graph Γ is called **locally** Δ, for some graph Δ, if $\Gamma(x) \cong \Delta$ for some, whence all, vertices x of Γ. It is easily seen (cf. Proposition 3.3) that the graph $\mathbf{G}(U_n)$ is locally $\mathbf{G}(U_{n-2})$. Conversely, this property is characteristic for this graph for sufficiently large n:

Theorem 1 *Let $n \geq 7$, let q be a prime power, and let Γ be a connected graph that is locally $\mathbf{G}(U_n)$. Then Γ is isomorphic to $\mathbf{G}(U_{n+2})$, unless $(n, q) = (7, 2)$.*

The requirement in the preceding theorem that Γ be connected comes from the fact that a graph is locally Δ if and only if each of its connected components is locally Δ. So its primary role is to provide irreducibility. We do not know whether the case $(n, q) = (7, 2)$ provides an actual counter example.

For $n \geq 8$ this result has been stated without proof in the second author's PhD thesis [7, Theorem 4.5.3]. Since counter examples to the local recognition are only known for $n = 6$—they come from the exceptional groups of type ${}^2E_6(q^2)$, see [13]—publication of this result was deferred until the case $n = 7$ could be proved. This has finally been achieved during the preparation of the first author's PhD thesis. Comparing the proofs of Lemmata 5.5, 5.6 and 5.7 with the proof of Lemma 5.8, the reader will understand why the case $n = 7$ is so much more difficult than the case $n \geq 8$.

As mentioned before, the motivation of our research was of group-theoretic nature. If the field $\mathbb{F}$ has characteristic distinct from 2, translating Theorem 1 into the language of group theory yields the following.

Theorem 2 *Let $n \geq 7$ and let q be an odd prime power. Let G be a group with subgroups A and B isomorphic to $SL_2(q)$, and denote the central involution of A by x and the central involution of B by y. Furthermore, assume the following holds:*

- *$C_G(x) = A \times K$ with $K \cong SU_n(q^2)$;*
- *$C_G(y) = B \times J$ with $J \cong SU_n(q^2)$;*
- *A is a fundamental SL_2 of J;*
- *B is a fundamental SL_2 of K;*
- *there exists an involution in $J \cap K$ that is the central involution of a fundamental SL_2 of both J and K.*

If $G = \langle J, K\rangle$, then $G/Z(G) \cong PSU_{n+2}(q^2)$.

This article is organised as follows: In Sections 2 and 3 we study properties of the hyperbolic line graph $\mathbf{G}(U_n)$ for $n \geq 5$. Section 4 deals with the relation of the graph $\mathbf{G}(U_n)$ with the hyperbolic unitary geometry. In that section we also study embeddings of $\mathbf{G}(U_{n-2})$ in $\mathbf{G}(U_n)$, which provides us with valuable information for the proof of Theorem 1 that we give in Section 5. Most of our arguments are based on counting in subspaces of U_n of various dimensions and ranks. For the convenience of the reader we include a collection of results on the number of subspaces of various types in Appendix A. For quick reference we also give some tables containing the necessary information in Table 1. A proof of Theorem 2 is not included in this article, because the problem of how to deduce a result like Theorem 2 from a result like Theorem 1 has been thoroughly studied in [3, Section 6], [7] and, thus, is well-understood.

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2 The hyperbolic line graph of U_5

Let $q \geq 3$ be a prime power and let U_5 be a five-dimensional non-degenerate unitary vector space over $\mathbb{F}_{q^2}$ with polarity π. Define the graph $\mathbf{G}(U_5)$ with the set of non-degenerate two-dimensional subspaces of U_5 as the set of vertices in which two vertices l and m are adjacent if and only if $l \subset m^\pi$. The aim of this section is to reconstruct the unitary vector space U_5 from the graph $\mathbf{G}(U_5)$. To this end we will define a point-line geometry $G = (\mathcal{I}, \mathcal{L}, \supset)$ using intrinsic properties of the graph $\mathbf{G}(U_5)$ and establish an isomorphism between G and the geometry on arbitrary points and hyperbolic lines of U_5. From there U_5 is easily recovered.

We first determine the diameter of $\mathbf{G}(U_5)$.

Lemma 2.1 *Let l and m be distinct hyperbolic lines of U_5. Then l and m have distance two in $\mathbf{G}(U_5)$ if and only if the subspace $\langle l, m\rangle$ is a non-degenerate plane in U_5.*

Proof Let l and m be two hyperbolic lines of U_5 which have distance two in $\mathbf{G}(U_5)$. That is, the graph $\mathbf{G}(U_5)$ contains a vertex z, which is a hyperbolic line in U_5, adjacent to the vertices l and m. Its perpendicular space z^π, a non-degenerate plane of U_5, contains the distinct hyperbolic lines l and m. Hence the hyperbolic lines l and m span the non-degenerate plane z^π.

Conversely, suppose that $\langle l, m\rangle$ is a non-degenerate three-dimensional subspace of U_5. Since U_5 is a five-dimensional non-degenerate unitary vector space, the pole of $\langle l, m\rangle$ is a hyperbolic line $h = \langle l, m\rangle^\pi$ of U_5. By definition the vertex h is adjacent to the vertices l and m in $\mathbf{G}(U_5)$. Since the hyperbolic lines l and m intersect in U_5, it follows that $l \not\perp m$. Therefore the vertices l and m have distance two in $\mathbf{G}(U_5)$. □

Lemma 2.2 *Let l and m be distinct hyperbolic lines of U_5. Then l and m have distance three in $\mathbf{G}(U_5)$ if and only if l and m are two non-intersecting hyperbolic lines such that $l^\pi \cap m$ is a one-dimensional subspace of U_5.*

Proof Suppose the vertices l and m have distance three in the graph $\mathbf{G}(U_5)$. Then by Lemma 2.1 we find a vertex z in the graph $\mathbf{G}(U_5)$ adjacent to l such that $\langle z, m\rangle$ is a non-degenerate plane of U_5. The intersection $p := m \cap z$ is a one-dimensional. As $z \subseteq l^\pi$, the hyperbolic line m intersects the subspace l^π in at least the point p. Since the vertices l and m are not adjacent in $\mathbf{G}(U_5)$, we have $m \not\subseteq l^\pi$, so $m \cap l^\pi = p$.

In order to prove the first implication of the statement it is left to show that the hyperbolic lines l and m do not intersect in U_5. By way of contradiction we assume that $\langle m, l\rangle$ is a three-dimensional subspace. The plane $\langle l, m\rangle$ is degenerate by Lemma 2.1, thus $l^\pi \cap m^\pi$ is a singular two-dimensional subspace of U_5. Since p, the intersection point of m and l^π, is incident to the hyperbolic line m, we have $p \notin \mathrm{rad}(\langle m, l\rangle)$ and $m^\pi \subseteq p^\pi$, whence $m^\pi \cap l^\pi \subseteq p^\pi \cap l^\pi$. Of course, p is either singular or non-degenerate. Furthermore $\dim(m^\pi \cap l^\pi) = 2 = \dim(p^\pi \cap l^\pi)$. Consequently $m^\pi \cap l^\pi = p^\pi \cap l^\pi$.

If p is a non-degenerate point, then $p^\pi \cap l^\pi$ is a non-degenerate line, contradicting the fact that $m^\pi \cap l^\pi$ is degenerate. If p is a singular point, then of course $p^\pi \cap l^\pi$ is a singular two-dimensional subspace s of rank one containing the point p itself and the radical of $p^\pi \cap l^\pi$. Therefore $p = \mathrm{rad}(p^\pi \cap l^\pi) = \mathrm{rad}(m^\pi \cap l^\pi) = \mathrm{rad}(\langle m, l\rangle) \neq p$, a contradiction. Thus $\langle m, l\rangle$ has to be a four-dimensional space and the two hyperbolic lines l and m have a trivial intersection in U_5.

Now for the other implication. If l and m are two non-intersecting hyperbolic lines in U_5 such that $l^\pi \cap m$ is a one-dimensional subspace p, then, by Lemma 2.1, the vertices l and m have not distance one or two in the graph $\mathbf{G}(U_5)$. To prove the statement, we construct a hyperbolic line z in the subspace l^π with the property that the subspace $\langle m, z\rangle$ is a non-degenerate plane in U_5, implying that $l \perp z$ and that the distance between the vertices z and m in $\mathbf{G}(U_5)$ is two, by Lemma 2.1.

Consider the subspace l^π perpendicular to l and two points $p \in l^\pi \cap m$ and $x \in l^\pi \cap m^\pi$. Note that p and x are uniquely determined by the assumptions that $\dim(l^\pi \cap m) = 1$ and $\dim(\langle l, m\rangle) = 4$ in U_5. Moreover $p \in x^\pi$ since $p \in m$ and $x \in m^\pi$.

If both the point p and the point x are non-degenerate, then $z = \langle p, x\rangle$ is a hyperbolic line contained in l^π, since p and x are perpendicular to each other as noted before. Furthermore $\langle m, x\rangle$ is a non-degenerate plane of U_5 due to the fact that $x \in m^\pi$, proving the statement in this special case.

If p is singular and x is non-degenerate, then $\langle p, x\rangle$ is a singular line of rank one, because $p \in x^\pi$ as mentioned above. We consider the $q^2 - q$ hyperbolic lines h_i, $1 \le i \le q^2 - q$, in l^π incident to x, cf. Table 1, entry $n + l = 2$, $n = 2$, $r = 1$. Any two different hyperbolic lines h_i and h_j span the plane l^π and each subspace h_i^π is a non-degenerate plane in x^π for $1 \le i < j \le q^2 - q$. Moreover, the intersection of h_i^π with the hyperbolic line m is a point $r_i = h_i^\pi \cap m$ in x^π with $r_i \ne r_j$ for $1 \le i < j \le q^2 - q$. Indeed, $m \not\subseteq h_i^\pi$, because $h_i \cap m^\pi$ is the one-dimensional subspace x for each hyperbolic line h_i. If $r_i = r_j$ for $i \ne j$, then we obtain $r_i = h_i^\pi \cap m = r_j = h_j^\pi \cap m = h_i^\pi \cap h_j^\pi \cap m = \langle h_i, h_j\rangle^\pi \cap m = l \cap m = \{0\}$, a contradiction. Thus we have $q^2 - q$ different one-dimensional subspaces r_i on the hyperbolic line m. Hence the line m contains a non-degenerate point $r = r_k$ for some $k \in \{1, \dots, q^2 - q\}$, because $q^2 - q > q + 1$ for $q \ge 3$, where $q + 1$ is the number of singular points on a hyperbolic line (cf. Table 1 on page 581). Note that the points r and p span together the hyperbolic line m. Note also that $r^\pi \cap l^\pi = h_k$. For, $r_i^\pi = (h_i^\pi \cap m)^\pi = \langle h_i, m^\pi\rangle$, so r^π contains h_k; since l^π contains h_k as well and since $r^\pi \cap l^\pi$ is two-dimensional, we have $r^\pi \cap l^\pi = h_k$. As follows from Table 1 all points on the hyperbolic line h_k different from the point x generate together with the point p a non-degenerate two-dimensional subspace of l^π. Therefore the hyperbolic line h_k contains $q^2 - q - 1$ different non-degenerate points y_i such that $\langle y_i, p\rangle$ is a hyperbolic line. Furthermore the span of the two hyperbolic lines m and h_k is a four-dimensional space of rank at least three, since $\langle r, h_k\rangle \subseteq \langle m, h_k\rangle$ and $\mathrm{rk}(\langle r, h_k\rangle) = \mathrm{rk}(\langle r, r^\pi \cap l^\pi\rangle) = 3$.

If the four-dimensional space $\langle m, h_k\rangle$ is non-degenerate, Table 1 implies that the hyperbolic line h_k contains at least $q^2 - 2q - 2 > 0$ (recall that $q \ge 3$) different non-degenerate points z_i such that $\langle z_i, p\rangle = z$ is a hyperbolic line and $\langle m, z\rangle = \langle r, p, z_i\rangle$ is a non-degenerate plane. Alternatively, if the rank of the four-dimensional space $\langle m, h_k\rangle$ is three then, by the information from Table 1, the hyperbolic line h_k contains at least $q^2 - q - 2 > 0$ different non-degenerate points z_i, which satisfy the conditions that $\langle z_i, p\rangle = z$ is a hyperbolic line and $\langle m, z\rangle = \langle r, p, z_i\rangle$ is a non-degenerate plane and we are done in this case.

Next we assume the point p to be non-degenerate and the point x to be singular. Then the hyperbolic line $h = l^\pi \cap p^\pi$ is incident to the singular point $x = l^\pi \cap m^\pi$, because p is incident to m. Moreover the non-degenerate point $r = p^\pi \cap m$ and the hyperbolic line h span a plane P of rank two or three. As follows from the information from Table 1 the plane P contains at least $q^2 - q$ different hyperbolic lines incident to the point r. Certainly, the intersections of these $q^2 - q$ hyperbolic lines with h

are pairwise distinct as follows by arguments similar to the ones used above. At least q^2-2q-1 of those intersection points are non-degenerate. Choosing one of those, say a, the line $z=\langle a,p\rangle \subset l^\pi$ is a hyperbolic line, as $a\in p^\pi$. The plane $\langle m,z\rangle = \langle r,p,a\rangle$ has a Gram matrix (with respect to some suitably chosen basis in r, p, and a) of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \overline{\gamma} \\ 0 & \gamma & 1 \end{pmatrix}$. This matrix has a non-zero determinant as follows from the fact that z is a hyperbolic line, so $\langle m,z\rangle$ is non-degenerate. Again, by Lemma 2.1 we are finished in this case.

The case that both points x and p are singular does not occur, as otherwise the non-degenerate plane l^π would contain the totally singular line $\langle x,p\rangle$, a contradiction. □

Lemma 2.3 *Let l and m be two different hyperbolic lines of U_5. Then l and m have distance four in $\mathbf{G}(U_5)$ if and only if either*

- *l and m are two non-intersecting lines such that $l^\pi \cap m$ is trivial in U_5, or*
- *l and m are two intersecting lines spanning a degenerate plane in U_5.*

Proof Let l and m be two vertices in the graph $\mathbf{G}(U_5)$ at mutual distance four. If the subspace $\langle l,m\rangle$ is a non-degenerate plane, then l and m have distance two by Lemma 2.1. Therefore, if $\langle l,m\rangle$ is a plane, then $\langle l,m\rangle$ is a degenerate subspace of U_5. Alternatively, if $\langle l,m\rangle$ is a four-dimensional subspace in U_5 and $l^\pi\cap m\neq\{0\}$, then l and m have distance one in $\mathbf{G}(U_5)$ by definition or distance three in $\mathbf{G}(U_5)$ by Lemma 2.2, a contradiction again. It follows that, if the subspace $\langle l,m\rangle$ is of dimension four, then $l^\pi\cap m$ is trivial.

In order to show the converse implication of the statement, let $\langle l,m\rangle$ be either a degenerate plane or a four-dimensional subspace such that $l^\pi\cap m=\{0\}$. By the Lemma 2.1 and Lemma 2.2 the vertices l and m do not have distance one, two, or three in $\mathbf{G}(U_5)$. Therefore it is enough to find a path of length four in $\mathbf{G}(U_5)$ between the vertices l and m to finish the proof of this lemma.

We choose a hyperbolic line z in l^π intersecting the space m^π in a point. Such a choice is possible, because l^π is non-degenerate and the intersection $l^\pi\cap m^\pi$ is non-trivial and properly contained in l^π. By construction the vertices l and z are adjacent in $\mathbf{G}(U_5)$. By the above $m\cap l^\pi$ is trivial. Hence m and z do not intersect, but satisfy the condition $\dim(z\cap m^\pi)=1$. So m and z have distance three in the hyperbolic line graph $\mathbf{G}(U_5)$ by Lemma 2.2 and, thus, the distance between the vertices l and m is four in $\mathbf{G}(U_5)$. □

Proposition 2.4 *The graph $\mathbf{G}(U_5)$ is a connected locally $\mathbf{G}(U_3)$ graph of diameter four.*

Proof For any singular point p in the orthogonal space l^π of a hyperbolic line l in U_5, the subspace $\langle l,p\rangle$ is of dimension three and rank two. As follows from Lemma A.1 it is possible to choose a hyperbolic line m different from l in the plane $\langle l,p\rangle$. Thus l and m span the degenerate plane $\langle l,p\rangle$ and hence the vertices l and m have distance four in $\mathbf{G}(U_5)$ by Lemma 2.3. The statement about the diameter follows now from the fact that two hyperbolic lines cannot form a configuration other than adjacency and the ones described in 2.1 to 2.3. The local property is obvious. □

Remark 2.5 Let l and m be two arbitrary vertices of the hyperbolic line graph $\mathbf{G}(U_5)$. An important induced subgraph of $\mathbf{G}(U_5)$ is the common perp of the vertices l and m. If the induced subgraph $\{l,m\}^\perp$ is not empty, then the subspace $\langle l,m\rangle^\pi$ of U_5 contains some hyperbolic line. We observe that $\{l,m\}^\perp \neq \{0\}$ in $\mathbf{G}(U_5)$ if and only if l and m have distance two in $\mathbf{G}(U_5)$. Indeed if l and m are at distance two in $\mathbf{G}(U_5)$, then the hyperbolic lines l and m span a non-degenerate plane in U_5 and $\langle l,m\rangle^\pi$ is a hyperbolic line, by Lemma 2.1. In all other cases $\langle l,m\rangle$ is a four-dimensional subspace (and therefore $\langle l,m\rangle^\pi$ is a single point of U_5) or the hyperbolic lines l and m span a degenerate planes (which implies that $\langle l,m\rangle^\pi$ is a rank one line). Thus in these cases the subgraph $\{l,m\}^\perp$ is the empty graph. Of course, if the vertices l and m have distance one in $\mathbf{G}(U_5)$, then $\langle l,m\rangle$ is a four dimensional non-degenerate space in the unitary vector space U_5 and $l^\pi \cap m^\pi = \langle l,m\rangle^\pi$ is a non-degenerate point of U_5.

Definition 2.6 Let W be a subspace of U_5. The set of all hyperbolic lines of W is denoted by $\mathbf{L}(W)$.

Lemma 2.7 *Let l and m be two distinct vertices of $\mathbf{G}(U_5)$ with $\{l,m\}^\perp \neq \emptyset$. Then $\{l,m\}^{\perp\perp} = \mathbf{L}(\langle l,m\rangle)$.*

Proof Let l and m be two distinct vertices in $\mathbf{G}(U_5)$ such that $\{l,m\}^\perp$ is not empty. Due to Remark 2.5 the vertices l and m have distance two in $\mathbf{G}(U_5)$ and it follows that the graph $\{l,m\}^\perp$ is the single vertex $\langle l,m\rangle^\pi$. Thus we obtain the equalities $\{l,m\}^{\perp\perp} = (\{l,m\}^\perp)^\perp = \bigcap_{z\in\{m,l\}^\perp} z^\perp = (\langle l,m\rangle^\pi)^\perp = \mathbf{L}((\langle l,m\rangle^\pi)^\pi) = \mathbf{L}(\langle l,m\rangle)$. □

It will prove useful to know whether two hyperbolic lines intersect in the projective space (i.e., the two hyperbolic lines span a plane in the projective space) or not (i.e., they span a four-dimensional space in the projective space). Lemmas 2.1 to 2.3 show that in order to distinguish the above two cases, we have to study vertices of $\mathbf{G}(U_5)$ at mutual distance three or four more thoroughly.

Lemma 2.8 *If l and m are two non-intersecting hyperbolic lines of U_5 such that $l^\pi \cap m$ is a point p, then in the graph $\mathbf{G}(U_5)$ the number of different paths of length three between l and m is at most q^2. On the other hand, this number is at least q^2-q-1 if p is a singular point and at least q^2-2q-1 if p is a non-degenerate point.*

Proof Let h be an arbitrary neighbor of l in $\mathbf{G}(U_5)$, i.e., $h \subset l^\pi$. By Lemma 2.1 there exists a common neighbor k of h and m (and, thus, a path of length three from l to m through h) if and only if $\langle h,m\rangle$ is a non-degenerate plane. In fact, if $\langle h,m\rangle$ is a non-degenerate plane, then k is uniquely determined as $\langle h,m\rangle^\pi$. Therefore it suffices to study all non-degenerate planes E with $m \subseteq E \subseteq \langle m, l^\pi\rangle$ such that $E \cap l^\pi$ is a non-degenerate line.

Let us first deduce the upper bound in the statement of the lemma from the observations made in the above paragraph. If $p = l^\pi \cap m$ is a singular point, then q^2

different hyperbolic lines and exactly one singular line of the orthogonal space l^π run through the point p by Table 1, entry $n+l=2$, $n=1$, $r=1$ and $r=0$. If $p=l^\pi\cap m$ is a non-degenerate point, then q^2-q different hyperbolic lines and $q+1$ distinct singular lines are incident to the point p in the subspace l^π. Hence there are at most q^2 paths from l to m.

