The local recognition of reflection graphs of spherical Coxeter groups

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Abstract Based on the third author's thesis (arXiv:0805.2403) in this article we complete the local recognition of commuting reflection graphs of spherical Coxeter groups arising from irreducible crystallographic root systems.

Keywords Local recognition of graphs · Coxeter groups

1 Introduction

Given a connected graph one may ask to which extent it is determined by its local graphs, that is, by the induced subgraphs on the vertices adjacent to a particular vertex. This local recognition of graphs has been studied extensively in the literature, for

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instance in [4, 9, 18, 22, 30, 31] to mention a few; see also [3, 6, 15]. A particularly guiding example for the topic of the present article is the local recognition of the Kneser graphs studied in [14] and [16].

We are interested in the local recognition of Weyl graphs, i.e., graphs on the reflections of Coxeter groups with the commutation relation as adjacency. A combination of our findings with results from [4, 16, 18] yields the following recognition result.

Main Theorem *The following are true up to isomorphism.*

- *A Weyl graph of type A_n ($n \geqslant 8$), type B_n, C_n ($n = 3$ or $n \geqslant 5$), type D_n ($n \geqslant 9$), or type E_7 is uniquely determined, as a connected graph, by its local graphs.*
- *A Weyl graph of type $A_6, A_7, D_7, D_8, E_6, E_8$ is uniquely determined by its local graphs and its size.*
- *The Weyl graph W of type F_4 and its twisted copy (defined at the end of section 4.1) are the only bichromatic graphs of size 24 with local graphs like W.*

The remaining small Weyl graphs of type A_n as well as those of types $I_2(m)$, G_2, H_3, H_4 are locally a disjoint union of complete graphs. The graphs of type D_n are obtained as doubles of those of type A_{n-1}, so that local recognition results for type A_n transfer to D_n. Finally, types B_4 and C_4 are treated in Remark 12.

The local recognition of the Weyl graphs of type A_7, E_6 and E_8 has been established in the fundamental work [4]. The case of A_6, for which the Weyl graph is locally the Petersen graph, has been studied in [14]. Weyl graphs of types A_n and E_n which are locally cotriangular have been treated in [18]. The local recognition of types B_n and C_n is proved in Theorem 5. The Weyl graph of type F_4 is not uniquely determined by its local graphs (Corollary 11). We nevertheless characterize this Weyl graph as one of two tightest graphs with the prescribed local structure (Theorem 16). In the last section we turn to group theoretical applications of local recognition results for Weyl graphs.

2 Local recognition of graphs

All graphs considered in this text are simple and undirected. We use $\perp$ to denote adjacency, and our notation for operations on graphs like the Cartesian product or joins follows [17]. Let Γ be a graph, and $x \in \Gamma$ a vertex. We write $x^\perp$ to denote the set of neighbors of x, that is, the set of vertices adjacent to x. Likewise, for $X \subseteq \Gamma$ we write $X^\perp = \bigcap_{x \in X} x^\perp$. The induced subgraph on $x^\perp$ is called the *local graph* at x. A graph Γ is said to be *locally homogeneous*, if there exists a graph Δ such that each local graph of Γ is isomorphic to Δ. In this case, Γ is said to be locally Δ, and Δ is referred to as the local graph of Γ. If Γ is locally homogeneous, then we denote its local graph by $\Delta(\Gamma)$.

In this article we are interested in the problem of characterizing a connected locally homogeneous graph in terms of its local graph. We say that a connected locally homogeneous graph Γ is *locally recognizable*, if up to isomorphism Γ is the only connected graph that is locally $\Delta(\Gamma)$. In case Λ is another locally homogeneous graph such that $\Delta(\Lambda) \cong \Delta(\Gamma)$ we say that Λ is *locally like* Γ.

The above terminology naturally extends to bichromatic graphs. For reasons that become clear later, we distinguish the vertices of a bichromatic graph as *short* versus *long*. All morphisms between bichromatic graphs are understood to preserve this distinction. We say that a bichromatic graph is *locally homogeneous*, if the local graphs at short vertices are all isomorphic to some bichromatic graph Δ_s and the local graphs at long vertices are all isomorphic to some bichromatic graph Δ_ℓ. In this case we say that Δ_s is the *short local graph* of Γ and that Δ_ℓ is the *long local graph* of Γ. If Γ is a bichromatic locally homogeneous graph, then we denote its short local graph by $\Delta_s(\Gamma)$ and its long local graph by $\Delta_\ell(\Gamma)$. If Λ is another bichromatic locally homogeneous graph such that the short as well as the long local graphs of Λ and Γ are isomorphic as bichromatic graphs, then we say that Λ is *locally like* Γ. Finally, given a graph Γ we denote with Γ^s and Γ^ℓ the bichromatic graphs obtained from Γ with all vertices treated as short respectively long.