Next we want to establish the respective lower bounds. Regard the four-dimensional subspace $W=\langle m,l^\pi\rangle$, which is of rank three or four. In the subspace W the hyperbolic line m is contained in q^2+1 different planes E_i by Lemma A.1. Each plane E_i of W intersects the non-degenerate plane l^π in a line, by the dimension formula and because $m\not\subseteq l^\pi$. Since $p\in l^\pi$ is incident to each plane E_i, every line $h_i=E_i\cap l^\pi$ runs through p. Moreover the lines h_i are mutually distinct, because the identity $h_i=h_j$ implies $E_i=\langle h_i,m\rangle=\langle h_j,m\rangle=E_j$.

If the subspace W is of rank four, then the hyperbolic line m lies on q^2-q different non-degenerate planes E_i^m by Table 1. Therefore we obtain q^2-q different lines $E_i^m\cap l^\pi=h_i^m$ incident to the point p in the subspace l^π. At least q^2-q-1 lines of the q^2-q lines h_i^m are hyperbolic lines, if p is a singular point, due to Table 1. On the other hand, if p is a non-degenerate point, then at least $q^2-q-(q+1)=q^2-2q-1$ lines of the q^2-q lines h_i^m are hyperbolic lines by Table 1 again. Alternatively if W is of rank three, then exactly q^2 different non-degenerate planes E_i^m are incident to the hyperbolic line m by Table 1. Hence we obtain q^2 different lines $E_i^m\cap l^\pi=h_i^m$ in the non-degenerate plane l^π containing the point p. By Table 1, at least q^2-1 of these q^2 lines h_i^m are non-degenerate, if p is a singular point and at least $q^2-(q+1)=q^2-q-1$ lines are non-degenerate, if p is a non-degenerate point. □

Lemma 2.9 *If l and m are two non-intersecting hyperbolic lines of U_5 which are at distance four in the graph $\mathbf{G}(U_5)$, then there are at most q^4 different paths of length four from l to m.*

Proof By Lemma 2.3 we have $\dim(\langle l,m\rangle)=4$ with $l^\pi\cap m=\{0\}$. A neighbor h of l in $\mathbf{G}(U_5)$ is at distance three from the vertex m if and only if $\dim(\langle h,m\rangle)=4$ and $\dim(h\cap m^\pi)=1$ by Lemma 2.2. Thus h is a hyperbolic line in l^π passing through the point $x:=\langle l,m\rangle^\pi$. If the one-dimensional subspace x is singular, then l^π contains q^2 different hyperbolic lines h_i^l incident with x by Table 1. If x is non-degenerate point, then, by Table 1 again, there are q^2-q hyperbolic lines through x in l^π. By Lemma 2.2 the vertices m and h_i^l are at distance three in $\mathbf{G}(U_5)$. Combining the above numbers with Lemma 2.8 we obtain at most $q^2\cdot q^2=q^4$ paths from l to m. □

Lemma 2.10 *If l and m are two intersecting hyperbolic lines spanning a degenerate plane, then the hyperbolic line graph $\mathbf{G}(U_5)$ contains at least $q^6-3q^5+2q^4-q^2$ different paths of length four from l to m.*

Proof If h is a neighbor of l, then the vertex h is at distance three from m in $\mathbf{G}(U_5)$ if and only if $\dim(\langle h,m\rangle)=4$ and $\dim(h\cap m^\pi)=1$ in U_5 by Lemma 2.2. Consequently h is a hyperbolic line in the polar space l^π of l such that $\langle l,m\rangle^\pi\cap h$ is a one-dimensional subspace. Since the rank one line $\langle l,m\rangle^\pi$ contains exactly one singular point x and q^2 non-degenerate points p_i (see Table 1), the non-degenerate plane

l^π contains q^2 hyperbolic lines h_{x_i} incident to the point x. Each non-degenerate point p_i admits q^2-q incident hyperbolic lines $h_{p_i,j}$ of l^π. Certainly, all those hyperbolic lines h_{x_i} and $h_{p_i,j}$ are pairwise distinct as otherwise they would coincide with the line $\langle l,m\rangle^\pi$. By Lemma 2.8 we have at least q^2-q-1 different paths of length three in $\mathbf{G}(U_5)$ between each vertex h_{x_i} and the vertex m and not less than q^2-2q-1 different paths of length three in $\mathbf{G}(U_5)$ from each vertex $h_{p_i,j}$ to the vertex m, again. Accordingly we obtain at least $q^2(q^2-q-1)+q^2(q^2-q)(q^2-2q-1)=q^6-3q^5+2q^4-q^2$ different paths of length four from l and m in the graph $\mathbf{G}(U_5)$. □

Lemma 2.11 *Two different vertices l and m of distance four in $\mathbf{G}(U_5)$ intersect in a point in the vector space U_5 if and only if the number of different paths of length four between l and m in $\mathbf{G}(U_5)$ is greater than q^4.*

Proof Since $q\geq 3$, we have $q^6-3q^5+2q^4-q^2>q^4$, so the claim follows from Lemma 2.9 and Lemma 2.10. □

Lemma 2.12 *Two distinct vertices l and m of the hyperbolic line graph $\mathbf{G}(U_5)$ intersect in a common point in U_5 if and only if either*

- *the subgraph $\{l,m\}^\perp$ is not empty, or*
- *the vertices l and m have distance four in $\mathbf{G}(U_5)$ and there are more than q^4 different paths of length four from l to m.*

Proof This is an immediate consequence of Lemma 2.1 to Lemma 2.3 together with statements of Lemma 2.11, Lemma 2.7 and Remark 2.5. □

In the next step we want to recover all points of the space U_5 as pencils of hyperbolic lines. Therefore we need a construction to check in the graph $\mathbf{G}(U_5)$ whether three distinct lines of U_5 intersect in one point or not. Therefore take the following characterisation: three different hyperbolic lines k_1,k_2 and k_3 of U_5 intersect in one point if we can find a hyperbolic line s in U_5 such that

- the plane $\langle s,k_i\rangle$ is non-degenerate for $1\leq i\leq 3$ and $s\neq k_i$,
- $\langle s,k_1,k_2\rangle$ is a four-dimensional space in U_5.

The same statement in terms of graph language is that three different vertices k_1, k_2 and k_3 of $\mathbf{G}(U_5)$ intersect in one point if we can find a vertex s of $\mathbf{G}(U_5)$ with the following properties:

- the induced subgraph $\{s,k_i\}^\perp$ is not empty for $i\in\{1,2,3\}$ and $s\neq k_i$,
- $\{s,k_1,k_2\}^\perp$ is the empty graph.

To verify the claim that every one-dimensional subspace of the U_5 can be detected by three pairwise intersecting distinct vertices k_1,k_2 and k_3 of $\mathbf{G}(U_5)$ as stated above, we have to show that we can find a vertex s in $\mathbf{G}(U_5)$ such that $\{s,k_1,k_2\}^\perp=\emptyset$ and $\{s,k_i\}^\perp\neq\emptyset$ for $i=1,2,3$ and $s\neq k_i$. This will be proved in the next lemma.

Lemma 2.13 *Let k_1, k_2 and k_3 be three distinct hyperbolic lines of U_5, which intersect in a one-dimensional subspace p. Then the unitary polar space U_5 contains a* *hyperbolic line l with the properties that $\langle k_1, k_2, l\rangle$ is a four-dimensional space and that $\langle l, k_i\rangle$ is a non-degenerate plane for $i = 1, 2, 3$ and $l \neq k_i$.*

Proof In the unitary polar space U_5 every hyperbolic line k is incident to $q^4 - q^3 + q^2$ different non-degenerate planes E_j^k and to $q^3 + 1$ different singular planes S_i^k in U_5 as follows from Lemma A.1. For the hyperbolic line k_1 we obtain the $q^4 - q^3 + q^2$ different non-degenerate planes $E_j^{k_1}$ and consider in each of these the hyperbolic lines $h_r^{E_j^{k_1}}$ containing the point p. If p is a singular point then in each plane $E_j^{k_1}$ there are $q^2 - 1$ different hyperbolic lines $h_r^{E_j^{k_1}}$ incident to p and different from the hyperbolic line k_1 by Table 1. Alternatively, if p is a non-degenerate point, then in each plane $E_j^{k_1}$ we find $q^2 - q - 1$ different hyperbolic lines $h_r^{E_j^{k_1}}$ passing through the point p, which are different from the hyperbolic line k_1, using Table 1 again. Recall that $E_i^{k_1} \cap E_j^{k_1} = k_1$ if and only if the planes are different, which leads to the fact that a hyperbolic lines $h_r^{E_j^{k_1}}$ is not incident to the non-degenerate plane $E_i^{k_1}$ if and only if $i \neq j$. Therefore, if p is non-degenerate point, then in the unitary vector space U_5 there are $(q^4 - q^3 + q^2 - 1)(q^2 - q - 1) = q^6 - 2q^5 + q^4 - 2q^2 + q + 1$ different hyperbolic lines $h_r^{E_j^{k_1}}$ incident to the point p, different from the hyperbolic line k_1 and not lines of the plane $\langle k_1, k_2\rangle$. Alternatively, if p is a singular point, then the polar space U_5 contains $(q^4 - q^3 + q^2 - 1)(q^2 - 1) = q^6 - q^5 + q^3 - 2q^2 - 1$ different hyperbolic lines $h_r^{E_j^{k_1}}$ with the same properties as above.

Next we consider the singular planes $S_j^{k_2}$ and $S_i^{k_3}$ in U_5. The point p is not contained in the radicals of the planes $S_j^{k_i}$, because the hyperbolic lines k_2 and k_3 are passing through the point p, and thus in each rank two plane $S_j^{k_i}$ are $q^2 - 1$ different hyperbolic lines $l_r^{S_j^{k_i}}$ incident to p and different from the hyperbolic line k_i. Therefore in the planes $S_{k_2,j}$ and $S_{k_3,j}$ are together at most $2(q^2 - 1)(q^3 + 1) = 2q^5 - 2q^3 + 2q^2 - 2$ different hyperbolic lines $l_r^{S_j^{k_i}}$ with the assumed properties.

If p is a non-degenerate point, then $q^6 - 2q^5 + q^4 - 2q^2 + q + 1 - (2q^5 - 2q^3 + 2q^2 - 2) = q^6 - 4q^5 + q^4 + 2q^3 - 4q^2 + 3 > 0$. This implies that U_5 contains a hyperbolic line s which intersects each hyperbolic line k_i for $i = 1, 2, 3$ and such that the planes $\langle s, k_i\rangle$ are non-degenerate for $i = 1, 2, 3$ and $\langle s, k_1, k_2\rangle$ is a four-dimensional space.In the other case, if p is singular, then $q^6 - q^5 + q^3 - 2q^2 - 1 - (2q^5 - 2q^3 + 2q^2 - 2) = q^6 - 3q^5 + 3q^3 - 4q^2 + 1 > 0$, and such a hyperbolic line s exists as well. □

Definition 2.14 Let Γ be graph isomorphic to $\mathbf{G}(U_5)$. Two different vertices k and l of Γ are defined to **intersect** if

- either the induced subgraph $\{k, l\}^\perp$ is not empty, or
- the vertices k and l have distance four in Γ and the number of different paths of length four between l and m in $\mathbf{G}(U_5)$ is greater than q^4.

Three distinct pairwise intersecting vertices k_1, k_2 and k_3 of Γ are defined to **intersect in one point** if there is a vertex s of Γ with the following properties:

- the induced subgraph $\{s, k_i\}^\perp$ is not empty for $i \in \{1, 2, 3\}$ and $s \neq k_i$,
- $\{s, k_1, k_2\}^\perp$ is the empty graph.

An **interior point** of the graph Γ is a maximal set p of distinct pairwise intersecting vertices of Γ such that any three elements of p intersect in one point. We denote the set of all interior points of Γ by $\mathcal{I}$. Moreover, an **interior line** of the graph Γ is a vertex of the graph Γ. The set of all interior lines of Γ is denoted by $\mathcal{L}$.

The discussions in this section imply the following result.

Proposition 2.15 *Let Γ be a graph isomorphic to $\mathbf{G}(U_5)$. Then the geometry $(\mathcal{I}, \mathcal{L}, \supset)$ is isomorphic to the geometry on arbitrary one-dimensional subspaces and non-degenerate two-dimensional subspaces of the unitary polar space U_5.*

3 The hyperbolic line graph of U_n, $n \geq 6$

Let q be a prime power, let $n \geq 6$, and let U_n be an n-dimensional non-degenerate unitary vector space over $\mathbb{F}_{q^2}$ with polarity π. Let $\mathbf{G}(U_n)$ be the graph with the set of non-degenerate two-dimensional subspaces of U_n as set of vertices in which two vertices l and m are adjacent if and only if $l \subset m^\pi$. In anology to the preceding section, the aim of this section is to reconstruct the unitary vector space U_n from the hyperbolic line graph $\mathbf{G}(U_n)$.

Proposition 3.1 *Let $n \geq 8$. Then $\mathbf{G}(U_n)$ is a connected graph of diameter two.*

Proof Let l and k be two distinct vertices of the graph $\mathbf{G}(U_n)$. The space $H = \langle l, k\rangle$ has dimension three or four. Since it contains the hyperbolic lines l and m, the rank of H is at least two. Hence the radical of H has dimension at most two. The space H^π has dimension at least four and rank at least two, since $\mathrm{rad}(H^\pi) = \mathrm{rad}(H)$. Therefore $H^\pi = \langle k, l\rangle^\pi = k^\pi \cap l^\pi$ contains a hyperbolic line h, so that the distance between the vertices l and k is at most two. As $\mathbf{G}(U_n)$ obviously admits non-adjacent vertices, the diameter of $\mathbf{G}(U_n)$ is two. □

Proposition 3.2 *The graphs $\mathbf{G}(U_6)$ and $\mathbf{G}(U_7)$ are connected of diameter three.*

Proof We first study the graph $\mathbf{G}(U_6)$. Let l and m be distinct vertices of $\mathbf{G}(U_6)$. Then $P = \langle l, m\rangle$ is a three- or four-dimensional subspace of U_6.

Assume first that $P = \langle l, m\rangle$ is a plane. Then the planes P and P^π have rank two or three, because the hyperbolic line l and m are proper subspaces of P. Therefore the plane P^π contains a hyperbolic line h and thus the vertices l and m have distance two in $\mathbf{G}(U_6)$.

If $P = \langle l, m\rangle$ is a four-dimensional subspace of U_6, then P is of rank two, three, or four. In the case that P is a non-degenerate subspace, then of course P^π is a hyperbolic line and the vertices l and m have distance two. Finally, we assume that the four-dimensional space P is a singular subspace of U_6. We fix a point x in the radical of P. Then x is incident to the perpendicular space l^π of the hyperbolic line l, which is a non-degenerate four-dimensional subspace of U_6. We choose a hyperbolic line h in l^π passing through the point x in l^π and certainly the vertex h is adjacent to l in the hyperbolic line graph $\mathbf{G}(U_6)$. If $\langle h, m\rangle$ is a plane, then there exists a common neighbor of h and m by the above, yielding a path of length three from l to m in $\mathbf{G}(U_6)$. Hence we can assume that subspace of $\langle h, m\rangle$ is of dimension four. The rank of this space is four as well. Indeed, the Gram matrix of $\langle m, h\rangle$ is $G = \begin{pmatrix} 0 & 1 & 0 & \alpha \\ 1 & 0 & 0 & \beta \\ 0 & 0 & 0 & \delta \\ \overline{\alpha} & \overline{\beta} & \overline{\delta} & \gamma \end{pmatrix}$ with respect to a basis v_1^m, v_2^m, x^h, v_2^h of $\langle m, h\rangle$ such that the pair of vectors v_1^m, v_2^m is a hyperbolic pair of is line m, the vector x^h is some non-trivial vector of the point x and v_2^h is a non-trivial vector of a non-degenerate point of the line h. So $(x^h, v_2^h) = \delta \neq 0$. But that implies that the Gram matrix has determinant $\delta\overline{\delta} \neq 0$ and, hence, $\langle m, h\rangle$ is of dimension four. By the above h and m have distance two, so the vertices l and m are at mutual distance at most three in $\mathbf{G}(U_6)$.

We now turn our attention to the graph $\mathbf{G}(U_7)$. Let l and m be distinct vertices of $\mathbf{G}(U_7)$. Since the subspace $\langle l, m\rangle$ has dimension at most four and rank at least two, there exists a non-degenerate six-dimensional subspace W of U_7 containing l and m. By the above, the vertices l and m have distance at most three in the hyperbolic line graph $\mathbf{G}(W)$, which is a subgraph of $\mathbf{G}(U_7)$. Whence the diameter of $\mathbf{G}(U_7)$ is at most three.

In order to establish that the diameter of the graphs $\mathbf{G}(U_6)$ and $\mathbf{G}(U_7)$ is three, we have to find vertices that are not at mutual distance one or two. Choose a four-dimensional rank two subspace H of U_6 respectively of U_7. By Table 1 the subspace H contains q^8 hyperbolic lines and any point of this space is incident to q^4+q^2+1 different lines. Since $q^8 \geq (q^2+1)\cdot(q^4+q^2+1) = q^6+q^4+q^2+1$ we find two non-intersecting hyperbolic lines l and m of U_6 resp. U_7 spanning the subspace H. The pole $\langle l, m\rangle^\pi = H^\pi$ has dimension two or three, respectively, and rank zero or one, respectively. Hence $\mathbf{G}(U_6)$ resp. $\mathbf{G}(U_7)$ do not contain a common neighbor of l and m. Therefore the diameter of $\mathbf{G}(U_n)$ with $6 \leq n \leq 7$ is three. □

The next proposition describes two key properties of the hyperbolic line graph $\mathbf{G}(U_n)$ which will turn out to characterise $\mathbf{G}(U_n)$ for $n \geq 9$ (cf. Theorem 1).

Proposition 3.3 *Let $n \geq 5$. The hyperbolic line graph $\mathbf{G}(U_n)$ is connected, unless $(n, q) = (5, 2)$, and locally $\mathbf{G}(U_{n-2})$.*

Proof See Propositions 2.4, 3.1, 3.2. The local property is obvious. □

Definition 3.4 Let W be a subspace of U_n. The set of all hyperbolic lines of W is denoted by $\mathbf{L}(W)$.

Lemma 3.5 *Let $n \geq 6$ and let l and m be two distinct vertices of the graph $\mathbf{G}(U_n)$ such that $\{l, m\}^\perp \neq \emptyset$. Then $\{l, m\}^{\perp\perp} = \mathbf{L}(\langle l, m\rangle)$.*

Proof Since $\{l,m\}^{\perp\perp} = (\{l,m\}^{\perp})^{\perp} = \bigcap_{z\in\{l,m\}^{\perp}} z^{\perp} = \bigcap_{z\in\{l,m\}^{\perp}} \mathbf{L}(z^{\pi})$, obviously $\mathbf{L}(\langle l,m\rangle) \subseteq \{l,m\}^{\perp\perp}$.

Conversely, let k be a hyperbolic line of U_n not incident to the subspace $\langle l,m\rangle$. Then, of course, $\langle l,m\rangle^{\pi} \not\subseteq k^{\pi}$. The statement is proved, if we can find a hyperbolic line $h \subseteq \langle l,m\rangle^{\pi}$, which is not incident to the perpendicular space k^{π}. From the assumption that the induced subgraph $\{l,m\}^{\perp}$ is not empty it follows that $\mathrm{rad}(\langle l,m\rangle^{\pi})$ is properly contained in the subspace $\langle l,m\rangle^{\pi}$. We claim that the unitary space U_n contains some point y in the set $\langle l,m\rangle^{\pi}\backslash(k^{\pi} \cup \mathrm{rad}(\langle l,m\rangle^{\pi}))$. If $\mathrm{rad}(\langle l,m\rangle^{\pi}) \subseteq k^{\pi}$ then by De Morgan's laws $\langle l,m\rangle^{\pi}\backslash(k^{\pi} \cup \mathrm{rad}(\langle l,m\rangle^{\pi})) = \langle l,m\rangle^{\pi}\backslash k^{\pi} \cap \langle l,m\rangle^{\pi}\backslash\mathrm{rad}(\langle l,m\rangle^{\pi}) = \langle l,m\rangle^{\pi}\backslash\mathrm{rad}(\langle l,m\rangle^{\pi})$ and, of course, the set $\langle l,m\rangle^{\pi}\backslash\mathrm{rad}(\langle l,m\rangle^{\pi})$ contains some point y. On the other hand, if $\mathrm{rad}(\langle l,m\rangle^{\pi}) \not\subseteq k^{\pi}$, then $\mathrm{rad}(\langle l,m\rangle^{\pi}) \cup k^{\pi}$ is not a subspace of the vector space U_n and $\langle l,m\rangle^{\pi}$ is neither a subspace of $\mathrm{rad}(\langle l,m\rangle^{\pi})$ nor a subspace of k^{π}, thus the set $\langle l,m\rangle^{\pi}\backslash(k^{\pi} \cup \mathrm{rad}(\langle l,m\rangle^{\pi}))$ contains a point y.

An arbitrary two-dimensional subspace g of U_n containing the point y intersects the set $k^{\pi} \cup \mathrm{rad}(\langle l,m\rangle^{\pi})$ in at most two points by the fact that $\dim(k^{\pi} \cap g)$ as well as $\dim(\mathrm{rad}(\langle l,m\rangle^{\pi}) \cap g)$ is at most one. Therefore, we choose a hyperbolic line passing through y in $\langle l,m\rangle^{\pi}$ and find a singular point $x \in \langle l,m\rangle^{\pi}\backslash(k^{\pi} \cup \mathrm{rad}(\langle l,m\rangle^{\pi}))$. Using $x \not\subseteq \mathrm{rad}(\langle l,m\rangle^{\pi})$ we obtain a hyperbolic line h in $\langle l,m\rangle^{\pi}$ incident to the point x which is not contained in the subspace k^{π}. The lemma is now proved. □

A similar conclusion can be shown for three different vertices in the graph $\mathbf{G}(U_n)$.

Lemma 3.6 *Let $n \geq 6$ and k, l and m be three distinct vertices in $\mathbf{G}(U_n)$. Suppose the hyperbolic lines k, l, m intersect in a common point of U_n and satisfy $\{k,l,m\}^{\perp} \neq \emptyset$. Then $\mathbf{L}(\langle k,l,m\rangle) = \{k,l,m\}^{\perp\perp}$.*

Proof By assumption the subspace spanned by the hyperbolic lines k, l, m is of dimension three or four. Denote the common intersection of the three hyperbolic lines by p.

Suppose $\langle l,k,m\rangle$ is a plane. Then m is a hyperbolic line of $\langle l,k\rangle$ and, thus, $\langle l,k,m\rangle = \langle l,k\rangle$. Using Lemma 3.5 we obtain that $\mathbf{L}(\langle l,k,m\rangle) = \mathbf{L}(\langle l,k\rangle) = \{l,k\}^{\perp\perp}$.

If $\langle l,k,m\rangle$ is a four-dimensional subspace, we want to find a hyperbolic line h such that $\langle l,k,m\rangle = \langle l,h\rangle$. In case $\langle l,k,m\rangle$ has rank four, we choose $h = l^{\pi} \cap \langle l,k,m\rangle$. If the subspace $\langle l,k,m\rangle$ has rank two, take as h an arbitrary line in the complement of both l and $\mathrm{rad}(\langle l,k,m\rangle)$. Indeed, we can find such a line h in $\langle l,k,m\rangle$ by the fact that at most $2q^6 + 4q^4 + 4q^2 + 2$ of the $q^8 + q^6 + 2q^4 + q^2 + 1$ different lines of $\langle l,k,m\rangle$ intersect l or $\mathrm{rad}(\langle l,k,m\rangle)$. Certainly, h is a hyperbolic line since every complement of the radical of $\langle l,k,m\rangle$ is non-degenerate. Finally, if $\langle l,k,m\rangle$ has rank three, then consider the rank two plane $P = \langle k, \mathrm{rad}(\langle l,k,m\rangle)\rangle$. Since the hyperbolic lines k and l are distinct and intersect in a common point we have $\dim(l \cap P) = 1$. Moreover the radical of P coincides with the point $\mathrm{rad}(\langle l,k,m\rangle)$. In the plane P we choose the line h in the complement of both $\mathrm{rad}(\langle l,k,m\rangle)$ and $l \cap P$. Certainly the subspace h is non-degenerate. It follows from the construction that $\langle l,h\rangle = \langle l,P\rangle = \langle l,k,\mathrm{rad}(\langle l,k,m\rangle)\rangle = \langle k,l,m\rangle$ and, by Lemma 3.5, that $\mathbf{L}(\langle l,k,m\rangle) = \mathbf{L}(\langle l,h\rangle) = \{l,h\}^{\perp\perp}$.

For suitable $g \in \{h, k\}$, the equality between $\{l, k, m\}^{\perp\perp}$ and $\{l, g\}^{\perp\perp}$ follows from the fact that $\{l, k, m\}^{\perp\perp} = (\{l, k, m\}^{\perp})^{\perp} = \bigcap_{z \in \{l,k,m\}^{\perp}} z^{\perp} = \bigcap_{z \in l^{\perp} \cap k^{\perp} \cap m^{\perp}} z^{\perp} = \bigcap_{z \in \mathbf{L}(l^{\pi}) \cap \mathbf{L}(k^{\pi}) \cap \mathbf{L}(m^{\pi})} z^{\perp} = \bigcap_{z \in \mathbf{L}(\langle l,k,m\rangle^{\pi})} z^{\perp} = \bigcap_{z \in \mathbf{L}(\langle l,g\rangle^{\pi})} z^{\perp} = \bigcap_{z \in \{l,g\}^{\perp}} z^{\perp} = \{l, g\}^{\perp\perp}$. □

Our main goal in this section is to construct a point-line geometry from the graph $\mathbf{G}(U_n)$ which is isomorphic to the geometry on arbitrary one-dimensional subspaces and non-degenerate two-dimensional subspaces of U_n. We use the vertices of $\mathbf{G}(U_n)$ as lines. The points are going to be defined as pencils of lines. Therefore we now study properties of vertices of $\mathbf{G}(U_n)$ that allow us to characterise the situation when they correspond to intersecting hyperbolic lines of $\mathbf{G}(U_n)$.

Lemma 3.7 *Let $n \geq 6$. Two hyperbolic lines l and m intersect in a common point in the unitary polar space U_n if and only if $\{l, m\}^{\perp}$ is non-empty and $\{l, m\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_n)$ with respect to inclusion (i.e. for any pair of distinct hyperbolic lines $s_1, s_2 \in \{l, m\}^{\perp\perp}$ we have $\{s_1, s_2\}^{\perp\perp} = \{l, m\}^{\perp\perp}$).*

Proof Assume that two distinct hyperbolic lines l and m intersect in the point p in U_n, so that $\langle l, m\rangle$ is a plane of rank two or three. Since $n \geq 6$, the pole $\langle l, m\rangle^{\pi}$ of the plane $\langle l, m\rangle$ is a subspace of dimension at least three, which contains a hyperbolic line, since $\dim(\mathrm{rad}(\langle l, m\rangle^{\pi}) = \dim(\mathrm{rad}(\langle l, m\rangle)) \leq 1$. Hence $\{l, m\}^{\perp} \neq \emptyset$. Using Lemma 3.5, we have $\{l, m\}^{\perp\perp} = \mathbf{L}(\langle l, m\rangle)$. It follows immediately that $\{l, m\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_n)$ with respect to inclusion.

Conversely, assume l and m do not intersect. Therefore $\langle l, m\rangle$ is four-dimensional of rank two, three, or four. There exists a plane P of rank two in $\langle l, m\rangle$ containing distinct hyperbolic lines s_1 and s_2 spanning P. Lemma 3.5 yields $\{s_1, s_2\}^{\perp\perp} = \mathbf{L}(\langle s_1, s_2\rangle) \subsetneq \mathbf{L}(\langle l, m\rangle)$, thus $\{l, m\}^{\perp\perp}$ is not minimal in $\mathbf{G}(U_n)$ with respect to inclusion. □

We now generalise Lemma 3.7 to the situation of three lines. Three distinct pairwise intersecting hyperbolic lines k_1, k_2, k_3 intersect in one point in U_n, if we can find a hyperbolic line s such that:

- the hyperbolic line s intersects each hyperbolic line k_i with $s \neq k_i$ for $1 \leq i \leq 3$, and
- the space $\langle s, k_1, k_2\rangle$ is of dimension four.

Translated into graph language the above conditions say that three different mutually intersecting vertices k_1, k_2, k_3 intersect in one point if there exists a vertex s of $\mathbf{G}(U_n)$ such that:

- $\{s, k_i\}^{\perp} \neq \emptyset$ and $\{s, k_i\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_n)$ with respect to inclusion, if $k_i \neq s$, for $1 \leq i \leq 3$ (cf. Lemma 3.7), and
- $\{k_1, k_2\}^{\perp\perp} = \mathbf{L}(\langle k_1, k_2\rangle) \subsetneq \mathbf{L}(\langle k_1, k_1, s\rangle) = \{k_1, k_2, s\}^{\perp\perp}$.

Observe that for any two intersecting hyperbolic lines l and m, there indeed exists a hyperbolic line s in the vector space U_n such that $\langle l, m, s\rangle$ is a four-dimensional space and $\{l, m, s\}^{\perp} \neq \emptyset$.

Definition 3.8 Let $n \geq 6$ and let Γ be graph isomorphic to $\mathbf{G}(U_n)$. Two vertices k and l of Γ are defined to **intersect** if both $\{k,l\}^{\perp} \neq \emptyset$ and $\{k,l\}^{\perp\perp}$ is minimal in Γ with respect to inclusion. Three mutually intersecting vertices k_1, k_2, k_3 of Γ are defined to **intersect in one point** if there exists a vertex s in Γ with the following properties:

- the vertex s intersect with each vertex k_i, if $s \neq k_i$, for $1 \leq i \leq 3$,
- $\{k_1,k_2,s\}^{\perp} \neq \emptyset$ and $\{k_1,k_2\}^{\perp\perp} = \mathbf{L}(\langle k_1,k_2\rangle) \subsetneq \mathbf{L}(\langle k_1,k_1,s\rangle) = \{k_1,k_2,s\}^{\perp\perp}$.

An **interior point** of the graph Γ is a maximal set p of distinct pairwise intersecting vertices of Γ such that any three elements of p intersect in one point. We denote the set of all interior points of Γ by $\mathcal{I}$. Moreover, an **interior line** of the graph Γ is a vertex of the graph Γ. The set of all interior lines of Γ is denoted by $\mathcal{L}$.

The discussions in this section yield the following.

Proposition 3.9 *Let $n \geq 6$ and let Γ be a graph isomorphic to $\mathbf{G}(U_n)$. Then the point-line geometry $(\mathcal{I}, \mathcal{L}, \supset)$ is isomorphic to the geometry on arbitrary one-dimensional subspaces and non-degenerate two-dimensional subspaces of U_n.*

4 The hyperbolic geometry and its subspaces

We still have to distinguish singular one-dimensional subspaces from non-degenerate one-dimensional subspaces in the geometry $(\mathcal{I}, \mathcal{L})$ from Propositions 2.15 and 3.9.

Definition 4.1 Let p be an interior point of Γ. By Lemma A.1 the number $\mathcal{N}_p$ of hyperbolic lines incident to p is either equal to $q^{2(n-2)}$ or equal to $\frac{q^{n-2}\cdot(q^{n-1}-(-1)^{n-1})}{q+1}$. We call p an **interior singular point** of Γ, if $\mathcal{N}_p = q^{2(n-2)}$, and an **interior non-singular point** otherwise.

Notation We denote by $\mathbb{H}(U_n)$ the geometry of singular points and hyperbolic lines of an n-dimensional non-degenerate unitary polar space U_n over the field $\mathbb{F}_{q^2}$.

Proposition 4.2 *Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be the point-line geometry on interior singular points and on interior lines of Γ. Then $\mathcal{G} = (\mathcal{P}, \mathcal{L}) \cong \mathbb{H}(U_n)$.*

Proof This is obvious by Propositions 2.15 and 3.9. □

Corollary 4.3 *The automorphism group of $\mathbf{G}(U_n)$ is isomorphic to the automorphism group of the unitary space U_n.*

Proof Obviously, each automorphism of U_n induces an automorphism of $\mathbf{G}(U_n)$. □

Definition 4.4 Let $n \geq 5$ and let Γ be a graph isomorphic to $\mathbf{G}(U_n)$. Then the point-line geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ from Proposition 4.2 is called the **interior hyperbolic space** on Γ.

In Section 5 we will define geometry similar to the one in Proposition 4.2 on an arbitrary connected locally $\mathbf{G}(U_n)$ graph for $n \geq 7$. It is far from obvious how to determine the isomorphism type of that geometry, and accomplishing this task will take most of Section 5. The key tool will be [6, Theorem 1.2], which we restate below as Theorem 4.6 for the reader's convenience. Before doing so some explanation of notation and terminology in the context of point-line geometries is due.

Definition 4.5 Let $G = (P, L)$ be a point-line geometry. A **subspace** X of G is a subset of the point set P such that any line of L intersecting the set X in at least two points is completely contained in X. Using the observation that the intersection of subspaces again is a subspace, we define for each subset Y of the point set P the subspace $\langle Y\rangle$ generated by Y to be the intersection of all subspaces of G containing the set Y. Hence $\langle Y\rangle$ denotes the smallest subspace of G containing Y. A **plane** is a subspace of G generated by two intersecting lines. The point-line geometry G is called **planar** if any pair of intersecting lines are contained in a unique plane.

The **order** of a geometry G equals $k \in \mathbb{N}$, if all lines of G are incident with exactly $k+1$ points.

A **partially linear space** is a point-line geometry $G = (P, L)$ with the property that each line contains at least two different points and two different points are in at most one common line. We call two different points contained in a common line are **collinear**. A partial linear space is called **thick**, if all lines contain at least three points.

The **point graph** of G is the graph with vertex set P in which two different points are adjacent if and only if a, b are collinear. G is **connected**, if the point graph of G is a connected graph.

Moreover, in this paper non-collinearity is denoted by the symbol $\sim$. By convention, a point is non-collinear to itself.

Theorem 4.6 (Cuypers [6], Theorem 1.2) *Let $G = (P, L)$ be a non-linear, planar connected partially linear space of finite order $q \geq 3$. Suppose the following holds in G*:

1. *all planes are finite and either isomorphic to a dual affine plane or linear plane*;
2. *in a linear plane no four lines intersect in six points*;
3. *for all points x and y the inclusion $x^{\sim} \subseteq y^{\sim}$ implies $x = y$*;
4. *if E is a linear plane and x a point, then $x^{\sim} \cap E \neq \emptyset$.*

Then q is a prime power and G is isomorphic to the geometry of singular points and hyperbolic lines of a non-degenerate symplectic or unitary polar space over the field $\mathbb{F}_q$, respectively $\mathbb{F}_{q^2}$.

Recall that the geometries described in the conclusion of the preceding theorem are called the hyperbolic symplectic and hyperbolic unitary geometries.

Remark 4.7 As mentioned in the introduction, the hyperbolic unitary geometry of an n-dimensional finite hermitian space V is isomorphic to the geometry of long root subgroups (as points) and fundamental SL_2's (as lines) of the group $SU_n(q^2)$.

The long root subgroups of $SU_n(q^2)$ are abelian, conjugate in $SU_n(q^2)$ (as $SU_n(q^2)$ acts transitively on the set of isotropic one-dimensional subspaces of V), and generate $SU_n(q^2)$ (see, e.g., [2]). Moreover, depending on whether two isotropic one-dimensional subspaces a, b of V are perpendicular or not, the corresponding long root subgroups U_a and U_b commute or generate a (fundamental) SL_2. Hence the hyperbolic unitary geometry, and therefore also the geometry $\mathcal{G}$ studied in Proposition 4.2, is a geometry of transvection subgroups of $SU_n(q^2)$ in the sense of [5]. It follows from [5, Proposition 1.1] that $\mathcal{G}$ is a partially linear space satisfying assertions 1 and 3 of Theorem 4.6 (as $PSU_n(q^2)$ is a simple group). Since any linear plane of the geometry $\mathcal{G}$ in fact is isomorphic to a connected component of the geometry on the singular points and the hyperbolic lines of a classical hermitian unital, [10] implies that $\mathcal{G}$ also satisfies assertion 2. Assertion 4 is easily established and non-linearity of $\mathcal{G}$ is obvious. Finally, planarity of $\mathcal{G}$ follows from the fact that any two fundamental SL_2's of $SU_n(q^2)$ generate a unique subgroup of $SU_n(q^2)$.

It is clear from Theorem 4.6 that planes play a crucial role. Since in Section 5 we will prove and use that 'global' planes can seen 'locally', we require a concept of interior planes.

Definition 4.8 Distinct intersecting vertices k and l of $\Gamma \cong \mathbf{G}(U_n)$ are defined to **intersect in a singular point**, if their intersection is an interior singular point in the sense of Definitions 2.14 and 3.8. The **interior geometric plane** spanned by k and l is the smallest subspace of $\mathcal{G}$ containing k and l; this interior geometric plane is denoted by $\langle k, l\rangle_{\mathcal{G}}^{g}$.

Notation We always add the superscript ^{g} to each geometric plane of $\mathcal{G}$ in order to distinguish it from the projective span of k and l, i.e., the projective plane $\langle k, l\rangle$, in the ambient projective space $\mathbb{P}(U_n)$.

In the remainder of this section we will study subspaces of the interior hyperbolic space $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ on $\mathbf{G}(U_n)$ which are induced by the embedding $\mathbf{G}(U_{n-2}) \cong x^{\perp} \subset \mathbf{G}(U_n)$ for $x \in \mathbf{G}(U_n)$. By Proposition 4.2 we can construct the interior hyperbolic space $\mathcal{G}_x = (\mathcal{P}_x, \mathcal{L}_x)$ of the graph $x^{\perp} \cong \mathbf{G}(U_{n-2})$, which is isomorphic to the geometry of singular points and hyperbolic lines of the non-degenerate unitary space U_{n-2}. The corresponding non-degenerate unitary form $(\cdot,\cdot)_x$ of $\mathcal{G}_x$ can be identified with the restriction $(\cdot,\cdot)_{|x^{\pi}}$ of the unitary form $(\cdot,\cdot)$ on U_n. In this context the elements of the geometry $\mathcal{G}_x$ are called **local**.

Notation We index every local object of the interior hyperbolic space $\mathcal{G}_x$ with the vertex x. In particular, for vertices l, k, m of $x^{\perp}$ we use the notations $\{k, l, m\}_x^{\perp} = \{k, m, l\}^{\perp} \cap x^{\perp}$ and $\{k, l, m\}_x^{\perp\perp} = (\{k, l, m\}_x^{\perp})_x^{\perp} = (\{k, l, m\}_x^{\perp})^{\perp} \cap x^{\perp} = (\{k, m, l\}^{\perp} \cap x^{\perp})^{\perp} \cap x^{\perp}$. With $\langle l, k\rangle_x$ we denote the vector subspace of $x^{\pi} \cong U_{n-2}$ generated by the two interior lines l and k of $\mathcal{G}_x$.

We show that the interior hyperbolic space $\mathcal{G}_x$ is isomorphic to a subspace of codimension two of the interior hyperbolic space $\mathcal{G}$. We also prove that each singular interior point $p_x \in \mathcal{G}_x$ is contained in a unique singular interior point of $\mathcal{G}$ and, conversely, that for any singular interior point p of the geometry $\mathcal{G}$ either $p \cap \mathcal{L}_x$ is empty or a singular interior point of $\mathcal{G}_x$.

We concentrate on the case $n = 7$, the general case being left as an exercise for the reader.

Lemma 4.9 *Let p be a singular interior point in $\mathcal{G}$. If $l, m \in p \cap \mathcal{L}_x$ are distinct elements, then the interior lines l and m intersect in a singular interior point of $\mathcal{G}_{\mathbf{x}}$.*

Proof We need to establish the defining properties from Definition 2.14 for l and m. Therefore we have to verify that either $\{l, m\}_x^{\perp} \neq \emptyset$ or the vertices l and m have distance four in $x^{\perp}$ with more than q^4 different paths of length four between these two vertices in $x^{\perp}$. In $\mathbf{G}(U_7)$ both vertices l and m are adjacent to x as $l, m \in \mathcal{L}_x$. Furthermore $\langle l, m\rangle$ is a three-dimensional subspace and contained in x^{π}. If the plane $\langle l, m\rangle$ is non-degenerate, then $\{l, m\}_x^{\perp} \neq \emptyset$ by Remark 2.5. If on the other hand the subspace $\langle l, m\rangle$ is degenerate, then by Lemma 2.3 the vertices l and m have distance four in the induced subgraph $x^{\perp}$. By Lemma 2.11, the graph $x^{\perp}$ contains more than q^4 different paths of length four between l and m. Hence the interior lines l and m intersect in the interior hyperbolic space $\mathcal{G}_x$. □

Lemma 4.10 *Let p be a singular interior point of $\mathcal{G}$ and k_1, k_2, and k_3 be pairwise distinct elements of $p \cap \mathcal{L}_x$. Then the interior lines k_1, k_2, and k_3 intersect in one interior singular point of $\mathcal{G}_{\mathbf{x}}$.*

Proof In order to prove the claim we show that k_1, k_2, k_3 satisfy the properties of Definition 2.14. By Lemma 4.9 the interior lines k_1, k_2, k_3 intersect pairwise in a singular interior point of $\mathcal{G}_x$. Furthermore the vector subspace of U_7 spanned by the hyperbolic lines k_1, k_2 and k_3 is a subspace of x^{π} and since the vertices k_1, k_2, k_3 are elements of p, the one-dimensional subspace $d = k_1 \cap k_2 \cap k_3$ is contained in x^{π} as well. This setup satisfies the hypothesis of Lemma 2.13 implying that the subspace x^{π} contains a hyperbolic line s such that

- $\{s, k_i\}_x^{\perp} \neq \emptyset$, if $s \neq k_i$, for $i = 1, 2, 3$,
- $\{s, k_1, k_2\}_x^{\perp} = \emptyset$.