One easily verifies that the Kneser graph $K(n, k)$ is locally homogeneous with local graph $K(n - k, k)$. The second author proved in [16] that for n sufficiently large compared to k the Kneser graphs are locally recognizable; for $k = 2$, it suffices to require $n \geq 7$. In [14] he classified the three connected graphs which are locally the Petersen graph $K(5, 2)$. The classification of graphs that are locally $K(6, 2)$ is contained in [4].

Theorem 1 ([4, 14, 16])　*Let $k \geqslant 1$, and Γ be a connected graph that is locally $K(n, k)$.*

- *If $n \geqslant 3k + 1$ then $\Gamma \cong K(n + k, k)$.*
- *If $(n, k) = (5, 2)$ then Γ is isomorphic to one of the graphs $K(7, 2)$, $3 \cdot K(7, 2)$, or $\Sigma L_{2,25}$. In particular, $|\Gamma| \in \{21, 63, 65\}$.*
- *If $(n, k) = (6, 2)$ then Γ is isomorphic to one of the graphs $K(8, 2)$, $\mathcal{S}p_6(2)$ minus $\{x\} \cup x^\perp$ for some x, or $\mathcal{N}_6^-(2)$. In particular, $|\Gamma| \in \{28, 32, 36\}$.*

Here, the graph $3 \cdot K(7, 2)$ is the 3-fold cover of $K(7, 2)$, and $\Sigma L_{2,25}$ is the graph on the conjugates of the unique non-trivial field automorphism of $\mathbb{F}_{25}$ in the special semilinear group $\Sigma L(2, 25)$ with two elements adjacent whenever they commute. More details can be found in [14]. Further, the graph $\mathcal{S}p_{2n}(2)$ is the graph on the non-zero vectors of $V = \mathbb{F}_2^{2n}$ with two vectors adjacent whenever they are perpendicular with respect to a non-degenerate symplectic form B on V. Up to isomorphism there are only two quadratic forms Q^+ and Q^-, corresponding to maximal or minimal Witt index, on V that B is associated to, and the graph $\mathcal{N}_{2n}^\varepsilon(2)$ is the induced subgraph of $\mathcal{S}p_{2n}(2)$ on the vectors that are non-singular under Q^ε. For more details about these graphs we refer to [18].

Ernest E. Shult and the second author actually proved a lot more in [18]. They characterize the graphs that are locally cotriangular in the following sense. A graph is said to be *cotriangular*, if every pair x, y of non-adjacent vertices is contained in a cotriangle, that is, a 3-coclique $\{x, y, z\}$ such that every other vertex is adjacent to either all or exactly one of the vertices x, y, z. Observe that a join $\Gamma + \Lambda$ is cotriangular if and only if both Γ and Λ are. Denote with Γ^* the *reduced graph* of Γ, that is, the graph on the equivalence classes of vertices of Γ with the same closed neighborhood and two classes adjacent whenever some representatives are adjacent. Then Γ is

cotriangular if and only if Γ^* is. A graph Γ is called completely reduced in this context whenever $\Gamma^* = \Gamma$ and Γ can not be decomposed into $\Gamma_1 + \Gamma_2$ with non-empty Γ_1, Γ_2. A classification of all cotriangular graphs is given by the following theorem due to Ernest E. Shult.

Theorem 2 ([27]) *A finite completely reduced graph is cotriangular if and only if it is isomorphic to one of the graphs*

$$K(n, 2), \quad n \geqslant 2; \quad \mathcal{S}p_{2n}(2), \quad n \geqslant 2; \quad \mathcal{N}_{2n}^\varepsilon(2), \quad \varepsilon = \pm 1, n \geqslant 3.$$

The graphs $K(2, 2) \cong K_1$ and $K(3, 2) \cong \overline{K_3}$ are considered degenerate. Let $\mathcal{D}$ denote the set of graphs Γ such that Γ^* is a finite completely reduced cotriangular graph. If $\mathcal{G}$ is a collection of graphs, then we say that a graph Γ is locally $\mathcal{G}$ if for each $x \in \Gamma$ the local graph at x is isomorphic to some graph of $\mathcal{G}$.

Theorem 3 ([18], Main Theorem) *Let Γ be connected and locally $\mathcal{D}$. Then either Γ is locally $\{K_1, \overline{K_3}\}$ or Γ is isomorphic to one of the following graphs*

- $K(n, 2)$ *where* $n \geqslant 7$,
- $\mathcal{S}p_{2n}(2)$ *possibly with a polar subspace deleted,*
- $\mathcal{H}_{2n}^\varepsilon(T), \mathcal{G}_{2n}^\varepsilon$,
- $3 \cdot K(7, 2), \Sigma L_{2,25}$, *or* $\mathcal{N}_6^+(3)$.

The graphs $\mathcal{H}_{2n}^\varepsilon(T), \mathcal{G}_{2n}^\varepsilon$ are derived from the graph $\mathcal{S}p_{2n}(2)$; see [18]. Note that the case $k = 2$ of Theorem 1 can be regarded as a special case of the classification in Theorem 3. The following special case of Theorem 3 has already been established in [4] by Francis Buekenhout and Xavier Hubaut.