Hence by Definition 2.14 the three vertices k_1, k_2 and k_3 of $p \cap \mathcal{L}_x$ intersect in one interior point of $\mathcal{G}_x$, which is singular, cf. Definition 4.1. □

Proposition 4.11 *Let p be an interior singular point in $\mathcal{G}$. The interior line set $p \cap \mathcal{L}_x$ is either an interior singular point p_x in $\mathcal{G}_x$ or the empty set.*

Proof Suppose $p \cap \mathcal{L}_x \neq \emptyset$, then let l be some element of $p \cap \mathcal{L}_x$ and m be an interior line of the point p different from l. Since $l \perp x$, it follows that the hyperbolic line l is a subspace of x^{π} in $\mathcal{G}$ which intersects the hyperbolic line m in a one-dimensional singular subspace d. Hence the singular point d is also a subspace of x^{π}. Let p_x be the interior point of $\mathcal{G}_x$ containing all hyperbolic lines of x^{π} incident to the point d.

Let k be an arbitrary hyperbolic line of the interior point p_x. The proposition is proved, if the vertex k is an element of $p \cap \mathcal{L}_x$. Since $k \subseteq x^{\pi}$ it suffices to prove $k \in p$.

Any element n of the interior point p is a hyperbolic line of U_7 incident to the point d. Thus we choose a vertex $n \in p$ distinct from k and intend to prove that $\{k,n\}^{\perp} \neq \emptyset$ and that $\{k,n\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_7)$ with respect to inclusion, cf. Definition 3.8. Since both hyperbolic lines k and n contain the point d in U_7, the vector space spanned by both is a plane of rank at least two. Hence $\langle k,n\rangle^{\pi}$ is four-dimensional subspace of rank at least three, thus $\langle k,n\rangle^{\pi}$ contains a hyperbolic line. In particular, $\{k,n\}^{\perp} \neq \emptyset$ and due to Lemma 3.5 we have $\{k,n\}^{\perp\perp} = \mathbf{L}(\langle k,n\rangle)$. By Lemma 3.5 again and the fact that the span of two different hyperbolic lines s_1, s_2 of the three-dimensional subspace $\langle k,n\rangle$ again is this plane, we obtain the equality $\{k,n\}^{\perp\perp} = \mathbf{L}(\langle k,n\rangle) = \mathbf{L}(\langle s_1,s_2\rangle) = \{s_1,s_2\}^{\perp\perp}$. Therefore $\{k,n\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_7)$ with respect to inclusion.

Next, we choose two different elements n and m of p. By the argumentation above n, m and k are three mutually intersecting interior lines of $\mathcal{G}$ and the subspace $\langle n,m,k\rangle$ of U_7 is of dimension three or four. If $\langle n,m,k\rangle$ is a non-degenerate four-dimensional subspace, then $\langle n,m,k\rangle^{\pi}$ is a non-degenerate plane in $\mathcal{G}$ containing some hyperbolic line. Hence the subgraph $\{k,m,n\}^{\perp}$ is not empty and by Lemma 3.5 and Lemma 3.6 it follows directly that $\{k,n\}^{\perp\perp} = \mathbf{L}(\langle k,n\rangle) \subsetneq \mathbf{L}(\langle k,n,m\rangle) = \{k,m,n\}^{\perp\perp}$. If otherwise the subspace $\langle n,m,k\rangle$ is of dimension three or degenerate and of dimension four, then there exists a hyperbolic line s in the unitary vector space $\mathcal{G}$ intersecting the lines m and n (and consequently k) in d such that $\langle s,n,m\rangle$ is four-dimensional non-degenerate subspace. This implies that $\{s,n,m\}^{\perp} \neq \emptyset$ and again we get the inequality $\{m,n\}^{\perp\perp} = \mathbf{L}(\langle m,n\rangle) \subsetneq \mathbf{L}(\langle s,n,m\rangle) = \{s,m,n\}^{\perp\perp}$, thus $k \in p$ by Definition 3.8. □

Lemma 4.12 *Let p_x be a singular interior point of $\mathcal{G}_x$ for some vertex x in $\mathbf{G}(U_7)$ and l and m be distinct elements of p_x. Then l and m intersect in a singular interior point of $\mathcal{G}$.*

Proof By Definition 3.8 the vertices $l,m \in p_x$ intersect in $\mathbf{G}(U_7)$, if $\{l,m\}^{\perp} \neq \emptyset$ and $\{l,m\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_7)$ with respect to inclusion. Since l and m are vertices of the induced subgraph $x^{\perp}$ of $\mathbf{G}(U_7)$, we conclude that $x \in \{l,m\}^{\perp}$. By Lemma 3.5 we have $\{l,m\}^{\perp\perp} = \mathbf{L}(\langle l,m\rangle)$. The plane $\langle l,m\rangle$ is a subspace of x^{π}, since l and m are incident to x^{π}. This implies $\langle l,m\rangle = \langle l,m\rangle_x$ and $\mathbf{L}(\langle l,m\rangle) = \mathbf{L}(\langle l,m\rangle_x)$.

Next, let s and t be two different vertices of $\{l,m\}^{\perp\perp}$, by the identities above $s,t \in \{l,m\}^{\perp\perp} = \mathbf{L}(\langle l,m\rangle) = \mathbf{L}(\langle l,m\rangle_x) = \langle l,m\rangle_x$. In fact the interior lines s and t span the plane $\langle l,m\rangle$ in $\mathcal{G}$. Moreover $\{l,m\}^{\perp\perp} = \bigcap_{z\in\{k,l\}^{\perp}} z^{\perp}$, so $\{l,m\}^{\perp\perp} \subseteq x^{\perp}$, which implies that s and t are vertices of the subgraph $x^{\perp}$. Again, $\langle s,t\rangle = \langle s,t\rangle_x$ and $\{s,t\}^{\perp\perp} = \mathbf{L}(\langle s,t\rangle) = \mathbf{L}(\langle s,t\rangle_x)$. Therefore $\{s,t\}^{\perp\perp} = \mathbf{L}(\langle s,t\rangle) = \mathbf{L}(\langle l,m\rangle) = \{l,m\}^{\perp\perp}$, which shows that the double perp $\{l,m\}^{\perp\perp}$ is minimal in the graph $\mathbf{G}(U_7)$ with respect to inclusion. □

Lemma 4.13 *Let p_x be an interior point of $\mathcal{G}_x$. Any three distinct vertices k_1, k_2 and k_3 of p_x intersect in one point in $\mathcal{G}$.*

Proof By the previous Lemma 4.12 any three distinct lines k_1, k_2 and k_3 of an interior point $p_x \in \mathcal{P}_x$ are mutually intersecting interior lines in the interior hyperbolic space $\mathcal{G}$. Moreover the induced subgraph $x^\perp$ contains a vertex s with the properties that $\{s,k_i\}_x^\perp \neq \emptyset$ in $x^\perp$ if $k_i \neq s$ for $i \in \{1,2,3\}$ and $\{s,k_1,k_2\}_x^\perp = \emptyset$. Thus the plane $\langle k_1,k_2\rangle_x$ is properly contained in the four-dimensional subspace $\langle k_1,k_2,s\rangle_x$ of x^π in $\mathcal{G}$, so $\langle k_1,k_2\rangle = \langle k_1,k_2\rangle_x \subsetneqq \langle k_1,k_2,s\rangle_x = \langle k_1,k_2,s\rangle$.

Furthermore the vertex s is also an interior line of the space $\mathcal{G}$ and by Lemma 4.12 the interior line s intersects each interior line k_i different from s in $\mathcal{G}$ for $i \in \{1,2,3\}$. The proof of the statement is finished if we can show that $\{k_1,k_2\}^{\perp\perp} \subsetneqq \{k_1,k_2,s\}^{\perp\perp}$ in $\mathbf{G}(U_7)$. The interior lines k_1,k_2 and s are vertices of $x^\perp$ thus $\{k_1,k_2,s\}^{\perp\perp} = \mathbf{L}(\langle k_1,k_2,s\rangle) = \mathbf{L}(\langle k_1,k_2,s\rangle_x)$ by Lemma 3.6 and the fact that $x \in \{k_1,k_2,s\}^\perp$. Using Lemma 3.5 we get equality between the vertex set of the induced subgraph $\{k_1,k_2\}^{\perp\perp}$ and the hyperbolic lines set $\mathbf{L}(\langle k_1,k_2\rangle) = \mathbf{L}(\langle k_1,k_2\rangle_x)$. Finally we obtain the equalities $\{k_1,k_2\}^{\perp\perp} = \mathbf{L}(\langle k_1,k_2,\rangle_x) \subsetneq \mathbf{L}(\langle k_1,k_2,s\rangle_x) = \{k_1,k_2,s\}^{\perp\perp}$, and we are done. $\square$

Proposition 4.14 *Let p_x be an interior point of $\mathcal{G}_x$. There is an unique interior point p in the interior hyperbolic space of $\mathbf{G}(U_7)$ such that $p_x \subseteq p$.*

Proof Suppose the interior hyperbolic space $\mathcal{G}$ contains two different interior points p and g such that $p_x \subseteq p$ and $p_x \subseteq g$. Then let k be an interior line of p which is not incident to g and let l_1 and l_2 be two different interior lines of p_x. In the unitary polar space $\mathcal{G}$ the two different hyperbolic lines l_1 and l_2 intersect in the point p, but on the other hand $p = k \cap l_1 = l_2 \cap k = l_1 \cap l_2 = g$, contradiction. $\square$

The lines set $\mathcal{L}_x$ of the interior hyperbolic space $\mathcal{G}$ is a subset of the interior line set $\mathcal{L}$, also every interior point p_x of $\mathcal{P}_x$ is contained in an unique point p of the interior hyperbolic space $\mathcal{G}$, thus the interior hyperbolic space $\mathcal{G}_x$ is a subspace of the interior hyperbolic space $\mathcal{G}$. In the next proposition we also determine the dimension of the subspace $\mathcal{G}_x$ in the interior hyperbolic space $\mathcal{G}$.

Proposition 4.15 *Let $n \geq 7$ and let x be a vertex of the graph $\mathbf{G}(U_n)$. The interior hyperbolic space $\mathcal{G}_x$ on $x^\perp$ is isomorphic to a codimension two subspace of the interior hyperbolic space $\mathcal{G}$ on $\mathbf{G}(U_n)$.*

Proof Since $\mathcal{G}_x \cong \mathbb{H}(U_5)$ and $\mathcal{G} \cong \mathbb{H}(U_7)$, the claim follows for $n = 7$. The proof for general $n \geq 7$ is similar. The details are left to the reader as an exercise. $\square$

5 The global space

In this section we analyse the following situation. Let $n \geq 7$ and let Γ be a connected graph which is locally isomorphic to the hyperbolic line graph $\mathbf{G}(U_n)$. At the end of this section we prove Theorem 1, i.e., we prove that Γ is isomorphic to the hyperbolic line graph $\mathbf{G}(U_{n+2})$.

Due to the property that for every vertex $\mathbf{x}$ of Γ the induced subgraph $\mathbf{x}^\perp$ is isomorphic to $\mathbf{G}(U_n)$, we can construct the interior hyperbolic spaces $\mathcal{G}_\mathbf{x}$ on $\mathbf{x}^\perp$, see Proposition 3.9 and Proposition 4.2. We use this family $(\mathcal{G}_\mathbf{x})_{\mathbf{x}\in\Gamma}$ of local interior hyperbolic spaces to construct a global geometry $\mathcal{G}_\Gamma$ on Γ, which via Theorem 4.6 will turn out to be isomorphic to the geometry on the singular points and the hyperbolic lines of some unitary polar space. This enables us to identify $\mathcal{G}_\Gamma$ with $\mathbb{H}(U_{n+2})$ and Γ with $\mathbf{G}(U_{n+2})$.

Interior objects are a priori only defined in some interior hyperbolic space $\mathcal{G}_x$, $x \in \Gamma$. They are called **local objects**. Therefore one problem we have to tackle in this section is to introduce well-defined global points and lines for our point-line geometry $\mathcal{G}_\Gamma$. After that we will establish the validity of the hypotheses of Theorem 4.6 for $\mathcal{G}_\Gamma$.

Notation To avoid confusion, we will index every local object by the vertex $\mathbf{x}$ whose interior hyperbolic space it belongs to. For example, if $\mathbf{x} \perp \mathbf{y}$ in the graph Γ, then $\mathbf{y}$ is a vertex of the subgraph $\mathbf{x}^\perp$ corresponding to the local object $y_\mathbf{x}$, an interior line, in the space $\mathcal{G}_\mathbf{x}$. By $\mathbf{y}_\mathbf{x}$ we denote the vertex $\mathbf{y}$ considered as a vertex of the subgraph $\mathbf{x}^\perp$. With the symbol $\mathbf{y}_\mathbf{x}^\perp$ we denote the subgraph $\{\mathbf{x}, \mathbf{y}\}^\perp$ which is of course an induced subgraph of $\mathbf{x}^\perp$. The interior hyperbolic space obtained from the graph $\mathbf{y}_\mathbf{x}^\perp$ will be denoted with $\mathcal{G}_{\mathbf{y}_\mathbf{x}}$. Furthermore, by $\langle y_\mathbf{x}, z_\mathbf{x}\rangle$ we denote the projective space of the two interior lines $y_\mathbf{x}$ and $z_\mathbf{x}$ in $\mathcal{G}_\mathbf{x}$.

Definition 5.1 A **global line** of Γ is a vertex of the graph Γ. The set of all global lines of Γ is denoted by $\mathcal{L}_\Gamma$.

Lemma 5.2 *Let $n \geq 7$ and let $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be vertices of Γ with the property that $\mathbf{z}\perp\mathbf{x}\perp\mathbf{w}\perp\mathbf{y}\perp\mathbf{z}$. Assume that the vertices $\mathbf{w}$ and $\mathbf{z}$ are connected by a path in the induced subgraph $\{\mathbf{x}, \mathbf{y}\}^\perp$ of Γ. Then $\{\mathbf{x}_\mathbf{w}, \mathbf{y}_\mathbf{w}\}_\mathbf{w}^{\perp\perp} = \{\mathbf{x}_\mathbf{z}, \mathbf{y}_\mathbf{z}\}_\mathbf{z}^{\perp\perp}$. In particular, the spaces $\langle x_\mathbf{w}, y_\mathbf{w}\rangle$ and $\langle x_\mathbf{z}, y_\mathbf{z}\rangle$ have equal global line sets and can be identified.*

Proof By assumption there exist vertices $\mathbf{c}^1, \ldots, \mathbf{c}^n$ of the graph Γ such that $\mathbf{z} \perp \mathbf{c}^1 \perp \mathbf{c}^2 \perp \ldots \perp \mathbf{c}^n \perp \mathbf{w}$ is a path from $\mathbf{z}$ to $\mathbf{w}$ in $\{\mathbf{x}, \mathbf{y}\}^\perp$. Since $\mathbf{c}^1 \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}^\perp$ the hyperbolic lines $x_\mathbf{z}$ and $y_\mathbf{z}$ are perpendicular to the line $c_\mathbf{z}^1$ in the interior hyperbolic space $\mathcal{G}_\mathbf{z}$. Hence the projective span of $x_\mathbf{z}$ and $y_\mathbf{z}$ is perpendicular to the hyperbolic line $c_\mathbf{z}^1$ in $\mathcal{G}_\mathbf{z}$. In particular, all hyperbolic lines contained in $\langle x_\mathbf{z}, y_\mathbf{z}\rangle$ are adjacent to $\mathbf{c}_\mathbf{z}^1$ and $\langle x_\mathbf{z}, y_\mathbf{z}\rangle$ can be identified with a subspace of $\mathcal{G}_{\mathbf{c}_\mathbf{z}^1}$, whence with a subspace of $\mathcal{G}_{\mathbf{c}^1}$, cf. Propositions 2.15, 3.9, 4.15. Hence, by Lemma 3.5, we have $\{\mathbf{x}_\mathbf{z}, \mathbf{y}_\mathbf{z}\}_\mathbf{z}^{\perp\perp} = \{\mathbf{x}_{\mathbf{c}^1}, \mathbf{y}_{\mathbf{c}^1}\}_{\mathbf{c}^1}^{\perp\perp}$. Repeating the above argument along the path $\mathbf{z} \perp \mathbf{c}^1 \perp \ldots \perp \mathbf{c}^n \perp \mathbf{w}$, we obtain $\{\mathbf{x}_\mathbf{w}, \mathbf{y}_\mathbf{w}\}_\mathbf{w}^{\perp\perp} = \{\mathbf{x}_\mathbf{z}, \mathbf{y}_\mathbf{z}\}_\mathbf{z}^{\perp\perp}$. □

Notation Let $\mathbf{z} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$ be a chain of vertices of Γ. Consider the projective span $\langle z_\mathbf{x}, y_\mathbf{x}\rangle$ and its pole $H_{z_\mathbf{x}, y_\mathbf{x}}^\mathbf{x} = z_\mathbf{x}^\pi \cap y_\mathbf{x}^\pi = \langle z_\mathbf{x}, y_\mathbf{x}\rangle^\pi$ in $\mathcal{G}_\mathbf{x}$. Since this subspace $H_{z_\mathbf{x}, y_\mathbf{x}}^\mathbf{x}$ is perpendicular to the hyperbolic line $y_\mathbf{x}$ in $\mathcal{G}_\mathbf{x}$, it can be identified with a unique subspace of $\mathcal{G}_\mathbf{y}$, denoted by $H_{z_\mathbf{x}, y_\mathbf{x}}^\mathbf{y}$. We emphasise that this identification requires an application of Proposition 2.15 or 3.9 to the graph $\{\mathbf{x}, \mathbf{y}\}^\perp \cong \mathbf{G}(U_{n-2})$.

Lemma 5.3 *Let $n \geq 7$ and $\mathbf{z} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$ be a chain in Γ such that rank of the subspace $\langle H_{z_\mathbf{x}, y_\mathbf{x}}^\mathbf{y}, x_\mathbf{y}\rangle$ of $\mathcal{G}_\mathbf{y}$ is at least* $\max\{n-4, 6\}$. *Then there is a vertex $\mathbf{h} \in \{\mathbf{z}, \mathbf{y}, \mathbf{w}\}^\perp$ in the same connected component as $\mathbf{x}$ in $\{\mathbf{y}, \mathbf{z}\}^\perp$.*

Proof By hypothesis the rank of $W_\mathbf{y} := \langle x_\mathbf{y}, H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}\rangle \subseteq \mathcal{G}_\mathbf{y}$ is at least six. Thus $W_\mathbf{y}$ contains a six-dimensional non-degenerate space $V_\mathbf{y}$. It follows that the space $V_\mathbf{y} \cap H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}$ contains a four-dimensional subspace of rank at least two, as $H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}$ has codimension two in $W_\mathbf{y}$. Hence there exists a hyperbolic line $k_\mathbf{y}$ in $V_\mathbf{y} \cap H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}$. Also, the intersection $V_\mathbf{y} \cap w^\pi_\mathbf{y}$ contains a four-dimensional subspace of rank at least two, so there also exists a hyperbolic line $h_\mathbf{y}$ in $V_\mathbf{y} \cap w^\pi_\mathbf{y}$. The local line $h_\mathbf{y}$ leads to a vertex $\mathbf{h} \in \{\mathbf{y},\mathbf{w}\}^\perp$ and the local line $k_\mathbf{y}$ corresponds to a vertex $\mathbf{k} \in \{\mathbf{y},\mathbf{x}\}^\perp$. Local analysis of $\mathbf{x}^\perp$ and $\mathcal{G}_\mathbf{x}$ shows $\mathbf{k} \perp \mathbf{z}$. Indeed $k_\mathbf{y}$ is a hyperbolic line of $V_\mathbf{y} \cap H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}} \subseteq H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}$ and $\mathbf{k} \perp \mathbf{x}$ in Γ it follows that the hyperbolic line $k_\mathbf{x}$ is contained in $H^\mathbf{x}_{z_\mathbf{x},y_\mathbf{x}} \subseteq z^\pi_\mathbf{x}$, thus $\mathbf{k}$ and $\mathbf{z}$ are two adjacent vertices of Γ. By Proposition 3.3 we can find a path from $\mathbf{k}$ to $\mathbf{h}$ in the graph $\mathbf{G}(V_\mathbf{y}) \subseteq \mathbf{y}^\perp$. In particular, the vertex $\mathbf{h}$ lies in the same connected component of $\mathbf{y}^\perp$ as the vertex $\mathbf{x}$.