Theorem 4 ([4], Theorem 2 (3)) *Let Γ be connected and locally $\mathcal{S}p_{2n}(2)$ for some $n \geqslant 2$. Then Γ is isomorphic to one of the following graphs $\mathcal{N}_{2n+2}^+(2), \mathcal{N}_{2n+2}^-(2)$, or $\mathcal{S}p_{2n+2}(2)$ minus $\{x\} \cup x^\perp$ for some x.*

The preceding theorem has been generalized in [8, 9].

3 Local recognition of Weyl graphs

We assume that the reader is familiar with Coxeter groups and root systems as treated in [20] or [5]. The *commuting graph* of a group G on $X \subseteq G$ is the graph with vertex set X in which two vertices $g, h \in X$ are adjacent whenever g and h commute. We will study the commuting graphs of finite Coxeter groups on their reflections. Since we are interested in local recognition results we will focus on finite irreducible Coxeter groups for which the reflection graph is locally homogeneous. The graphs arising from the cases H_3, H_4 and $I_2(m)$ are locally disjoint unions of complete graphs and therefore not interesting for the purpose of local recognition. Hence, we further restrict to Coxeter groups which arise from irreducible crystallographic root

systems. These are those with Dynkin diagram equal to one of A_n ($n \geqslant 1$), B_n or C_n ($n \geqslant 2$), D_n ($n \geqslant 4$), E_6, E_7, E_8, F_4, or G_2.

Recall that each root of an irreducible crystallographic root system Φ is considered either short or long (with the convention that in the absence of two distinct root lengths every root is long). If M is the Dynkin diagram of Φ then we denote with $W(M)$ the Weyl group of Φ, i.e., the group generated by the reflections through the roots of Φ, together with the notion of a short (respectively long) root reflection by $W(M)$. The *Weyl graph* $\mathbb{W}(M)$ is the commuting graph of $W(M)$ on its reflections. If M is simply laced then all reflections in $W(M)$ are conjugate, which implies that the Weyl graph $\mathbb{W}(M)$ is locally homogeneous. On the other hand, if M is not simply laced then there are two conjugacy classes of reflections in $W(M)$, namely short and long root reflections, and we regard $\mathbb{W}(M)$ as a bichromatic graph. Instead of assigning arbitrary colors we accordingly refer to the vertices of $\mathbb{W}(M)$ corresponding to short (respectively long) root reflections as short (respectively long) vertices. As a bichromatic graph, the Weyl graph $\mathbb{W}(M)$ is locally homogeneous.

$\mathbb{W}(A_n)$ is the graph with vertices $y_{i,j}$, $1 \leqslant i < j \leqslant n+1$, such that $y_{i,j} \perp y_{k,l}$ if and only if $\{i,j\} \cap \{k,l\} = \emptyset$. Consequently, the Weyl graph $\mathbb{W}(A_n)$ is isomorphic to the Kneser graph $K(n+1, 2)$. Likewise, $\mathbb{W}(D_n)$ is the graph with vertices $y_{i,j}$, $1 \leqslant i \neq j \leqslant n$, such that $y_{i,j} \perp y_{k,l}$ if and only if $\{i,j\} \cap \{k,l\} = \emptyset$ or $(k,l) = (j,i)$. $\mathbb{W}(D_n)$ is therefore isomorphic to the composition graph $K(n,2)[K_2]$, that is, the graph arising from the Kneser graph $K(n,2)$ by replacing each vertex by an adjacent pair of vertices. Accordingly, Theorem 1 applies and yields the recognition results of the Main Theorem for types A_n and D_n. By [4] we have $\mathbb{W}(E_6) \cong \mathcal{N}_6^-(2)$, $\mathbb{W}(E_7) \cong \mathcal{S}p_6(2)$ and $\mathbb{W}(E_8) \cong \mathcal{N}_8^+(2)$. The corresponding recognition results of the Main Theorem follow from Theorems 1, 3 and 4.

$\mathbb{W}(B_n)$ is the bichromatic graph with vertices $y_{i,j}$, $1 \leqslant i, j \leqslant n$, where the $y_{i,i}$ are short and the $y_{i,j}$ with $i \neq j$ are long vertices, and $y_{i,j} \perp y_{k,l}$ if and only if $\{i,j\} \cap \{k,l\} = \emptyset$ or $(k,l) = (j,i)$. The Weyl graph $\mathbb{W}(C_n)$ is obtained from $\mathbb{W}(B_n)$ by exchanging the role of short and long vertices. The recognition results of the Main Theorem for types B_n and C_n are therefore contained in the following theorem.

Theorem 5 *Let $n = 3$ or $n \geqslant 5$, and let Γ be a connected bichromatic graph which is locally like $\mathbb{W}(B_n)$. Then $\Gamma \cong \mathbb{W}(B_n)$.*

Proof It is straightforward to check the case $n = 3$.

Next, let $n \geqslant 6$. Let X be a short component of Γ and $x \in X$ a short vertex. The short induced subgraph on $x^\perp$ is a clique on $n-1$ elements which implies that X is a clique on n elements. By assumption, the long neighbors of x induce a subgraph isomorphic to the long induced subgraph of $\mathbb{W}(B_{n-1})$. This subgraph is isomorphic to $\mathbb{W}(D_{n-1})$ and, in particular, is connected for $n \geqslant 6$. This implies that all long neighbors of x are contained in a single long component Y of Γ. Consider a short vertex $x_1 \in X$ adjacent to x. Again, all long neighbors of x_1 lie in one long component of Γ. But looking at $\{x, x_1\}^\perp \subset x^\perp$ we see that x and x_1 share long neighbors whence this component has to be Y as well. Since X is connected this shows that all long vertices adjacent to some vertex of X are contained in Y. Likewise, let $y \in Y$. The short induced subgraph of $y^\perp$ is a clique on n vertices and thus in particular connected.

Again, we see that for a long vertex y_1 adjacent to y the common neighbors $\{y, y_1\}^\perp$ contain a short vertex. Therefore the same argument as before shows that all short vertices adjacent to some vertex of Y are contained in X. Since Γ is connected this proves that X and Y are the only short respectively long components of Γ.

We count the number of long vertices by counting the long neighbors of the n short vertices of Γ. By assumption, a short vertex has $(n-1)(n-2)$ long neighbors. Further, two short vertices have $(n-2)(n-3)$ long neighbors in common, three short vertices have $(n-3)(n-4)$ long neighbors in common, and so on. Thus there are

$$\binom{n}{1}(n-1)(n-2) - \binom{n}{2}(n-2)(n-3) + \ldots + (-1)^{n-1}\binom{n}{n-2}2 = n(n-1)$$

long vertices in Γ. Note that for the above equation we exploited that the alternating sum of the binomial coefficients equals zero, that is, $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

Let $x_1, x_2, \ldots, x_n$ be the short vertices of Γ. Γ is locally $\mathbb{W}(B_{n-1})$ at short vertices which implies that for $1 \leqslant i \neq j \leqslant n$ the common neighborhood $\{x_r : r \notin \{i, j\}\}^\perp$ contains exactly two long vertices which we denote by $y_{i,j}$ and $y_{j,i}$. Since a long vertex is adjacent to exactly $n-2$ short vertices the $y_{i,j}$ thus defined are all distinct. By construction, $y_{i,j} \perp y_{j,i}$. Further, the $y_{i,j}$ exhaust Y because Γ contains exactly $n(n-1)$ long vertices. Given two vertices $y_{i,j}$ and $y_{k,l}$, we find $m \in \{1, 2, \ldots, n\}\setminus\{i, j, k, l\}$ whence $y_{i,j}$ and $y_{k,l}$ are both contained in $x_m^\perp \cong \mathbb{W}(B_{n-1})$. $y_{i,j}$ is characterized in $x_m^\perp$ as one of the two long vertices contained in $\{x_r : r \notin \{i, j, m\}\}^\perp$. Likewise, $y_{k,l}$ is characterized in $x_m^\perp$ as one of the two long vertices contained in $\{x_r : r \notin \{k, l, m\}\}^\perp$. Consequently, for $\{i, j\} \neq \{k, l\}$, $y_{i,j} \perp y_{k,l}$ if and only if $\{i, j\} \cap \{k, l\} = \emptyset$. Hence, $\Gamma \cong \mathbb{W}(B_n)$.

Finally, consider the case $n = 5$. We still find that each short component is a clique on 5 vertices. Let X be one such short component. We count that there are 20 long vertices neighbored to one of the vertices of X. On the other hand, we see again that each long component has short neighbors in only one short component. Accordingly, the 20 long neighbors of X constitute a union of long components. However, a long component is locally $K_1 \sqcup 3 \cdot K_2$ and therefore has at least 12 vertices. We conclude that there is only one long component Y with vertices neighbored to X. Now, the remainder of the preceding argument applies and shows that $\Gamma \cong \mathbb{W}(B_5)$ as claimed. $\square$

The case $n = 4$ of Theorem 5 is discussed in Remark 12 where it is shown that there are infinitely many finite connected bichromatic graphs that are locally like $\mathbb{W}(B_4)$. The case of type F_4 is discussed in detail in the next section. Note that the Weyl graph $\mathbb{W}(G_2)$ is isomorphic to three disjoint edges of mixed type.