Let $\mathbf{s}^0 \perp \mathbf{s}^1 \perp \cdots \perp \mathbf{s}^m$ be a path from $\mathbf{k} = \mathbf{s}^0$ to $\mathbf{h} = \mathbf{s}^m$ in $\mathbf{G}(V_\mathbf{y})$. To finish the proof it suffices to prove that $\mathbf{s}^0 \perp \mathbf{s}^1 \perp \cdots \perp \mathbf{s}^m$ is a path in the induced subgraph $\mathbf{z}^\perp$. We proceed by induction. The vertex $\mathbf{k}$ is adjacent to $\mathbf{z}$ by construction. We have $M_\mathbf{y} := k^\pi_\mathbf{y} \cap W_\mathbf{y} = k^\pi_\mathbf{y} \cap \langle x_\mathbf{y}, H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}\rangle = \langle x_\mathbf{y}, k^\pi_\mathbf{y} \cap H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}\rangle$, because $\mathbf{x} \perp \mathbf{k}$. Notice that $M_\mathbf{y} = \langle x_\mathbf{y}, k^\pi_\mathbf{y} \cap H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}\rangle$ is a $\dim(H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}})$-dimensional subspace of $k^\pi_\mathbf{y} \subseteq \mathcal{G}_\mathbf{y}$. Considering this space inside the interior hyperbolic space $\mathcal{G}_\mathbf{k}$, denoted by $M_\mathbf{k}$, we obtain $\dim(M_\mathbf{k}) = \dim(M_\mathbf{y}) = \dim(H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}) = \dim(H^\mathbf{x}_{z_\mathbf{x},y_\mathbf{x}}) = \dim(H^\mathbf{k}_{z_\mathbf{k},y_\mathbf{k}})$ by Lemma 5.2, where $H^\mathbf{k}_{z_\mathbf{k},y_\mathbf{k}} = z^\pi_\mathbf{k} \cap y^\pi_\mathbf{k} = \langle z_\mathbf{k}, y_\mathbf{k}\rangle^\pi$. Furthermore, $M_\mathbf{k} = \langle x_\mathbf{k}, (k^\pi_\mathbf{y} \cap H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}})^\mathbf{k}\rangle \subseteq H^\mathbf{k}_{z_\mathbf{k},y_\mathbf{k}}$, whence $M_\mathbf{k} = H^\mathbf{k}_{z_\mathbf{k},y_\mathbf{k}}$. Here $(k^\pi_\mathbf{y} \cap H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}})^\mathbf{k}$ denotes the subspace of $\mathcal{G}_\mathbf{k}$ corresponding to the subspace $k^\pi_\mathbf{y} \cap H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}$ of $\mathcal{G}_\mathbf{y}$. Consequently, $k^\pi_\mathbf{y} \cap W_\mathbf{y} = M_\mathbf{k} = H^\mathbf{y}_{z_\mathbf{k},y_\mathbf{k}}$ and in particular, $W_\mathbf{y} = \langle k_\mathbf{y}, H^\mathbf{y}_{z_\mathbf{k},y_\mathbf{k}}\rangle$.

By induction we assume that the vertices $\mathbf{s}^i$ with $i \leq n$, $n \in \mathbb{N}$, are adjacent to $\mathbf{z}$. Then a similar argument as in the paragraph above yields $W_\mathbf{y} = \langle s^i_\mathbf{y}, H^\mathbf{y}_{z_{\mathbf{s}^i},y_{\mathbf{s}^i}}\rangle$ and $(s^i_\mathbf{y})^\pi \cap W_\mathbf{y} = H^\mathbf{y}_{z_{\mathbf{s}^i},y_{\mathbf{s}^i}}$, whence $W_\mathbf{y} = \langle s^i_\mathbf{y}, H^\mathbf{y}_{z_{\mathbf{s}^i},y_{\mathbf{s}^i}}\rangle$ for $i = 1, \cdots, n$. The vertex $\mathbf{s}^{n+1}$ is adjacent to $\mathbf{y}$ and $\mathbf{s}_n$ in the graph Γ. Moreover, $s^{n+1}_\mathbf{y}$ is a hyperbolic line of the subspace $V_\mathbf{y}$ in the interior hyperbolic space $\mathcal{G}_\mathbf{y}$. Thus $s^{n+1}_\mathbf{y}$ is a hyperbolic line of the $(\dim(V_\mathbf{y})-2)$-dimensional subspace $(s^n_\mathbf{y})^\pi \cap V_\mathbf{y}$ in $\mathcal{G}_\mathbf{y}$. Since $(s^n_\mathbf{y})^\pi \cap V_\mathbf{y}$ is a subspace of $(s^n_\mathbf{y})^\pi \cap W_\mathbf{y} = H^\mathbf{y}_{z_{\mathbf{s}^n},y_{\mathbf{s}^n}}$ it follows that $s^{n+1}_\mathbf{y} \subseteq (s^n_\mathbf{y})^\pi \cap W_\mathbf{y} = H^\mathbf{y}_{z_{\mathbf{s}^n},y_{\mathbf{s}^n}}$. Therefore the vertex $\mathbf{s}^{n+1}$ is adjacent to $\mathbf{z}$. □

Lemma 5.4 *Let $n \in \{7, 8\}$ and let $\mathbf{z} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$ be a path in Γ such that*

- *the subspace $\langle H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}, x_\mathbf{y}\rangle$ of $\mathcal{G}_\mathbf{y}$ is of dimension six and of rank five, or*
- *the subspace $\langle H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}, x_\mathbf{y}\rangle$ of $\mathcal{G}_\mathbf{y}$ is a non-degenerate subspace of dimension five and $\langle z^\pi_\mathbf{x} \cap y^\pi_\mathbf{x}, x_\mathbf{y}\rangle \cap w^\pi_\mathbf{y}$ of rank at least two.*

Then there is a vertex $\mathbf{h} \in \{\mathbf{z},\mathbf{y},\mathbf{w}\}^\perp$ in the same connected component as $\mathbf{x}$ in $\{\mathbf{y},\mathbf{z}\}^\perp$.

Proof We will prove this statement in a way similar to the proof of Lemma 5.3, using the same notation.

First we assume that the subspace $W_\mathbf{y} = \langle H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}, x_\mathbf{y}\rangle$ is of dimension six and of rank five, which implies that $H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}$ is a four-dimensional subspace in $\mathcal{G}_\mathbf{y}$ of rank three. The radical of $H^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}$ coincides with the radical of $W_\mathbf{y}$. Furthermore $W_\mathbf{y} \cap w^\pi_\mathbf{y}$

is at least four-dimensional of rank at least two as $w_{\mathbf{y}}^{\pi}$ is a $(n-2)$-dimensional non-degenerate subspace of $\mathcal{G}_{\mathbf{y}}$. Thus we can fix a hyperbolic line $h_{\mathbf{y}}$ in $W_{\mathbf{y}} \cap w_{\mathbf{y}}^{\pi}$. In the case that $h_{\mathbf{y}}$ can be chosen to lie inside the subspace $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$, then there is nothing else to prove, so we may assume for the rest of this proof that $h_{\mathbf{y}} \not\subseteq H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$. Next we choose a non-radical point $s_{\mathbf{y}}$ of $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ in the subspace $h_{\mathbf{y}}^{\pi} \cap H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$, which is at least of dimension two. If possible, we choose $s_{\mathbf{y}}$ to be singular and fix a hyperbolic line $l_{\mathbf{y}}$ in $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ going through $s_{\mathbf{y}}$. This construction implies immediately that the hyperbolic lines $h_{\mathbf{y}}$ and $l_{\mathbf{y}}$ span a non-degenerate four-dimensional space inside the subspace $W_{\mathbf{y}}$, which is contained in some five-dimensional non-degenerate subspace $V_{\mathbf{y}}$ of $W_{\mathbf{y}}$.

If $s_{\mathbf{y}}$ has to be chosen non-degenerate, then we pick a hyperbolic line $l_{\mathbf{y}}$ incident to $s_{\mathbf{y}}$ and not intersecting the line $h_{\mathbf{y}}$ in $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ in such a way that the radical of $\langle l_{\mathbf{y}}, h_{\mathbf{y}}\rangle$ is different from the radical of $W_{\mathbf{y}}$. We can satisfy this requirement by the following argument. Let $l_{\mathbf{y}}$ and $\tilde{l}_{\mathbf{y}}$ be distinct hyperbolic lines in $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ containing the point $s_{\mathbf{y}}$ such that $\langle h_{\mathbf{y}}, l_{\mathbf{y}}\rangle \neq \langle h_{\mathbf{y}}, \tilde{l}_{\mathbf{y}}\rangle$. Since the non-degenerate plane $\langle h_{\mathbf{y}}, s_{\mathbf{y}}\rangle$ is contained in both, we have $\mathrm{rad}(\langle h_{\mathbf{y}}, l_{\mathbf{y}}\rangle) \neq \mathrm{rad}(\langle h_{\mathbf{y}}, \tilde{l}_{\mathbf{y}}\rangle)$. Now, $h_{\mathbf{y}}^{\pi} \cap l_{\mathbf{y}}^{\pi} \cap W_{\mathbf{y}} = \langle \mathrm{rad}(W_{\mathbf{y}}), \mathrm{rad}(\langle h_{\mathbf{y}}, l_{\mathbf{y}}\rangle)\rangle$, whence there is a point $r_{\mathbf{y}} \in W_{\mathbf{y}}$ not contained in $\langle h_{\mathbf{y}}, l_{\mathbf{y}}\rangle$ and not contained in $\langle \mathrm{rad}(W_{\mathbf{y}}), \mathrm{rad}(\langle h_{\mathbf{y}}, l_{\mathbf{y}}\rangle)\rangle$. Hence $V_{\mathbf{y}} = \langle r_{\mathbf{y}}, h_{\mathbf{y}}, l_{\mathbf{y}}\rangle$ is a five-dimensional non-degenerate space of $W_{\mathbf{y}}$ containing both hyperbolic lines $h_{\mathbf{y}}$ and $l_{\mathbf{y}}$.

The local hyperbolic line $h_{\mathbf{y}}$ yields a vertex $\mathbf{h} \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}^{\perp}$ and the local line $l_{\mathbf{y}}$ a vertex $\mathbf{l} \in \{\mathbf{y}, \mathbf{w}\}^{\perp}$. By Proposition 3.3 there exists a path from $\mathbf{h}$ to $\mathbf{l}$ inside $\mathbf{G}(V_{\mathbf{y}})$, so that $\mathbf{h}$ lies in the same connected component of $\mathbf{y}^{\perp}$ as the vertex $\mathbf{x}$. The vertex $\mathbf{h}$ is also adjacent to the vertex $\mathbf{z}$ by the same argument as in the proof of Lemma 5.3.

Alternatively, let $W_{\mathbf{y}} = \langle H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} x_{\mathbf{y}}\rangle$ be a non-degenerate five-dimensional subspace of $\mathcal{G}_{\mathbf{y}}$ and $W_{\mathbf{y}} \cap w_{y}^{\pi}$ be a subspace of rank at least two. Then $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ is a non-degenerate plane and $n = 7$. We choose a hyperbolic line $h_{\mathbf{y}} \in H^{\mathbf{y}}_{z_{\mathbf{x}} y_{\mathbf{x}}}$ and a non-degenerate two-dimensional subspace $l_{\mathbf{y}}$ in the plane $W_{\mathbf{y}} \cap w_{\mathbf{y}}^{\pi}$. Again, the local line $h_{\mathbf{y}}$ yields a vertex $\mathbf{h} \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}^{\perp}$ and the local line $l_{\mathbf{y}}$ belongs to a vertex $\mathbf{l} \in \{\mathbf{y}, \mathbf{w}\}^{\perp}$. Now the proof is identical to the first part with $V_{\mathbf{y}}$ replaced by $W_{\mathbf{y}}$. □

For the next few lemmata let $\mathbf{z}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{w}$ be vertices of Γ with $\mathbf{z} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$. In the interior hyperbolic space $\mathcal{G}_{\mathbf{x}}$ the vertices $\mathbf{z}$ and $\mathbf{y}$ belong to hyperbolic lines $z_{\mathbf{x}}$ and $y_{\mathbf{x}}$ and $x_{\mathbf{y}}$ and $w_{\mathbf{y}}$ are the unique non-degenerate lines in $\mathcal{G}_{\mathbf{y}}$ of the vertices $\mathbf{x}$ and $\mathbf{w}$. Moreover, $H^{\mathbf{x}}_{z_{\mathbf{x}},y_{\mathbf{x}}} = z_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi}$ is a subspace of dimension $n-4$ or $n-3$ in $\mathcal{G}_{\mathbf{x}}$. Since $H^{\mathbf{x}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ is contained in $y_{\mathbf{x}}^{\pi}$, this subspace can also be identified with a unique subspace of $\mathcal{G}_{\mathbf{y}}$, denoted by $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$. Similarly, $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}} = x_{\mathbf{y}}^{\pi} \cap w_{\mathbf{y}}^{\pi}$ is an $(n-4)$- or $(n-3)$-dimensional subspace of $\mathcal{G}_{\mathbf{y}}$, corresponding to the subspace $H^{\mathbf{x}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ in $\mathcal{G}_{\mathbf{x}}$.

Lemma 5.5 *Let $n \geq 10$. Then the graph Γ has diameter two.*

Proof The space $W_{\mathbf{y}} = \langle x_{\mathbf{y}}, H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}\rangle$ is of dimension at least $n-2$ and of rank at least $n-4 \geq 6$. Thus, by Lemma 5.3, the space $W_{\mathbf{y}}$ contains a hyperbolic line $h_{\mathbf{y}}$, which corresponds to a vertex $\mathbf{h} \in \{\mathbf{z}, \mathbf{y}, \mathbf{w}\}^{\perp}$. It follows that $\mathbf{z}$ and $\mathbf{w}$ have distance two. Hence by induction each connected component of Γ has diameter two, and the claim results from the connectedness of Γ. □

Lemma 5.6 *Let $n = 9$. Then the graph Γ has diameter two.*

Proof If the subspace $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ is either of dimension six and of rank at least five or of dimension five and of rank at least four, then $W_{\mathbf{y}} = \langle H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}, x_{\mathbf{y}}\rangle$ is an eight-dimensional subspace of rank at least seven or a seven-dimensional subspace of rank at least six. In both cases by Lemma 5.3 the subspace $W_{\mathbf{y}}$ contains a hyperbolic line $h_{\mathbf{y}}$, such that the corresponding vertex $\mathbf{h}$ is an element of $\{\mathbf{z}, \mathbf{y}, \mathbf{w}\}^{\perp}$, yielding diameter two by induction.

The remaining possibility is that $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ is a five-dimensional subspace of rank three in $\mathcal{G}_{\mathbf{y}}$. In this case we choose a hyperbolic line $h_{\mathbf{y}}$ in $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ intersecting $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ in a one-dimensional subspace. This choice is possible, because the subspaces $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ and $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ are both contained in $x_{\mathbf{y}}^{\pi}$, which implies that $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}} \cap H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ has dimension at least three and so this intersection subspace contains an one-dimensional space which is not contained in the radical of $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$. This hyperbolic line $h_{\mathbf{y}}$ yields a vertex $\mathbf{h} \in \{\mathbf{x}, \mathbf{y}, \mathbf{w}\}^{\perp}$. Furthermore the subspace $\langle h_{\mathbf{x}}, z_{\mathbf{x}}\rangle$ in $\mathcal{G}_{\mathbf{x}}$ is four-dimensional and of rank at least three. Hence $H^{\mathbf{y}}_{z_{\mathbf{x}},h_{\mathbf{x}}}$ is a five-dimensional subspace of rank five or four. Applying the argumentation from above to the path $\mathbf{z} \perp \mathbf{x} \perp \mathbf{h} \perp \mathbf{w}$, it follows that the vertices $\mathbf{z}$ and $\mathbf{w}$ have distance two in Γ, again yielding diameter two by induction. □

Lemma 5.7 *Let $n = 8$. Then the graph Γ has diameter two.*

Proof We will prove the statement by induction, therefore let $\mathbf{z}, \mathbf{x}, \mathbf{y}$ and $\mathbf{w}$ be four different vertices of Γ such that $\mathbf{z} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$. The subspaces $H^{\mathbf{x}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ and $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ are four- or five-dimensional and of rank at least four, so we can distinguish the following cases:

case	$\dim(H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}})$	$\dim(H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}})$	$\dim(H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}})$
one	5	5	≥ 4
two	5	4	≥ 3
three	4	5	≥ 3
four	4	4	≥ 2

Suppose we are in case one or two, i.e., $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ is a five-dimensional subspace of rank at least four and the subspace $W_{\mathbf{y}} = \langle H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}, x_{\mathbf{y}}\rangle$ is of dimension seven and of rank at least six. Using Lemma 5.3 we obtain a vertex $\mathbf{h}$ in Γ adjacent to the vertices $\mathbf{z}$, $\mathbf{y}$, $\mathbf{w}$, whence the distance between the vertices $\mathbf{z}$ and $\mathbf{w}$ are at most two in Γ. Symmetry handles case three.

Assume we are in the final case, i.e., $\dim(H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}) = \dim(H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}) = 4$. We will proceed by another case distinction depending on the rank of $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ and the rank $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$.

case	$4-4$	$4-3$	$4-2$	$3-3$	$3-2$	$2-2$
$\mathrm{rank}(H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}})$	4	4	4	3	3	2
$\mathrm{rank}(H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}})$	4	3	2	3	2	2

cases 4-*: If $\mathrm{rank}(H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}) = 4$ then $W_{\mathbf{y}} = \langle x_{\mathbf{y}}, H^{\mathbf{y}}_{y_{\mathbf{x}},z_{\mathbf{x}}}\rangle$ is a non-degenerate subspace of dimension six. By Lemma 5.3 the subspace $W_{\mathbf{y}}$ contains a hyperbolic line $h_{\mathbf{y}}$

yielding a unique vertex $\mathbf{h} \in \{\mathbf{z}, \mathbf{y}, \mathbf{w}\}^\perp$, so the vertices $\mathbf{z}$ and $\mathbf{w}$ are at distance most two in Γ.

cases 3-*: In these two cases the subspace $W_{\mathbf{y}} = \langle x_{\mathbf{y}}, H^{\mathbf{y}}_{y_{\mathbf{x}}, z_{\mathbf{x}}} \rangle$ has dimension six and rank five. By Lemma 5.4 there exists again vertex $\mathbf{h} \in \{\mathbf{z}, \mathbf{y}, \mathbf{w}\}^\perp$. Thus $\mathbf{z}$ and $\mathbf{w}$ are at distance most two in Γ.

case 2-2: In the last case we assume that the subspaces $H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}}$ and $H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$ are of dimension four and of rank two. Note that in this case the hyperbolic line $w_{\mathbf{y}}$ does not intersect the subspace $x^\pi_{\mathbf{y}}$. The intersection $H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$ may have rank zero, one, or two.

If $H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$ has rank two, then $H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$ equals some hyperbolic line $h_{\mathbf{y}}$ and we are done, because the corresponding vertex $\mathbf{h}$ is adjacent to the vertices $\mathbf{z}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{w}$ in Γ implying that the distance between $\mathbf{z}$ and $\mathbf{w}$ is at most two in Γ.

Suppose $H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$ has rank one. Then we can find a hyperbolic line $l_{\mathbf{y}}$ in $H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}}$, which intersects the subspace $H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$ in an one-dimensional subspace. The four-dimensional space $\langle l_{\mathbf{y}}, w_{\mathbf{y}} \rangle$ has rank three or four, thus the path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ from $\mathbf{z}$ to $\mathbf{w}$ in Γ belongs either to case $4-2$ or to case $3-2$, and we are done.

If $H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$ is a totally singular subspace then we define the two set of points

$$S^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}} := \{p_{\mathbf{y}} \in H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}} \mid p_{\mathbf{y}} \notin \mathrm{rad}(H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}}),\ p_{\mathbf{y}} \text{ a singular point}\}$$

and

$$S^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}} := \{p_{\mathbf{y}} \in H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}} \mid p_{\mathbf{y}} \notin \mathrm{rad}(H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}),\ p_{\mathbf{y}} \text{ a singular point}\}.$$

If either of $S^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}}$ and $S^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$ is not empty, then with out loss of generality we assume (after relabeling) that $S^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}} \neq \emptyset$ and choose a point $p_{\mathbf{y}} \in S^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}}$ as well as a hyperbolic line $l_{\mathbf{y}}$ in $H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}}$ containing the point $p_{\mathbf{y}}$. The subspace $\langle w_{\mathbf{y}}, l_{\mathbf{y}} \rangle$ is non-degenerate and of dimension four, moreover the hyperbolic line $l_{\mathbf{y}}$ corresponds to a vertex $\mathbf{l} \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}^\perp$. The resulting path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ belongs either to the case $4-3$ or to the case $4-2$, and again we are done.