4 Local recognition of $\mathbb{W}(F_4)$

4.1 Graphs locally like $\mathbb{W}(F_4)$

The Weyl graph $\mathbb{W}(F_4)$ is a connected bichromatic locally homogeneous graph on 24 vertices with short local graph $\mathbb{W}(B_3)$ and long local graph $\mathbb{W}(C_3)$. As we will

see shortly, $\mathbb{W}(F_4)$ is not locally recognizable. Before we turn to investigating additional constraints under which we seek to recognize $\mathbb{W}(F_4)$ nonetheless, we study connected bichromatic graphs Γ which are locally like $\mathbb{W}(F_4)$. The results we obtain then guide our way in determining appropriate conditions under which we are able to recognize $\mathbb{W}(F_4)$ alongside its twisted copy. An easy but crucial observation to start with is the following.

Proposition 6 *Let Γ be locally like $\mathbb{W}(F_4)$. The short (respectively long) induced subgraph of Γ is isomorphic to a disjoint union of 4-cliques.*

Let Γ be a bichromatic graph that is locally like $\mathbb{W}(F_4)$. Observe that the graph obtained from Γ by exchanging the roles of short and long vertices is locally like $\mathbb{W}(F_4)$ as well. Results that we obtain for short vertices of graphs locally like $\mathbb{W}(F_4)$ are therefore also true for long vertices.

Paraphrasing Proposition 6, the vertices of Γ come in 4-cliques of the same type. In order to simplify things it is natural to collapse these 4-cliques into single vertices.

Definition 7 Let Λ be a graph and Π a partition of its vertices. The *contraction* Λ/Π is the graph on Π such that two sets $A, B \in \Pi$ are adjacent whenever there is $a \in A$ and $b \in B$ which are adjacent in Λ. If Λ is bichromatic then Π is required to partition into sets of short and long vertices and Λ/Π is a bichromatic graph in the natural way.

In this language, we thus investigate the collapsed graph Γ/Π where Π is the partition of Γ into short and long 4-cliques. To this end, we analyze how these 4-cliques relate to each other.

Proposition 8 *Let Γ be locally like $\mathbb{W}(F_4)$, and x_1, x_2, x_3, x_4 a short 4-clique in Γ. Let $i \neq j$ and $k \neq l$.*

- $\{x_i, x_j\}^\perp$ *is locally $K_2^s \sqcup K_2^\ell$. In particular, for any pair x_i, x_j there exist unique long vertices $y_{i,j}, y_{j,i}$ contained in $\{x_i, x_j\}^\perp$.*
- $\{x_i, x_j, x_k\}^\perp$ *contains no long vertex if i, j, k are distinct. In particular, the vertices $y_{i,j}$ are all distinct.*
- *There are exactly 12 long vertices adjacent to at least one of the x_i, namely the above vertices $y_{i,j}$.*
- $y_{i,j} \perp y_{k,l}$ *implies that $\{k, l\} = \{i, j\}$ or $\{k, l\} \cap \{i, j\} = \emptyset$.*

Proof Exploiting the local structure at x_i we see that every short adjacent pair x_i, x_j has exactly two long neighbors in common which we will (arbitrarily) denote by $y_{i,j}$ and $y_{j,i}$. Accordingly, $y_{i,j} \perp y_{j,i}$. Looking at the neighbors of a vertex $y_{i,j}$ reveals that x_i and x_j are the only short vertices among x_1, x_2, x_3, x_4 which are adjacent to $y_{i,j}$. Consequently, the $y_{i,j}$ are 12 distinct vertices. Since three adjacent short vertices share no long neighbors we count that exactly

$$\binom{4}{1}6 - \binom{4}{2}2 = 12$$

long vertices are neighbored to at least one of the vertices x_1, x_2, x_3, x_4. Consequently, the long neighbors of the x_i are precisely the vertices $y_{i,j}$. For the last claim, assume that $y_{i,j} \perp y_{k,l}$ and $\{k, l\} \cap \{i, j\} = \{i_0\}$. A look at the neighbors of x_{i_0} shows that this is a contradiction.   □

If Γ is locally like $\mathbb{W}(F_4)$ and Π is the partition of Γ into short and long 4-cliques, then we add the following extra structure to the collapsed graph Γ / Π. Two vertices $X, Y \in \Gamma / \Pi$ are said to be *strongly connected* if every $x \in X$ is at distance 1 from Y in Γ and vice versa. In this case, we think of X and Y as being connected by two edges, the reason of which will be clear from the next proposition. The number of neighbors of X where we count those neighbors twice that are strongly connected to X is said to be the *bivalency* of X.

Proposition 9 *Let Γ be locally like $\mathbb{W}(F_4)$, and let Π be the partition of Γ into short and long 4-cliques. The contraction Γ / Π is bipartite of bivalency 6.*

Proof Let $X \in \Gamma / \Pi$ be a short vertex. By Proposition 6, X has only long neighbors. $X = \{x_1, x_2, x_3, x_4\}$ is a 4-clique of Γ and according to Proposition 8 there are 12 long vertices $y_{i,j}$ at distance 1 from X in Γ. Each pair of vertices $y_{i,j}, y_{j,i}$ is contained in exactly one long neighbor $Y_{\{i,j\}}$ of X. Let k, l be the indices such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then, by Proposition 8, either $Y_{\{k,l\}} \neq Y_{\{i,j\}}$, in which case both long vertices are connected to X by just one edge, or $Y_{\{k,l\}} = Y_{\{i,j\}}$, in which case both long vertices are connected to X by two edges. In any case, we count that the bivalency of X is 6.   □

We now do the reverse and prove that every bipartite graph of bivalency 6 is the contraction of some graph which is locally like $\mathbb{W}(F_4)$. Note, however, that non-isomorphic graphs locally like $\mathbb{W}(F_4)$ can have isomorphic contractions.