In the final step we assume $S^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}} = \emptyset = S^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$, which implies $\mathrm{rad}(H^{\mathbf{x}}_{z_{\mathbf{x}}, y_{\mathbf{x}}}) = H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}} = \mathrm{rad}(H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}})$. In other words the intersection $H^{\mathbf{y}}_{z_{\mathbf{x}}, y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$ is a totally singular radical two-dimensional subspace of $H^{\mathbf{x}}_{z_{\mathbf{x}}, y_{\mathbf{x}}}$ and of $H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$. For an arbitrary hyperbolic line $l_{\mathbf{y}}$ in $H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$ the subspace $l^\pi_{\mathbf{y}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}$ concides with $\mathrm{rad}(H^{\mathbf{y}}_{x_{\mathbf{y}}, w_{\mathbf{y}}})$ and, thus, $\mathrm{rad}(H^{\mathbf{x}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}) = \mathrm{rad}(H^{\mathbf{x}}_{z_{\mathbf{x}}, y_{\mathbf{x}}}) \subseteq l^\pi_{\mathbf{x}} \cap z^\pi_{\mathbf{x}} = H^{\mathbf{x}}_{z_{\mathbf{x}}, l_{\mathbf{x}}}$. Furthermore $\langle x_{\mathbf{l}}, w_{\mathbf{l}} \rangle = \langle x_{\mathbf{y}}, w_{\mathbf{y}} \rangle$ by Lemma 5.2, which implies that $\mathrm{rad}(H^{\mathbf{x}}_{x_{\mathbf{y}}, w_{\mathbf{y}}}) = \mathrm{rad}(H^{\mathbf{x}}_{x_{\mathbf{l}}, w_{\mathbf{l}}})$ and so every point of $\mathrm{rad}(H^{\mathbf{x}}_{z_{\mathbf{x}}, y_{\mathbf{x}}})$ is contained in $H^{\mathbf{x}}_{x_{\mathbf{l}}, w_{\mathbf{l}}} \cap H^{\mathbf{x}}_{z_{\mathbf{x}}, l_{\mathbf{x}}}$. As $\langle z_{\mathbf{x}}, y_{\mathbf{x}} \rangle \cap \langle z_{\mathbf{x}}, l_{\mathbf{x}} \rangle = z_{\mathbf{x}}$ it follows that $\mathrm{rad}(\langle z_{\mathbf{x}}, y_{\mathbf{x}} \rangle) \neq \mathrm{rad}(\langle z_{\mathbf{x}}, l_{\mathbf{x}} \rangle)$ and therefore not every point of $\mathrm{rad}(H^{\mathbf{x}}_{z_{\mathbf{x}}, y_{\mathbf{x}}})$ is a point of $\mathrm{rad}(H^{\mathbf{x}}_{z_{\mathbf{x}}, l_{\mathbf{x}}})$. Consequently the path $\mathbf{z} \perp \mathbf{x} \perp \mathbf{l} \perp \mathbf{w}$ belongs to some case already dealt with, because $S^{\mathbf{l}}_{z_{\mathbf{x}}, l_{\mathbf{x}}}$ is not empty.

Since the vertices $\mathbf{z}$ and $\mathbf{w}$ have at most distance two in Γ, as before by induction the graph Γ has diameter two. □

Lemma 5.8 *Let $n=7$. Then the graph Γ has diameter two.*

Proof As before we will use induction to prove the claim, therefore let $\mathbf{z}$, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{w}$ be four different vertices of Γ forming the path $\mathbf{z} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$. The subspaces $H^{\mathbf{x}}_{z_\mathbf{x}, y_\mathbf{x}}$ and $H^{\mathbf{y}}_{x_\mathbf{y}, w_\mathbf{y}}$ of $\mathcal{G}_\mathbf{x}$ resp. of $\mathcal{G}_\mathbf{y}$ have dimension three or four. We will distinguish the following four cases:

case	$\dim(H^{\mathbf{x}}_{z_\mathbf{x}, y_\mathbf{x}})$	$\dim(H^{\mathbf{y}}_{x_\mathbf{y}, w_\mathbf{y}})$	$\dim(H^{\mathbf{x}}_{z_\mathbf{x}, y_\mathbf{x}} \cap H^{\mathbf{x}}_{x_\mathbf{y}, w_\mathbf{y}})$
one	4	4	≥ 3
two	3	4	≥ 2
three	4	3	≥ 2
four	3	3	≥ 1

First we consider case one and two, and also case three by symmetry. Since $n=7$ and the dimension of $H^{\mathbf{x}}_{z_\mathbf{x}, y_\mathbf{x}}$ is four, the hyperbolic lines $y_\mathbf{x}$ and $z_\mathbf{x}$ span a three-dimensional space, whence $H^{\mathbf{x}}_{z_\mathbf{x}, y_\mathbf{x}}$ has a radical of dimension at most one. Thus the subspace $W_\mathbf{y} = \langle H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}, x_\mathbf{y}\rangle$ is of dimension six and rank at least five. By Lemma 5.3 and Lemma 5.4 there exists a vertex $\mathbf{h} \in \{\mathbf{w}, \mathbf{y}, \mathbf{z}\}^\perp$, yielding distance two between $\mathbf{z}$ and $\mathbf{w}$ in Γ.

It remains to prove the claim in the case that $H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$ and $H^{\mathbf{y}}_{x_\mathbf{y}, w_\mathbf{y}}$ are planes. We split up this setting into six different cases depending on the rank of the planes $H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$ and $H^{\mathbf{y}}_{x_\mathbf{y}, w_\mathbf{y}}$:

case	$3-3$	$3-2$	$3-1$	$2-2$	$2-1$	$1-1$
$\operatorname{rank}(H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}})$	3	3	3	2	2	1
$\operatorname{rank}(H^{\mathbf{y}}_{x_\mathbf{y}, w_\mathbf{y}})$	3	2	1	2	1	1

case 3-3: If $H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}} \cap H^{\mathbf{y}}_{x_\mathbf{y}, w_\mathbf{y}}$ is a three-dimensional subspace of $\mathcal{G}_\mathbf{y}$, then $H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}} \cap H^{\mathbf{y}}_{x_\mathbf{y}, w_\mathbf{y}} = H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$. Thus $H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}} \cap H^{\mathbf{y}}_{x_\mathbf{y}, w_\mathbf{y}}$ is a non-degenerate plane, which of course contains some hyperbolic line $h_\mathbf{y}$. The hyperbolic line corresponds to a vertex $\mathbf{h}$ in the subgraph $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}^\perp$, finishing the proof.

Therefore we assume that the intersection $H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}} \cap H^{\mathbf{y}}_{x_\mathbf{y}, w_\mathbf{y}}$ is of dimension one or two. Under this condition we regard the five-dimensional non-degenerate space $W_\mathbf{y} = \langle x_\mathbf{y}, H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}\rangle$, which intersects $w^\pi_\mathbf{y}$ in a three-dimensional space of rank at least one. Moreover, for each hyperbolic line $l_\mathbf{y}$ in the non-degenerate plane $H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$, the non-degenerate plane $H^{\mathbf{y}}_{z_\mathbf{l}, y_\mathbf{l}} = \langle x_\mathbf{y}, l^\pi_\mathbf{y} \cap H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}\rangle$ is a subspace of $W_\mathbf{y}$ and intersects $w^\pi_\mathbf{y}$ in a one- or two-dimensional subspace. As $x_\mathbf{y} \cap H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}} = \emptyset$, for different hyperbolic lines $h_\mathbf{y}$ and $l_\mathbf{y}$ in $H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$, the subspaces $H^{\mathbf{y}}_{z_\mathbf{l}, y_\mathbf{l}} \cap w^\pi_\mathbf{y}$ and $H^{\mathbf{y}}_{z_\mathbf{h}, y_\mathbf{h}} \cap w^\pi_\mathbf{y}$ are different.

By Lemma A.1 the non-degenerate plane $H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$ contains $q^4 - q^3 + q^2$ hyperbolic lines, while the plane $W_\mathbf{y} \cap w^\pi_\mathbf{y}$ contains at most $q^3 + q^2 + 1$ different singular points. Hence we find a hyperbolic line $l_\mathbf{y}$ in the plane $H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$ such that $H^{\mathbf{y}}_{z_\mathbf{l}, y_\mathbf{l}} \cap w^\pi_\mathbf{y}$ contains some non-degenerate point $p_\mathbf{y}$. The hyperbolic line $l_\mathbf{y}$ determines a vertex $\mathbf{l}$ in Γ adjacent to the vertices $\mathbf{x}, \mathbf{y}, \mathbf{z}$. Furthermore, we choose a hyperbolic line $n_\mathbf{y}$ in the non-degenerate plane $H^{\mathbf{y}}_{z_\mathbf{l}, y_\mathbf{l}}$ incident to the non-degenerate point $p_\mathbf{y}$. The hyperbolic line $n_\mathbf{y}$ yields a vertex $\mathbf{n}$ of $\{\mathbf{z}, \mathbf{x}, \mathbf{l}\}^\perp$. If the subspace $H^{\mathbf{n}}_{n_{\mathbf{y}},w_{\mathbf{y}}}$ is of dimension four, then the path $\mathbf{z} \perp \mathbf{n} \perp \mathbf{y} \perp \mathbf{z}$ of Γ belongs to one of the cases one, two, or three, so we may assume that $H^{\mathbf{n}}_{n_{\mathbf{y}},w_{\mathbf{y}}}$ is a three-dimensional subspace. Since the non-degenerate point $p_{\mathbf{y}}$ is perpendicular to $w_{\mathbf{y}}$, the four-dimensional subspace $\langle n_{\mathbf{y}}, w_{\mathbf{y}}\rangle$ has rank at least three and we conclude that $H^{\mathbf{n}}_{n_{\mathbf{y}},w_{\mathbf{y}}}$ has rank at least two. Thus, there exists a hyperbolic line $h_{\mathbf{y}}$ in the plane $H^{\mathbf{n}}_{n_{\mathbf{y}},w_{\mathbf{y}}}$ in such a way that $H^{\mathbf{n}}_{n_{\mathbf{h}},w_{\mathbf{h}}} \cap z^{\pi}_{\mathbf{n}}$ contains a non-degenerate point $d_{\mathbf{n}}$, which is possible by the argumentation above; certainly if $H^{\mathbf{n}}_{n_{\mathbf{y}},w_{\mathbf{y}}}$ happens to have rank two instead of rank three, then this subspace contains q^4 hyperbolic lines by Lemma A.1, and the above argument is still applicable. Moreover the vertex $\mathbf{h}$ corresponding to $h_{\mathbf{y}}$ is contained in the induced subgraph $\{\mathbf{w},\mathbf{n},\mathbf{y}\}^{\perp}$. The interior hyperbolic space $\mathcal{G}_{\mathbf{h}}$ contains the non-degenerate point $d_{\mathbf{h}}$ and the hyperbolic line $n_{\mathbf{h}}$, which in turn contains the non-degenerate point $p_{\mathbf{h}}$. Since the point $d_{\mathbf{h}}$ is contained in the subspace $n^{\pi}_{\mathbf{h}}$, the two non-degenerate points $p_{\mathbf{h}}$ and $d_{\mathbf{h}}$ span a hyperbolic line $k_{\mathbf{h}}$ in the space $\mathcal{G}_{\mathbf{h}}$, in particular $k_{\mathbf{h}}$ is a hyperbolic line of subspace $w^{\pi}_{\mathbf{h}}$. Indeed the hyperbolic line $n_{\mathbf{h}}$ intersects the subspace $w^{\pi}_{\mathbf{h}}$ in the non-degenerate point $p_{\mathbf{h}}$, while the non-degenerate point $d_{\mathbf{h}}$ is a point of $w^{\pi}_{\mathbf{h}}$ by construction. Thus we have determine a vertex $\mathbf{k}$ adjacent to $\mathbf{w}$ and $\mathbf{h}$. Furthermore, the two hyperbolic lines $n_{\mathbf{h}}$ and $k_{\mathbf{h}}$ generate a plane in $\mathcal{G}_{\mathbf{h}}$ implying that $\dim(H^{\mathbf{h}}_{n_{\mathbf{h}},k_{\mathbf{h}}}) = 4$. By these facts the path $\mathbf{z} \perp \mathbf{n} \perp \mathbf{h} \perp \mathbf{k}$ of Γ belongs to case two or three of this proof, so there exists a vertex $\mathbf{m} \in \{\mathbf{n},\mathbf{k},\mathbf{z}\}^{\perp}$ in the same connected component of the subgraph $\{\mathbf{n},\mathbf{k}\}^{\perp}$ as the vertex $\mathbf{h}$. Local analysis of the interior hyperbolic space $\mathcal{G}_{\mathbf{m}}$ shows that the perpendicular space $z^{\pi}_{\mathbf{m}}$ of $z_{\mathbf{m}}$ contains the two points $p_{\mathbf{m}}$ and $d_{\mathbf{m}}$, whence the hyperbolic line $k_{\mathbf{m}}$ spanned by $p_{\mathbf{m}}$ and $d_{\mathbf{m}}$. Consequently, the vertex $\mathbf{k}$ is adjacent to the vertices $\mathbf{z}$ and $\mathbf{w}$, so $\mathbf{z}$ and $\mathbf{w}$ have distance at most two in Γ.

case 3-2 and case 3-1: As before we study the intersection $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{w_{\mathbf{y}},x_{\mathbf{y}}}$. If the subspace $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{w_{\mathbf{y}},x_{\mathbf{y}}}$ has rank at least two, then it contains a hyperbolic line and we are done. Otherwise define $S^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}},w_{\mathbf{y}},x_{\mathbf{y}}}$ to be the set of all singular points incident to the subspace $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{w_{\mathbf{y}},x_{\mathbf{y}}}$. If $S^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}},w_{\mathbf{y}},x_{\mathbf{y}}} \neq \emptyset$, then let $p_{\mathbf{y}}$ be a point of $S^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}},w_{\mathbf{y}},x_{\mathbf{y}}}$ and we choose a hyperbolic line $l_{\mathbf{y}}$ in the non-degenerate plane $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ going through the singular point $p_{\mathbf{y}}$. The vertex $\mathbf{l}$ corresponding to $l_{\mathbf{y}}$ is contained in $\{\mathbf{z},\mathbf{x},\mathbf{y}\}^{\perp}$. Since the hyperbolic lines $l_{\mathbf{y}}$ and $w_{\mathbf{y}}$ span either a three-dimensional or a non-degenerate four-dimensional space, the path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ belongs to case four (3-3) or to case two. On the other hand, if $S^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}},w_{\mathbf{y}},x_{\mathbf{y}}} = \emptyset$, then we choose a non-degenerate point $r_{\mathbf{y}}$ in $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{w_{\mathbf{y}},x_{\mathbf{y}}}$ and a hyperbolic line $l_{\mathbf{y}}$ incident to the point $r_{\mathbf{y}}$ in the non-degenerate plane $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$, yielding the path $\gamma = (\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w})$ in Γ between $\mathbf{z}$ and $\mathbf{w}$. The subspace $\langle l_{\mathbf{y}}, w_{\mathbf{y}}\rangle$ is either of dimension three, in which case the path γ belongs to case two, or of dimension four. If this four-dimensional subspace is of rank four, then the path γ belongs to case 3-3. If the rank of $\langle l_{\mathbf{y}}, w_{\mathbf{y}}\rangle$ is strictly less than four, then we obtain the point set $S^{\mathbf{y}}_{z_{\mathbf{l}},y_{\mathbf{l}},w_{\mathbf{y}},l_{\mathbf{y}}} = \{s_{\mathbf{y}} \mid s_{\mathbf{y}} \in H^{\mathbf{y}}_{z_{\mathbf{l}},y_{\mathbf{l}}} \cap H^{\mathbf{y}}_{w_{\mathbf{y}},l_{\mathbf{y}}}, s_{\mathbf{y}} \text{ a singular point}\}$. If $S^{\mathbf{y}}_{z_{\mathbf{l}},y_{\mathbf{l}},w_{\mathbf{y}},l_{\mathbf{y}}} \neq \emptyset$, then the path γ satisfies the conditions of the previous paragraph, which leads to the fact that the path γ can be transformed to a path between the vertices $\mathbf{z}$ and $\mathbf{w}$ of length three belonging to case two or case four (3-3). If $S^{\mathbf{y}}_{z_{\mathbf{l}},y_{\mathbf{l}},w_{\mathbf{y}},l_{\mathbf{y}}}$ is also empty, then we choose a non-degenerate point $d_{\mathbf{y}}$ in $H^{\mathbf{y}}_{z_{\mathbf{l}},y_{\mathbf{l}}} \cap H^{\mathbf{y}}_{w_{\mathbf{y}},l_{\mathbf{y}}}$ and consider the two-dimensional space $h_{\mathbf{y}}$ spanned by the two different point $r_{\mathbf{y}}$ and $d_{\mathbf{y}}$. Since $d_{\mathbf{y}}$ is a non-degenerate point in $l_{\mathbf{y}}^{\pi} \subseteq r_{\mathbf{y}}^{\pi}$, the space $h_{\mathbf{y}}$ is a hyperbolic line, contained in $w_{\mathbf{y}}^{\pi}$. Thus the corresponding vertex $\mathbf{h}$ is adjacent to the vertices $\mathbf{y}$ and $\mathbf{w}$. The hyperbolic lines $l_{\mathbf{y}}$ and $h_{\mathbf{y}}$ span a plane implying $\dim(H^{\mathbf{y}}_{l_{\mathbf{y}},h_{\mathbf{y}}}) = 4$. Hence there exists a vertex $\mathbf{m} \in \{\mathbf{z},\mathbf{l},\mathbf{h}\}^{\perp}$ in the same connected component of $\{\mathbf{l},\mathbf{h}\}^{\perp}$ as $\mathbf{y}$, because the path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{h}$ in Γ belongs to case two or three. Hence, by local analysis of the space $\mathcal{G}_{\mathbf{m}}$, the vertex $\mathbf{h}$ is also adjacent to $\mathbf{z}$, as the hyperbolic line $h_{\mathbf{m}}$ is contained in the subspace $z_{\mathbf{m}}^{\pi}$ by construction. Therefore $\mathbf{w}$ and $\mathbf{z}$ have a common neighbor $\mathbf{h}$ and therefore have mutual distance at most two in Γ.

case 2-2 and case 2-1: Again we will analyse the subspace $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{w_{\mathbf{y}},x_{\mathbf{y}}}$ and the set of singular points $S^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}},w_{\mathbf{y}},x_{\mathbf{y}}} = \{s_{\mathbf{y}} \mid s_{\mathbf{y}} \in H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{w_{\mathbf{y}},x_{\mathbf{y}}}, s_{\mathbf{y}}$ a singular point$\}$. Suppose $p_{\mathbf{y}}$ is an element of $S^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}},w_{\mathbf{y}},x_{\mathbf{y}}}$ not contained in the radical of $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$. In this case we choose a hyperbolic line $l_{\mathbf{y}}$ in $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ incident to the point $p_{\mathbf{y}}$ and obtain the subspace $\langle w_y, l_y\rangle$, which is of dimension three or four and of rank at least three. As before the path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ belongs to case two or to case four (3-2).

If on the other hand $S^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}},w_{\mathbf{y}},x_{\mathbf{y}}} = \emptyset$, then every point of $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{w_{\mathbf{y}},x_{\mathbf{y}}}$ is non-degenerate. Note that $d_{\mathbf{y}} = H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{w_{\mathbf{y}},x_{\mathbf{y}}}$ is a unique point, because anisotropic two-dimensional unitary spaces over a finite field do not exist. Recall also that the non-degenerate point $d_{\mathbf{y}}$ is contained in q^2 hyperbolic lines and one singular line of $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$. Therefore the hyperbolic line $w_{\mathbf{y}}$ contains a singular point $s_{\mathbf{y}}$ such that $s_{\mathbf{y}}^{\pi} \cap H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ is a hyperbolic line $l_{\mathbf{y}}$, containing $d_{\mathbf{y}}$. The subspace $\langle w_{\mathbf{y}}, l_{\mathbf{y}}\rangle$ is non-degenerate of dimension three or four, so the path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ belongs either to case two or to case four (3-2).

It remains to deal with the case that each point of $S^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}},w_{\mathbf{y}},x_{\mathbf{y}}}$ is contained in the radical of $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$. Since $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ is a rank two plane, the point set $S^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}},w_{\mathbf{y}},x_{\mathbf{y}}}$ consists of a unique singular point. If the intersection $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ is a one-dimensional subspace, then it equals the radical of $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$. In this situation $x_{\mathbf{y}}^{\pi} = \langle H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}, H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}\rangle$, so that $S^{\mathbf{y}} z_{\mathbf{x}}, y_{\mathbf{x}}, w_{\mathbf{y}}, x_{\mathbf{y}}$ cannot be contained in the radical of $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$, as otherwise $S^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}},w_{\mathbf{y}},x_{\mathbf{y}}}$ is contained in the radical of $x_{\mathbf{y}}^{\pi}$, a contradiction. Since every singular point of a rank one plane is contained in the radical of that plane, the plane $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ necessarily has rank two. By symmetry, working with the singular points $S^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}},w_{\mathbf{y}},x_{\mathbf{y}}}$ not in the radical of $H^{\mathbf{y}}_{w_{\mathbf{y}},x_{\mathbf{y}}}$, we are done. Now we assume that the intersection $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ is a two-dimensional subspace. Any hyperbolic line $l_{\mathbf{y}}$ of $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ has the property that $l_{\mathbf{y}}^{\pi} \cap H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ equals the radical of $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$. Hence the subspace $\langle l_{\mathbf{y}}, w_{\mathbf{y}}\rangle^{\pi} \cap H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ is of dimension one. Since $\mathbf{l} \perp \mathbf{x}$, the path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ by Lemma 5.2 belongs to the situation that $H^{\mathbf{y}}_{z_{\mathbf{l}},y_{\mathbf{l}}} \cap H^{\mathbf{y}}_{l_{\mathbf{y}},w_{\mathbf{y}}}$ is of dimension one, that we have just dealt with.

case 1-1: In this final case we assume that $H^{\mathbf{x}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ and $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ are planes of rank one. Let $p_{\mathbf{x}}$ be some point in $H^{\mathbf{x}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{x}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$. Since $z_{\mathbf{x}}^{\pi}$ is a five-dimensional non-degenerate subspace, there exists a hyperbolic line $m_{\mathbf{x}}$ in $z_{\mathbf{x}}^{\pi}$ incident to $p_{\mathbf{x}}$. The hyperbolic line $m_{\mathbf{x}}$ corresponds to a vertex $\mathbf{m} \in \{\mathbf{z}, \mathbf{x}\}^{\perp}$. Moreover, the subspace $\langle m_{\mathbf{x}}, y_{\mathbf{x}} \rangle$ is either three-dimensional or four-dimensional and of rank at least three. Hence the path $\mathbf{m} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$ belongs to one of the above cases. Thus the graph Γ contains a vertex $\mathbf{n} \in \{\mathbf{m}, \mathbf{w}\}^{\perp}$. The resulting path $\mathbf{z} \perp \mathbf{m} \perp \mathbf{n} \perp \mathbf{w}$ from $\mathbf{z}$ to $\mathbf{w}$ has the property that $\langle m_{\mathbf{n}}, w_{\mathbf{n}} \rangle$ is a plane or a four-dimensional subspace of rank at least three, because the hyperbolic line $m_{\mathbf{n}}$ intersects $w_{\mathbf{n}}^{\pi}$ the point in $p_{\mathbf{n}}$. Thus this path belongs to one of the cases above.

As the vertices $\mathbf{z}$ and $\mathbf{w}$ have at most distance two in Γ, the claim follows by induction. □

Altogether, we have proved the following.

Proposition 5.9 *The graph Γ has diameter two.*

In fact, the proofs of Lemma 5.5 to Lemma 5.8 also imply that Γ is simply connected.

Next we want to construct a global point-line geometry on the graph Γ that will allow us to determine the isomorphism type of Γ. Recall the notation introduced for local objects in the beginning of this section. The following observation will play an important role by the definition of global points.

Lemma 5.10 *Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be distinct vertices of Γ satisfying $\mathbf{x} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x}$ and let $p_{\mathbf{x}}$ be a local point of $\mathcal{G}_{\mathbf{x}}$ such that $y_{\mathbf{x}}, z_{\mathbf{x}} \in p_{\mathbf{x}}^{\pi}$. Then the unique local point $p_{\mathbf{y}} \in \mathcal{G}_{\mathbf{y}}$ induced by the point $p_{\mathbf{y}_{\mathbf{x}}} \in \mathcal{G}_{\mathbf{y}_{\mathbf{x}}}$ and the unique local point $p_{\mathbf{z}} \in \mathcal{G}_{\mathbf{z}}$ induced by the point $p_{\mathbf{z}_{\mathbf{x}}} \in \mathcal{G}_{\mathbf{z}_{\mathbf{x}}}$ satisfy $z_{\mathbf{y}} \in p_{\mathbf{y}}^{\pi}$ and $y_{\mathbf{z}} \in p_{\mathbf{z}}^{\pi}$. Moreover, the unique local point in $\mathcal{G}_{\mathbf{z}}$ induced by $p_{\mathbf{z}_{\mathbf{y}}}$ is equal to the local point $p_{\mathbf{z}}$.*

Proof This lemma is proved using the results from Section 4. As by assumption $y_{\mathbf{x}}, z_{\mathbf{x}} \in p_{\mathbf{x}}^{\pi}$, the local point $p_{\mathbf{x}} \in \mathcal{G}_{\mathbf{x}}$ gives rise to a point $p_{\mathbf{x}} \cap \mathbf{y}^{\perp} = p_{\mathbf{y}_{\mathbf{x}}} = p_{\mathbf{x}_{\mathbf{y}}}$ of $\mathcal{G}_{\mathbf{y}_{\mathbf{x}}} = \mathcal{G}_{\mathbf{x}_{\mathbf{y}}}$ and to a point $p_{\mathbf{x}} \cap \mathbf{z}^{\perp} = p_{\mathbf{z}_{\mathbf{x}}} = p_{\mathbf{x}_{\mathbf{z}}}$ of $\mathcal{G}_{\mathbf{z}_{\mathbf{x}}} = \mathcal{G}_{\mathbf{x}_{\mathbf{z}}}$. Consider the unique local point $p_{\mathbf{y}}$ of $\mathcal{G}_{\mathbf{y}}$ which contains the point $p_{\mathbf{y}_{\mathbf{x}}}$ and the unique local point $p_{\mathbf{z}}$ of $\mathcal{G}_{\mathbf{z}}$ which contains the point $p_{\mathbf{z}_{\mathbf{x}}}$. Since $y_{\mathbf{x}}^{\pi} \cap z_{\mathbf{x}}^{\pi}$ is a non-degenerate subspace of dimension at least $n-4$ incident to the point $p_{\mathbf{x}}$, it also contains two hyperbolic lines $g_{\mathbf{x}}^{1}$ and $g_{\mathbf{x}}^{2}$, which are elements of $p_{\mathbf{x}}$. By construction the vertices $\mathbf{g}^1, \mathbf{g}^2$ belong to unique interior lines of the local points $p_{\mathbf{x}}, p_{\mathbf{y}}, p_{\mathbf{z}}, p_{\mathbf{x}_{\mathbf{y}}} = p_{\mathbf{y}_{\mathbf{x}}}, p_{\mathbf{x}_{\mathbf{z}}} = p_{\mathbf{z}_{\mathbf{x}}}, p_{\mathbf{y}_{\mathbf{z}}} = p_{\mathbf{z}_{\mathbf{y}}}$. Hence $z_{\mathbf{y}} \in p_{\mathbf{y}}^{\pi}$ and $y_{\mathbf{z}} \in p_{\mathbf{z}}^{\pi}$ and, by partial linearity of $\mathcal{G}_{\mathbf{z}}$ the unique local point in $\mathcal{G}_{\mathbf{z}}$ induced by $p_{\mathbf{z}_{\mathbf{y}}}$ is equal to the local point $p_{\mathbf{z}}$. □

For the construction of global points let $p_{\mathbf{x}}$ be a local singular point in the interior hyperbolic space $\mathcal{G}_{\mathbf{x}}$ for some vertex $\mathbf{x}$ of the graph Γ and consider the set of vertices $p_0 = p_{\mathbf{x}} \cup \bigcup_{\mathbf{l} \in \mathbf{L}(p_{\mathbf{x}}^{\pi})} \{p_{\mathbf{l}} \mid p_{\mathbf{x}_{\mathbf{l}}} \subseteq p_{\mathbf{l}}\}$ in Γ. Furthermore we define inductively the set of vertices $p_i = \bigcup_{p_{\mathbf{l}} \in p_{i-1}} (\bigcup_{\mathbf{k} \in \mathbf{L}(p_{\mathbf{l}}^{\pi})} \{p_{\mathbf{k}} \mid p_{\mathbf{l}_{\mathbf{k}}} \subseteq p_{\mathbf{k}}\})$ for $i \in \mathbb{N}$. Certainly $p_0 \subseteq p_1$ using the fact that for each local point $p_{\mathbf{l}}$ of p_0, which is different from the local point p_x, the local hyperbolic line $x_{\mathbf{l}}$ is an element of $\mathbf{L}(p_{\mathbf{l}}^{\pi})$. Thus $p_{\mathbf{x}} \subseteq p_1$. Moreover, since $p_{\mathbf{x}} \subseteq p_0$, so by construction $\bigcup_{\mathbf{l} \in \mathbf{L}(p_{\mathbf{x}}^{\pi})} \{p_{\mathbf{l}} \mid p_{\mathbf{x}_{\mathbf{l}}} \subseteq p_{\mathbf{l}}\} \subseteq p_1$.

Suppose there exists a vertex $\mathbf{k}$ in $p_1 \backslash p_0$. Then again by construction of the set p_1 there is a path $\mathbf{x} \perp \mathbf{y} \perp \mathbf{w} \perp \mathbf{k}$ in Γ from $\mathbf{x}$ to $\mathbf{k}$ such that $y_{\mathbf{x}}$ is a hyperbolic line contained in $p_{\mathbf{x}}^{\pi}$ and $w_{\mathbf{y}}$ is a hyperbolic line in the subspace $p_{\mathbf{y}}^{\pi}$ and $k_{\mathbf{w}}$ is a hyperbolic line going through the local point $p_{\mathbf{w}}$. Without loss of generality we may assume that $w_{\mathbf{y}}$ is a hyperbolic line of the subspace $p_{\mathbf{y}}^{\pi}$ which is not contained in $x_{\mathbf{y}}^{\pi}$ and that $k_{\mathbf{w}}$ is a hyperbolic line of the local point $p_{\mathbf{w}}$ but not of the local point $p_{\mathbf{y}_{\mathbf{w}}}$, as otherwise $\mathbf{k}$ is a vertex of p_0. Because of these assumptions $k_{\mathbf{w}}$ is not a hyperbolic line of the perpendicular subspace $y_{\mathbf{w}}^{\pi}$, but intersects the $y_{\mathbf{w}}^{\pi}$ in the singular point $p_{\mathbf{w}}$. We conclude that $\langle k_{\mathbf{w}}, y_{\mathbf{w}} \rangle$ is a four-dimensional non-degenerate space.

By Lemmata 5.3 and 5.4 there exists a vertex $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}, \mathbf{k}\}^{\perp}$ and a path $\mathbf{w} \perp \mathbf{c}^1 \perp \cdots \perp \mathbf{c}^n \perp \mathbf{z}$ in $\{\mathbf{y}, \mathbf{k}\}^{\perp}$. Since the local hyperbolic line $c_{\mathbf{w}}^1$ is incident to the subspace $k_{\mathbf{w}}^{\pi}$, it is also contained in $p_{\mathbf{w}}^{\pi}$, whence there is a local point $p_{\mathbf{c}^1} \supseteq p_{\mathbf{w}_{\mathbf{c}^1}}$ containing the local hyperbolic line $k_{\mathbf{c}_1}$. By Lemma 5.10 we have $p_{\mathbf{y}_{\mathbf{c}^1}} \subseteq p_{\mathbf{c}^1}$ and $c_{\mathbf{y}}^1 \in p_{\mathbf{y}}^{\pi}$. Repeating this argument along the path $\mathbf{w} \perp \mathbf{c}^1 \perp \ldots \perp \mathbf{c}^n \perp \mathbf{z}$, we end up with $p_{\mathbf{y}_{\mathbf{z}}} \subseteq p_{\mathbf{z}}$, that $k_{\mathbf{z}}$ is a hyperbolic line of the local point $p_{\mathbf{z}}$ and also that $z_{\mathbf{y}}$ is contained in the subspace $p_{\mathbf{y}}^{\pi}$. This implies that the hyperbolic line $z_{\mathbf{y}}$ is incident to the subspace $p_{\mathbf{x}_{\mathbf{y}}}^{\pi}$, and, thus, a subspace of $p_{\mathbf{x}}^{\pi}$, in particular $p_{\mathbf{x}_{\mathbf{z}}} \subseteq p_{\mathbf{z}}$. Whence the global line $\mathbf{k}$ is an element of the vertex set p_0, implying that $p_0 = p_i$ for each $i \in \mathbb{N}$. This construction leads to a well-behaved set of vertices $p := p_0 = p_{\mathbf{x}} \cup \bigcup_{\mathbf{l} \in \mathbf{L}(p_{\mathbf{x}}^{\pi})} \{p_{\mathbf{l}} : p_{\mathbf{x}_{\mathbf{l}}} \subseteq p_{\mathbf{l}}\}$ such that the local singular point $p_{\mathbf{x}} \subseteq p$.

Definition 5.11 A **global point** p of Γ equals $p_{\mathbf{x}} \cup \bigcup_{\mathbf{l} \in \mathbf{L}(p_{\mathbf{x}}^{\pi})} \{p_{\mathbf{l}} \mid p_{\mathbf{x}_{\mathbf{l}}} \subseteq p_{\mathbf{l}}\}$ for some vertex $\mathbf{x} \in \Gamma$ and some local singular point $p_{\mathbf{x}}$ of the interior hyperbolic space $\mathcal{G}_{\mathbf{x}}$. The set of all global points of Γ is denoted by $\mathcal{P}_{\Gamma}$.

Notice that the definition of a global point p does not depend on the starting local point $p_{\mathbf{x}} \subseteq p$ because $p = p_0 = p_i$ for all $i \in \mathbb{N}$. The next proposition follows immediately from the construction of a global point p.

Proposition 5.12 *Let p be a global point and $\mathbf{x}$ be vertex of Γ. Then $p \cap \mathcal{L}_{\mathbf{x}}$ is either empty or a local singular point of $\mathcal{G}_{\mathbf{x}}$.*

The pair $\mathcal{G}_{\Gamma} = (\mathcal{P}_{\Gamma}, \mathcal{L}_{\Gamma})$ with symmetrized inclusion as incidence is a point-line geometry called the **global space** on Γ.

Lemma 5.13 *The point-line geometry $\mathcal{G}_{\Gamma}$ is a connected partially linear space.*

Proof Let p and d be two different global points of $\mathcal{P}_{\Gamma}$ and suppose the vertex set $p \cap d$ contains two distinct vertices $\mathbf{x}$ and $\mathbf{y}$. Since the graph Γ has diameter two by Proposition 5.9, there exists a vertex $\mathbf{z}$ in the induced subgraph $\{\mathbf{x}, \mathbf{y}\}^{\perp}$. It follows that the two different local points $p_{\mathbf{z}} = p \cap \mathcal{L}_{\mathbf{z}}$ and $d_{\mathbf{z}} = d \cap \mathcal{L}_{\mathbf{z}}$ are incident to the two local lines $x_{\mathbf{z}}$ and $y_{\mathbf{z}}$ in $\mathcal{G}_{\mathbf{z}}$, thus $p_{\mathbf{z}} = d_{\mathbf{z}}$ by the partial linearity of $\mathcal{G}_{\mathbf{z}}$, whence $p = d$. Hence $\mathcal{G}_{\Gamma}$ is partially linear.

In order to prove connectedness of $\mathcal{G}_{\Gamma}$ let again p and d be two different global points. Choose $\mathbf{l} \in p$ and $\mathbf{m} \in d$. Using once again that the diameter of Γ is two, there is a vertex $\mathbf{k} \in \{\mathbf{m}, \mathbf{l}\}^{\perp}$. The interior hyperbolic space $G_{\mathbf{k}}$ contains the interior points $p_{\mathbf{k}} = p \cap \mathcal{L}_{\mathbf{k}}$ and $d_{\mathbf{k}} = d \cap \mathcal{L}_{\mathbf{k}}$. Hence connectedness of $\mathcal{G}_\Gamma$ follows from the connectedness of $\mathcal{G}_{\mathbf{k}}$. □

We intend to use Theorem 4.6 to identify the geometry $\mathcal{G}_\Gamma$. Therefore we need to define and study planes of $\mathcal{G}_\Gamma$.

Definition 5.14 Two global lines $\mathbf{k}$ and $\mathbf{l}$ are defined to span a **global plane** $\langle \mathbf{k}, \mathbf{l}\rangle_g$ with respect to $\mathbf{z} \in \{\mathbf{k}, \mathbf{l}\}^\perp$, if $\langle k_{\mathbf{z}}, l_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{z}}}$ is a local geometric plane of $\mathcal{G}_{\mathbf{z}}$. The global plane $\langle \mathbf{k}, \mathbf{l}\rangle_g$ consists of all global lines $\mathbf{m}$ such that $\mathbf{m} \in \mathbf{z}^\perp$ and $m_{\mathbf{z}}$ is an interior line of the local geometric plane $\langle x_{\mathbf{z}}, y_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{z}}}$ and contains all global points p with the property that $p_{\mathbf{z}} = p \cap \mathcal{L}_{\mathbf{z}}$ is an interior singular point of the local geometric plane $\langle x_{\mathbf{z}}, y_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{z}}}$.

The next step is to prove that the definition of a global plane is independent of the vertex $\mathbf{z}$ used in the definition. To this end let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$ be vertices of Γ such that $\mathbf{z}\perp\mathbf{x}\perp\mathbf{w}\perp\mathbf{y}\perp\mathbf{z}$. Since $x_{\mathbf{z}}$ and $y_{\mathbf{z}}$ are interior lines of the space $\mathcal{G}_{\mathbf{z}}$, the span of $x_{\mathbf{z}}$ and $y_{\mathbf{z}}$ is either a three-dimensional or a four-dimensional subspace in $\mathcal{G}_{\mathbf{z}}$. We want to prove that $\mathbf{x}$ and $\mathbf{y}$ span a global plane with respect to $\mathbf{z}$ if and only if they span a global plane with respect to $\mathbf{w}$. In view of Lemma 5.2 it suffices to show that $\mathbf{w}$ and $\mathbf{z}$ can be connected via a path in $\{\mathbf{x}, \mathbf{y}\}^\perp$.

Lemma 5.15 *Let $n \geq 7$ and let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$ be four vertices of Γ satisfying $\mathbf{z}\perp\mathbf{x}\perp\mathbf{w}\perp\mathbf{y}\perp\mathbf{z}$. If $\dim(\langle x_{\mathbf{z}}, y_{\mathbf{z}}\rangle) = 3$, then there exists a path from $\mathbf{z}$ to $\mathbf{w}$ in $\{\mathbf{x}, \mathbf{y}\}^\perp$. In particular, if $\mathbf{x}$, $\mathbf{y}$ span a global plane with respect to $\mathbf{z}$ if and only if they span a global plane with respect to $\mathbf{w}$ and those two global planes are equal.*

Proof The subspace $H^{\mathbf{z}}_{x_{\mathbf{z}},y_{\mathbf{z}}} = x^\pi_{\mathbf{z}} \cap y^\pi_{\mathbf{z}}$ has dimension $n-3$ and rank at least $n-4$ implying that the subspace $W_{\mathbf{y}} := \langle H^{\mathbf{y}}_{x_{\mathbf{z}},y_{\mathbf{z}}}, z_{\mathbf{y}}\rangle$ has dimension $n-1$ and rank at least $n-2$. This setting satisfies the assumption of Lemma 5.3, if $n \geq 8$, and the assumption of Lemma 5.4, if $n = 7$, thus the graph Γ contains a vertex $\mathbf{h} \in \{\mathbf{x}, \mathbf{y}, \mathbf{w}\}^\perp$ in the same connected component of $\{\mathbf{x}, \mathbf{y}\}^\perp$ as the vertex $\mathbf{z}$. The claim follows now from Lemma 5.2. □

Proposition 5.16 *Any global plane of $\mathcal{G}_\Gamma$ is finite and isomorphic to a linear plane or a symplectic plane.*

Proof Let E_g be a global plane of $\mathcal{G}_\Gamma$, i.e., $E_g = \langle \mathbf{x}, \mathbf{y}\rangle_g$ for some global lines $\mathbf{x}, \mathbf{y}$ of Γ. By Definition 5.14, the global plane E_g consists of all global lines $\mathbf{m}$ and all global points p such that the interior lines $m_{\mathbf{z}}$ and the interior points $p_{\mathbf{z}} = p \cap \mathcal{L}_{\mathbf{z}}$ are incident to the geometric plane $\langle x_{\mathbf{z}}, y_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{z}}}$ for some $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}^\perp$. Since the interior space $\mathcal{G}_{\mathbf{z}}$ is isomorphic to a subspace of $\mathcal{G}_\Gamma$ and since the global plane $E_g = \langle \mathbf{x}, \mathbf{y}\rangle_g$ is isomorphic to the local geometric plane $\langle x_{\mathbf{z}}, y_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{z}}}$, the claim follows from the fact that each plane of $\mathcal{G}_{\mathbf{z}}$ is linear or symplectic. □

Corollary 5.17 *The point-line geometry $\mathcal{G}_\Gamma$ is a non-linear space, i.e., the geometry $\mathcal{G}_\Gamma$ contains two distinct global points not incident to a common global line.*

Proof As every interior hyperbolic space $\mathcal{G}_{\mathbf{z}}$ for $\mathbf{z}$ be a vertex of Γ contains some local geometric plane isomorphic to a symplectic plane, which is a non-linear subspace of $\mathcal{G}_{\mathbf{z}}$, it follows that the geometry $\mathcal{G}$ yields some global plane, which are isomorphic to a symplectic plane by Proposition 5.16. □

Lemma 5.18 *The point-line geometry $\mathcal{G}_\Gamma = (\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ is a planar space, i.e., any two distinct intersecting global lines are contained in a unique plane.*

Proof Let $\mathbf{k}$ and $\mathbf{l}$ be two global lines contained in the global planes P_g and E_g. By Definition 5.14 we obtain that $P_g = \langle \mathbf{m}, \mathbf{n}\rangle_g = \langle m_{\mathbf{z}}, n_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{z}}}$ for some vertices $\mathbf{m}, \mathbf{z}, \mathbf{n}$ satisfying $\mathbf{m} \perp \mathbf{z} \perp \mathbf{n}$ and that $E_g = \langle \mathbf{s}, \mathbf{t}\rangle_g = \langle s_{\mathbf{x}}, t_{\mathbf{x}}\rangle^g_{\mathcal{G}_{\mathbf{x}}}$ for some vertices $\mathbf{s}, \mathbf{x}, \mathbf{t}$ of Γ such that $\mathbf{s} \perp \mathbf{x} \perp \mathbf{t}$. As the global line $\mathbf{k}$ and $\mathbf{l}$ are elements of P_g as well as elements of E_g it follows that $k_{\mathbf{z}}$ and $l_{\mathbf{z}}$ are two different interior lines of the geometric plane $\langle m_{\mathbf{z}}, n_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{z}}}$ thus $\langle k_{\mathbf{z}}, l_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{z}}} = \langle m_{\mathbf{z}}, n_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{z}}}$ as well as $k_{\mathbf{x}}$ and $l_{\mathbf{x}}$ are different interior line of $\langle s_{\mathbf{x}}, t_{\mathbf{x}}\rangle^g_{\mathcal{G}_{\mathbf{x}}}$ implying $\langle k_{\mathbf{x}}, l_{\mathbf{x}}\rangle^g_{\mathcal{G}_{\mathbf{x}}} = \langle s_{\mathbf{x}}, t_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{x}}}$. We conclude that $P_g = \langle \mathbf{m}, \mathbf{n}\rangle_g = \langle m_{\mathbf{z}}, n_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{z}}} = \langle k_{\mathbf{z}}, l_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{z}}} = \langle \mathbf{k}, \mathbf{l}\rangle_g = \langle k_{\mathbf{x}}, l_{\mathbf{x}}\rangle^g_{\mathcal{G}_{\mathbf{x}}} = \langle s_{\mathbf{x}}, t_{\mathbf{z}}\rangle^g_{\mathcal{G}_{\mathbf{x}}} = \langle \mathbf{s}, \mathbf{t}\rangle_g = E_g$. □

We will need the following notation for the last part of this section.