Lemma 10 *For every connected bipartite graph Λ of bivalency 6 there is a connected bichromatic graph Γ that is locally like $\mathbb{W}(F_4)$ such that $\Gamma / \Pi = \Lambda$ where Π is the partition of Γ into short and long 4-cliques.*

Proof Let Λ be a bipartite graph of bivalency 6. Exploiting that Λ is 2-colorable, we may identify Λ with a bichromatic graph such that no two vertices of the same type are adjacent. For any vertex x of Λ choose an injection

$$x^\perp \to \binom{4}{2}, \quad y \mapsto a(x, y)$$

from its neighbors to the six 2-subsets of $\{1, 2, 3, 4\}$ such that for strongly connected vertices x, y the complement of $a(x, y)$ is not attained. This is always possible since Λ has bivalency 6. To every directed edge (x, y) we thus assigned the 2-subset $a(x, y)$ of $\{1, 2, 3, 4\}$. Construct the bichromatic graph Γ from Λ as follows. For every vertex $x \in \Lambda$ add a 4-clique x_1, x_2, x_3, x_4 of the same type as x to Γ. Then, for $x, y \in \Lambda$ let x_i and y_j be adjacent in Γ if and only if x and y are adjacent in Λ and the following holds: either $(i, j) \in a(x, y) \times a(y, x)$, or x and y are strongly

connected and $(i, j) \in \overline{a(x, y)} \times \overline{a(y, x)}$. By construction, contracting the 4-cliques of Γ produces Λ. It is straightforward to check that Γ is locally like $\mathbb{W}(F_4)$. □

Corollary 11 *There exist infinitely many non-isomorphic finite connected bichromatic graphs that are locally like $\mathbb{W}(F_4)$.*

Proof We claim that there are infinitely many finite connected bipartite graphs Λ that are locally $\overline{K_6}$ and hence of bivalency 6 (if no vertices are assumed to be strongly connected). To this end, note that the graphs $C_k \times C_m \times C_n$ are connected and locally $\overline{K_6}$ for $k, m, n \geqslant 4$. Since cycles C_n are 2-colorable whenever n is even, the graphs $C_k \times C_m \times C_n$ are 2-colorable and hence bipartite whenever k, m, n are all even. The claim follows from Lemma 10. □

Remark 12 Analogous to Lemma 10 one shows that for every connected bipartite graph Λ of bivalency $(2, 6)$ (meaning that vertices of one type have valency 2 and vertices of the other type have valency 6) there is a connected bichromatic graph Γ that is locally like $\mathbb{W}(B_4)$ such that $\Gamma/\Pi = \Lambda$ where Π is again the partition of Γ into short and long 4-cliques. This easily implies that there are infinitely many finite connected bichromatic graphs that are locally like $\mathbb{W}(B_4)$.

Let Γ be locally like $\mathbb{W}(F_4)$ and assume that the collapsed graph Γ/Π contains strongly connected vertices X and Y. This means that, say, $X = x_1, x_2, x_3, x_4$ are short vertices, $Y = y_1, y_2, y_3, y_4$ are long vertices, and we have the adjacencies

$$x_1, x_2 \perp y_1, y_2, \qquad x_3, x_4 \perp y_3, y_4.$$

It is straightforward to check that replacing these by

$$x_1, x_2 \perp y_3, y_4, \qquad x_3, x_4 \perp y_1, y_2$$

produces a graph Γ' which is also locally like $\mathbb{W}(F_4)$. We say that Γ' is a *twisted copy* of Γ. In particular, for $\Gamma = \mathbb{W}(F_4)$ these twisted copies are all isomorphic and we denote the resulting graph by $\mathbb{W}(F_4)'$.

4.2 Recognition results

We now discuss further properties of the Weyl graph $\mathbb{W}(F_4)$ in order to characterize $\mathbb{W}(F_4)$ among the connected bichromatic graphs that are locally like $\mathbb{W}(F_4)$. For more details we refer to the thesis [28] of the third author. We start with some easy observations.

Proposition 13 *Let Γ be a finite bichromatic graph that is locally like $\mathbb{W}(F_4)$. Then the numbers of short and long vertices in Γ are the same, $|\Gamma|$ is divisible by 8 and $|\Gamma| \geqslant 24$.*

Since $|\mathbb{W}(F_4)| = 24$ we see that, in a sense, $\mathbb{W}(F_4)$ is maximally tight among the graphs that are locally like $\mathbb{W}(F_4)$. There are several further properties of a graph, for instance its diameter, that describe its tightness. The following notion of tight connectedness is another way to express tightness of a bichromatic graph.

Definition 14 A bichromatic graph is said to be *tightly connected* if every long vertex has a neighbor in every short component and vice versa.

These three notions of tightness, however, are not local in nature (where a local property is meant to be one which can be expressed in terms of the neighbors of each vertex). In order to find a more local notion to describe the tightness of $\mathbb{W}(F_4)$ we investigate the relation of vertices at distance 2. The following is straightforward to check.

Proposition 15 *Let Γ be locally like $\mathbb{W}(F_4)$, and let $x, y \in \Gamma$ be at distance 2.*

- *If x, y are both short (respectively long) vertices then $\{x, y\}^{\perp} \cong \mu(x, y) \cdot K_1^{\ell}$ (respectively K_1^s) for some $\mu(x, y) \in \{1, 2, 3\}$.*
- *If x, y are of mixed type then $\{x, y\}^{\perp} \cong \mu_s(x, y) \cdot K_2^s \sqcup \mu_{\ell}(x, y) \cdot K_2^{\ell}$ for some $\mu_s(x, y) + \mu_{\ell}(x, y) \in \{1, 2\}$.*

For the Weyl graph $\mathbb{W}(F_4)$ the parameters μ, μ_s, μ_{ℓ} defined in Proposition 15 are constant and take the maximum possible values $\mu = 3$ and $\mu_s = \mu_{\ell} = 1$ which is another, more local, instantiation of the tightness of $\mathbb{W}(F_4)$. Notice that the condition $\mu_s = \mu_{\ell} = 1$ is equivalent to the contraction Γ/Π, studied in Proposition 9, being locally homogeneous with $\Delta_s(\Gamma/\Pi) \cong \overline{K_3}^{\ell}$ and $\Delta_{\ell}(\Gamma/\Pi) \cong \overline{K_3}^s$.

The following theorem summarizes our recognition results for the Weyl graph $\mathbb{W}(F_4)$. Note that all of the provided conditions under which a graph Γ is almost recognized as $\mathbb{W}(F_4)$ are statements which describe the tightness of Γ.

Theorem 16 *Let Γ be a connected bichromatic graph that is locally like $\mathbb{W}(F_4)$. Assume that*

- *$|\Gamma| = 24$, or*
- *Γ is tightly connected, or*
- *Γ has diameter 2, or*
- *$\mu = 3$.*

If one of these conditions holds then Γ is isomorphic to $\mathbb{W}(F_4)$ or to its twisted copy $\mathbb{W}(F_4)'$. In particular, $\mathrm{Aut}(\Gamma) \cong W(F_4)/Z$ where Z denotes the center of $W(F_4)$.

We prove Theorem 16 by a series of propositions. The proof of the case $\mu = 3$ is similar in spirit to the previous ones. It is therefore omitted; the interested reader is referred to [28] for the details.

Proposition 17 *Let Γ be a connected bichromatic graph that is locally like $\mathbb{W}(F_4)$. If $|\Gamma| = 24$ then $\Gamma \cong \mathbb{W}(F_4)$ or $\Gamma \cong \mathbb{W}(F_4)'$. Further, $\mathrm{Aut}(\Gamma) \cong W(F_4)/Z$.*

Proof As observed in Proposition 13, every graph that is locally like $\mathbb{W}(F_4)$ has at least 12 short and 12 long vertices. Γ therefore consists of exactly 12 vertices of each type.

Let x_1, x_2, x_3, x_4 be a short 4-clique. Adopting the notation of Proposition 8, let $y_{i,j}$ and $y_{j,i}$ be the long vertices adjacent to both x_i and x_j. The $y_{i,j}$ are 12 distinct

vertices and therefore constitute the long vertices of Γ. It follows from Proposition 8 that the three long 4-cliques are given by $y_{i,j}, y_{j,i}, y_{k,l}, y_{l,k}$ for disjoint $\{i, j\}$ and $\{k, l\}$.

Each of the remaining eight short vertices has exactly two long neighbors in each of the three long 4-cliques. Let x_5 be one of remaining short vertices. The two neighbors of x_5 in a 4-clique $y_{i,j}, y_{j,i}, y_{k,l}, y_{l,k}$ are one of $y_{i,j}, y_{j,i}$ along with one of $y_{k,l}, y_{l,k}$. We ambiguously defined the vertices $y_{i,j}, y_{j,i}$ as the long vertices contained in $\{x_i, x_j\}^{\perp}$ so we may as well assume that x_5 is adjacent to $y_{i,j}$ and $y_{k,l}$ with $i < j$ and $k < l$. Let x_6 be the unique short vertex also adjacent to $y_{1,2}, y_{3,4}$. Likewise, let x_7 be the short vertex also adjacent to $y_{1,3}, y_{2,4}$, and x_8 the short vertex also adjacent to $y_{1,4}, y_{2,3}$. By construction, x_5, x_6, x_7, x_8 is a 4-clique. Notice that for instance $x_5, x_6 \in \{y_{1,2}, y_{3,4}\}^{\perp}$ implies that $x_7, x_8 \in \{y_{2,1}, y_{4,3}\}^{\perp}$. Altogether this determines the induced subgraph on $x_1, x_2, \ldots, x_8$ along with the vertices $y_{i,j}$.