Definition 5.19 Let p and d be two global points of $\mathcal{G}_\Gamma$. We say that p is **perpendicular** to d, in symbols $p \perp d$, if there is a global line $\mathbf{k}$ of p and a global line $\mathbf{m}$ if d satisfying $\mathbf{k} \perp \mathbf{m}$. We denote all global points perpendicular to a global point p by $\mathcal{P}^p$ and we define $p^\sim = \mathcal{P}^p \cup p$.

Recall from Definition 4.5 that $p^\sim$ contains all global points not collinear to the global point p.

Lemma 5.20 *Let p and d be distinct global points of $\mathcal{G}_\Gamma$. Then $p^\sim \not\subseteq d^\sim$.*

Proof Let $\mathbf{l}$ be global line of p and $\mathbf{m}$ be an element of d. By Proposition 5.9 there exists a vertex $\mathbf{z} \in \{\mathbf{l}, \mathbf{m}\}^\perp$. Since $p \cap \mathcal{L}_{\mathbf{z}} = p_{\mathbf{z}}$ is a local point of $\mathcal{G}_{\mathbf{z}}$ distinct from the local point $d \cap \mathcal{L}_{\mathbf{z}} = d_{\mathbf{z}}$, by Remark 4.7 we obtain that $p_{\mathbf{z}}^\sim \not\subseteq d_{\mathbf{z}}^\sim$. Since $\mathcal{G}_{\mathbf{z}}$ is isomorphic to a subspace of $\mathcal{G}_\Gamma$, the unique global point b containing the local point $b_{\mathbf{z}} \in p_{\mathbf{z}}^\sim$ is an element of $p^\sim$. This implies $p^\sim \not\subseteq d^\sim$. □

Lemma 5.21 *Let E_g be a linear global plane and let x be a global point. Then E_g and $x^\sim$ have a global point in common, so $E_g \cap x^\sim \neq \emptyset$.*

Proof If x is incident to E_g, then the property that $x \in x^\sim$ implies $x^\sim \cap E_g \neq \emptyset$. Hence we consider the setup that x is not contained in the plane E_g. The plane E_g is by definition spanned by two different intersecting global lines $\mathbf{k}$ and $\mathbf{l}$, i.e., $E_g = \langle \mathbf{k}, \mathbf{l}\rangle_g$. Let $\mathbf{h}$ be a global line of the x, Proposition 5.9 implies the existence of vertices $\mathbf{m}$, $\mathbf{n}$ and $\mathbf{z}$ such that $\mathbf{m} \in \{\mathbf{k}, \mathbf{h}\}^\perp$, of $\mathbf{n} \in \{\mathbf{h}, \mathbf{l}\}^\perp$, and of $\mathbf{z} \in \{\mathbf{l}, \mathbf{k}\}^\perp$. In the interior hyperbolic space $G_{\mathbf{m}}$, the subspace $k_{\mathbf{m}} \cap x_{\mathbf{m}}^\pi$ is of dimension at least one, so there local point $i_{\mathbf{m}}$ in the intersection $k_{\mathbf{m}} \cap x_{\mathbf{m}}^\pi$. We remark here that the interior point $i_{\mathbf{m}}$ is not necessarily singular. If the local point $i_{\mathbf{m}}$ is indeed singular, then $i_{\mathbf{m}} \in x_{\mathbf{m}}^\sim$, and therefore $i \in x^{\sim} \cap \mathbf{k} \subseteq x^{\sim} \cap E_{\Gamma}^{\mathbf{g}}$, where i is the unique global point containing $i_{\mathbf{m}}$, and we are done. Alternatively, we consider in the interior hyperbolic space $G_{\mathbf{n}}$ an local point $j_{\mathbf{n}}$ incident to the subspace $l_{\mathbf{n}} \cap x_{\mathbf{n}}^{\pi}$. Again, if $j_{\mathbf{n}}$ is a singular interior point, then we are done.

Hence we may assume that both subspaces $l_{\mathbf{n}} \cap x_{\mathbf{n}}^{\pi} = j_{\mathbf{n}}$ and $k_{\mathbf{m}} \cap x_{\mathbf{m}}^{\pi} = i_{\mathbf{m}}$ are non-degenerate interior points. By definition of a global plane, the global lines $\mathbf{k}$ and $\mathbf{l}$ intersect in a global point p, so $p_{\mathbf{m}} = p \cap \mathcal{L}_{\mathbf{m}}$ is a singular interior point in $\mathcal{G}_{\mathbf{m}}$ as well as the interior point $p_{\mathbf{n}} = p \cap \mathcal{L}_{\mathbf{n}}$ of the space $\mathcal{G}_{\mathbf{n}}$ is singular. We may also assume that neither $p_{\mathbf{m}}$ is incident to $x_{\mathbf{m}}^{\pi}$ nor $p_{\mathbf{n}}$ is incident to $x_{\mathbf{n}}^{\pi}$, as otherwise there is nothing to prove. It follows that the interior singular points $x_{\mathbf{m}}$ and $p_{\mathbf{m}}$ span an interior line $g_{\mathbf{m}}$, which corresponds to a vertex $\mathbf{g} \in \Gamma$. Moreover, as $\mathbf{g}$ and $\mathbf{k}$ intersect in the global point p, the lines $\mathbf{k}$ and $\mathbf{g}$ span the global plane $P_g = \langle \mathbf{k}, \mathbf{g} \rangle_g \subseteq \mathcal{G}_{\mathbf{m}}$. By construction of the interior line $g_{\mathbf{m}}$, the span $\langle k_{\mathbf{m}}, g_{\mathbf{m}} \rangle$ is a non-degenerate three-dimensional subspace of $\mathcal{G}_{\mathbf{m}}$, so $P_{\mathcal{G}_{\mathbf{m}}}^{g} = \langle k_{\mathbf{m}}, g_{\mathbf{m}} \rangle_{\mathcal{G}_{\mathbf{m}}}^{g}$ is a linear geometric plane.

Next we consider the path $\mathbf{g} \perp \mathbf{m} \perp \mathbf{k} \perp \mathbf{z} \perp \mathbf{l}$ between the vertices $\mathbf{g}$ and $\mathbf{l}$ in Γ. By assumption the global plane E_g is linear, thus $H_{k_{\mathbf{z}}, l_{\mathbf{z}}}^{\mathbf{z}} = k_{\mathbf{z}}^{\pi} \cap l_{\mathbf{z}}^{\pi}$ is an $(n-3)$-dimensional non-degenerate subspace in $\mathcal{G}_{\mathbf{z}}$. As P_g is also a linear plane the subspace $H_{k_{\mathbf{m}}, g_{\mathbf{m}}}^{\mathbf{m}} = k_{\mathbf{m}}^{\pi} \cap g_{\mathbf{m}}^{\pi}$ of $\mathcal{G}_{\mathbf{m}}$ is non-degenerate and of dimension $n-3$.

We will analyse the unique induced subspace $H_{k_{\mathbf{z}}, l_{\mathbf{z}}}^{\mathbf{k}}$ and $H_{k_{\mathbf{m}}, g_{\mathbf{m}}}^{\mathbf{k}}$ inside $\mathcal{G}_{\mathbf{m}}$ and claim the existence of a vertex $\mathbf{t} \in \{\mathbf{k}, \mathbf{l}, \mathbf{g}, \mathbf{z}\}^{\perp}$. Since $V_{\mathbf{k}} := \langle m_{\mathbf{k}}, H_{k_{\mathbf{m}}, g_{\mathbf{m}}}^{\mathbf{k}} \rangle$ is a non-degenerate $(n-1)$-dimensional subspace we obtain that $W_{\mathbf{k}} := V_{\mathbf{k}} \cap z_{\mathbf{k}}^{\pi}$ is at least of dimension $(n-3)$. Since the subspace $H_{k_{\mathbf{z}}, l_{\mathbf{z}}}^{\mathbf{k}} \subseteq z_{\mathbf{k}}^{\pi}$ is $(n-3)$-dimensional and non-degenerate, the intersection $W_{\mathbf{k}} \cap H_{k_{\mathbf{z}}, l_{\mathbf{z}}}^{\mathbf{k}}$ is at least $(n-4)$-dimensional of rank at least $n-5 \geq 2$. Therefore there is an interior line $t_{\mathbf{k}}$ in $W_{\mathbf{k}} \cap H_{k_{\mathbf{z}}, l_{\mathbf{z}}}^{\mathbf{k}}$. This interior line $t_{\mathbf{k}}$ corresponds to a vertex $\mathbf{t} \in \{\mathbf{k}, \mathbf{l}, \mathbf{g}, \mathbf{z}\}^{\perp}$, as claimed.

In the interior hyperbolic space $\mathcal{G}_{\mathbf{t}}$ the interior lines $k_{\mathbf{t}}$ and $l_{\mathbf{t}}$ span the linear geometric plane $E_{\mathcal{G}_{\mathbf{t}}}^{g} = \langle k_{\mathbf{t}}, l_{\mathbf{t}} \rangle_{\mathcal{G}_{\mathbf{t}}}^{g}$. Since $\mathbf{g}$ is a vertex of the set $x \cap \mathcal{L}_{\mathbf{t}}$, the intersection $x_{\mathbf{t}} = x \cap \mathcal{L}_{\mathbf{t}}$ is an interior singular point of $\mathcal{G}_{\mathbf{t}}$. Therefore the $(n-1)$-dimensional subspace $x_{\mathbf{t}}^{\pi}$ intersects the graphical plane $E_{\mathbf{G}(\mathcal{G}_{\mathbf{t}})}$ at least in a two-dimensional subspace, which contains an interior singular point $s_{\mathbf{t}}$. Certainly $s_{\mathbf{t}} \in x_{\mathbf{t}}^{\sim}$, which implies $s \in x^{\sim} \cap E_g$, where s is the unique global point containing $s_{\mathbf{t}}$. The claim is proved. □

We have now reached our goal.

Proposition 5.22 *The point-line geometry $\mathcal{G}_{\Gamma} = (\mathcal{P}_{\Gamma}, \mathcal{L}_{\Gamma})$ is isomorphic to the geometry of singular points and hyperbolic lines of an m-dimensional non-degenerate unitary polar space over $\mathbb{F}_{q^2}$.*

Proof By Lemma 5.13 the geometry $\mathcal{G}_{\Gamma}$ is a connected partially linear space. By Corollary 5.17 it is non-linear and by Lemma 5.18 planar. Since for every vertex $\mathbf{x}$ of Γ the interior hyperbolic space $\mathcal{G}_{\mathbf{x}}$ is of order q using that $\mathcal{G}_{\mathbf{z}} \cong \mathbb{H}(U_n)$ it follows by Lemma 5.12 and the property that the geometry $\mathcal{G}_{\mathbf{z}}$ is isomorphic to a subspace of $\mathcal{G}_{\Gamma}$ that the space $\mathcal{G}_{\Gamma}$ has order q. By Lemma 5.16, the space $\mathcal{G}_{\Gamma}$ satisfies Hypothesis 1 of Theorem 4.6. The validity of Hypothesis 2 has been addressed in Remark 4.7, Hypothesis 3 follows from Lemma 5.20, Hypothesis 4 from Lemma 5.21. Hence by

Theorem 4.6 the geometry $\mathcal{G}_\Gamma$ is isomorphic to the geometry of singular points and hyperbolic lines of a non-degenerate symplectic or unitary polar space over the field $\mathbb{F}_q$ respectively $\mathbb{F}_{q^2}$. Since $\mathcal{G}_\Gamma$ contains linear planes, it is isomorphic to the geometry of hyperbolic lines of some non-degenerate unitary polar space over the field $\mathbb{F}_{q^2}$. □

Corollary 5.23 *The graph Γ is isomorphic to the hyperbolic line graph of an m-dimensional non-degenerate unitary vector space over the field $\mathbb{F}_{q^2}$.*

Proof of Theorem 1 By Corollary 5.23, we have $\Gamma \cong \mathbf{G}(U_m)$ for some $m \in \mathbb{N}$. Since $\mathbf{G}(U_m)$ is locally $\mathbf{G}(U_n)$ if and only if $m = n + 2$, cf. Proposition 3.3, necessarily $\Gamma \cong \mathbf{G}(U_{n+2})$ □

Proof of Theorem 2 [3, Section 6] and [7] provide a standard method how to derive the claim from Theorem 1. □

Appendix A: Order formulae

In this appendix for convenience of the reader we collect a number of known results that will be used extensively throughout the paper. Let U be a finite dimensional vector space over the finite field $\mathbb{F}_{q^2}$. The finite field $\mathbb{F}_{q^2}$ has an automorphism of order two $\sigma : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ with $a \mapsto \overline{a} = \sigma(a) = a^q$. By $\mathbb{F}_0 = \{a \in \mathbb{F}_{q^2} \mid a = \overline{a}\}$ we denote the fixed field of order q of $\mathbb{F}_{q^2}$ under the automorphism σ. It is well-known, see [12] or [16], that for any non-zero scalar λ of $\mathbb{F}_0$ the equation $x \cdot \overline{x} = \lambda$ has exactly $q + 1$ solutions in $\mathbb{F}_{q^2}^\times$ and the equation $x + \overline{x} = \mu$ has precisely q solutions in $\mathbb{F}_{q^2}$ for any $\mu \in \mathbb{F}_0$.

Next we fix a non-degenerate sesquilinear form $(\cdot, \cdot)$ on the n-dimensional vector space U. The Gram matrix $G_\alpha = ((v_i, v_j))_{1 \le i,j \le n}$ has full rank with respect to any basis $\alpha : v_1, \dots, v_n$ of U. A vector v of U is said to be isotropic (degenerate) and non-isotropic (non-degenerate) respectively if $(v, v) = 0$ or $(v, v) \neq 0$. If the dimension of U is at least two then the unitary vector space U contains isotropic and nonisotropic vectors.

Lemma A.1 *A n-dimensional non-degenerate unitary vector space U_n contains*

$$q^{r(n+r-2m)} \frac{\prod_{i=n+r-2m+1}^{n} (q^i - (-1)^i)}{\prod_{i=1}^{r} (q^i - (-1)^i) \prod_{i=1}^{m-r} (q^{2i} - 1)}$$

different subspaces of dimension m and rank r for $2r \le 2m \le n + r$. Furthermore in a $(n + l)$-dimensional unitary vector space U_{n+l} of rank n are

$$\sum_{k=\max\{0, \frac{2m-n-r+1}{2}\}}^{\min\{l,m-r\}} q^{r(n+r-2m+2k)+2(m-k)(l-k)} \times \frac{\prod_{i=n+r-2m+2k+1}^{n} (q^i - (-1)^i) \prod_{i=l-k+1}^{l} (q^{2i} - 1)}{\prod_{i=1}^{r} (q^i - (-1)^i) \prod_{i=1}^{m-r-k} (q^{2i} - 1) \prod_{i=1}^{k} (q^{2i} - 1)}$$

different m-dimensional subspaces of rank r for $\max\{0, \frac{2m-n-r}{2}\} \le \min l, m - r$.

Table 1 Order formulae

		number of m-dimensional subspace of rank r in U_{n+l}				
		one-dimensional subspace		two-dimensional subspace		
$n+l$	n	$r=1$	$r=0$	$r=2$	$r=1$	$r=0$
2	2	q^2-q	$q+1$			
	1	q^2	1			
	0		q^2+1			
3	3	$q^4-q^3+q^2$	q^3+1	$q^4-q^3+q^2$	q^3-1	
	2	q^4-q^3	q^3+q^2+1	q^4	q^2-1	$q+1$
	1	q^4	q^2+1		q^4+q^2	1
	0		q^4+q^2+1			q^4+q^2+1
4	4	$q^6-q^5+a^4-q^3$	$q^5+q^3+q^2+1$	$q^8-q^7+2q^6-q^5+q^4$	$q^7-q^6+q^5-q^3+q^2-q$	q^4+q^3+q+1
	3	$q^6-q^5+q^4$	q^5+q^2+1	$q^8-q^7+q^6$	$q^7+2q^4-q^3+q^2$	q^3+1
	2	q^6-q^5	$q^5+q^4+q^2+1$	q^8	$q^6-q^5+q^4-q^3$	$q^5+q^4+q^3+q^2+1$
5	5	$q^8-q^7+q^6-q^5+q^4$	$q^7+q^5+q^2+1$	$q^{12}-q^{11}+2(q^{10}-q^9+q^8)-q^7+q^6$	$q^{11}-q^{10}+2q^9-q^8+q^7+q^6-q^5+2q^4-q^3+q^2$	$q^8+q^5+q^3+1$
	4	$q^8-q^7+q^6-q^5$	$q^7+q^5+q^4+q^2+1$	$q^{12}-q^{11}+2q^{10}-q^9+q^8$	$q^{11}-q^{10}+q^9-q^7+2q^6-2q^5+q^4-q^3$	$q^8+q^7+2q^5++q^4+q^3+q^2+1$
6	6	$q^{10}-q^9+q^8-q^7+q^6-q^5$	$q^9+q^7+q^5+q^4+q^2+1$	$q^{16}-q^{15}+2q^{14}-2q^{13}+3q^{12}-2q^{11}+2q^{10}-q^9+q^8$	$q^{15}-q^{14}+2q^{13}-2q^{12}+2q^{11}-q^{10}+q^8-2q^7+2q^6-2q^5+q^4-q^3$	$q^{12}+q^{10}+q^9+q^8+2q^7+2q^5+q^4+q^3+q^2+1$

Table 1 (*Continued*)

		number of m-dimensional subspace of rank r in U_{n+l}						
		three-dimensional subspace				four-dimensional subspace		
$n+l$	n	$r=3$	$r=2$	$r=1$	$r=0$	$r=4$	$r=3$	$r=2$
4	4	$q^6-q^5+q^4-q^3$	$q^5+q^3+q^2+1$					
	3	q^6	$q^4-q^3+q^2$	q^3+1				
	2		q^6+q^4	q^2-q	$q+1$			
5	5	$q^{12}-q^{11}+2q^{10}-2q^9+2q^8-q^7+q^6$	$q^{11}-q^{10}+2q^9-q^8+q^7+q^6-q^5+2q^4-q^3+q^2$	$q^8+q^5+q^3+1$		$q^8-q^7+q^6-q^5+q^4$	$q^7+q^5+q^2+1$	
	4	$q^{12}-q^{11}+2q^{10}-q^9$	$q^{11}+q^9+2q^8-q^7+3q^6-q^5+q^4$	$q^7-q^6+q^5-q^3+q^2-q$	q^4+q^3+q+1	q^8	$q^6-q^5+q^4-q^3$	$q^5+q^3+q^2+1$

Proof This is Lemma 5.19 of [16]. □

For quick reference, we list the possibilities for all m-dimensional subspace with the rank of a n-dimensional non-degenerate unitary vector space U_n, $m \leq n$, $1 \leq n \leq 6$ as well as of a $n+l$-dimensional rank n unitary vector space for $1 \leq n+l \leq 6$ in Table 1.

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