Let $x_9, x_{10}, x_{11}, x_{12}$ be the remaining short 4-clique. We may assume that x_9, x_{10} are the short vertices contained in $\{y_{1,2}, y_{4,3}\}^{\perp}$. Accordingly, $x_{11}, x_{12} \in \{y_{2,1}, y_{3,4}\}^{\perp}$. We may also assume that x_9 is contained in $\{y_{1,3}, y_{4,2}\}^{\perp}$ (because if both x_9 and x_{10} were not contained in $\{y_{1,3}, y_{4,2}\}^{\perp}$ then both $x_{11}, x_{12} \in \{y_{1,3}, y_{4,2}\}^{\perp}$ which contradicts $x_{11}, x_{12} \in \{y_{2,1}, y_{3,4}\}^{\perp}$). Further, we may assume that x_{11} is the second short vertex contained in $\{y_{1,3}, y_{4,2}\}^{\perp}$. Consider the two short vertices in $\{y_{1,4}, y_{3,2}\}^{\perp}$. These can be either x_9, x_{12} or x_{10}, x_{11}, and either choice determines Γ. Denote with Γ_1 the graph corresponding to the choice $x_9, x_{12} \in \{y_{1,4}, y_{3,2}\}^{\perp}$, and with Γ_2 the graph corresponding to the choice $x_{10}, x_{11} \in \{y_{1,4}, y_{3,2}\}^{\perp}$. The following table summarizes adjacency involving the vertices $x_9, x_{10}, x_{11}, x_{12}$.

by construction	$x_9, x_{10} \perp y_{1,2}, y_{4,3}$	$x_{11}, x_{12} \perp y_{2,1}, y_{3,4}$
	$x_9, x_{11} \perp y_{1,3}, y_{4,2}$	$x_{10}, x_{12} \perp y_{3,1}, y_{2,4}$
Γ_1	$x_9, x_{12} \perp y_{1,4}, y_{3,2}$	$x_{10}, x_{11} \perp y_{4,1}, y_{2,3}$
Γ_2	$x_9, x_{12} \perp y_{4,1}, y_{2,3}$	$x_{10}, x_{11} \perp y_{1,4}, y_{3,2}$

An implementation in the computer algebra system SAGE, see [26], of the graphs Γ_1 and Γ_2 can be found in the appendix of [28]. In particular, it is verified that Γ_1 and Γ_2 are non-isomorphic, that the automorphism group of both graphs is isomorphic to $W(F_4)/Z$, and that Γ_1 is isomorphic to $\mathbb{W}(F_4)$. Accordingly, Γ_2 is isomorphic to the twisted copy $\mathbb{W}(F_4)'$. □

Proposition 18 *Let Γ be a connected bichromatic graph that is locally like $\mathbb{W}(F_4)$. If Γ is tightly connected then $\Gamma \cong \mathbb{W}(F_4)$ or $\Gamma \cong \mathbb{W}(F_4)'$.*

Proof Fix a short 4-clique x_1, x_2, x_3, x_4. Because of tightness every long vertex is adjacent to one of the x_i, and by Proposition 8 there are exactly 12 such long vertices. Thus Γ consists of 12 long vertices. Likewise, Γ contains exactly 12 short vertices. Hence, $|\Gamma| = 24$, and the claim follows from Proposition 17. □

Proposition 19 *Let Γ be a connected bichromatic graph that is locally like $\mathbb{W}(F_4)$. If Γ has diameter 2 then $\Gamma \cong \mathbb{W}(F_4)$ or $\Gamma \cong \mathbb{W}(F_4)'$.*

Proof Let x_1, x_2, x_3, x_4 be a short 4-clique. As in Proposition 8 let $y_{i,j}, y_{j,i}$ be the long vertices adjacent to both x_i and x_j. Assume that there is a long vertex v which

is not among the 12 long vertices $y_{i,j}$. Because v is not adjacent to any of the x_i and since the diameter of Γ is 2, we find a long vertex that connects x_1 and v. Without loss of generality let this long vertex be $y_{1,2}$. This prevents $y_{1,2}, y_{2,1}, y_{3,4}, y_{4,3}$ from forming a long 4-clique. By Proposition 8 there are thus long vertices v_1, v_2 not among the $y_{i,j}$ such that $y_{3,4}, y_{4,3}, v_1, v_2$ form a long 4-clique. Again, v_1 is not adjacent to any of the x_i and hence is connected to x_1 by a long vertex. This is a contradiction since the long vertices adjacent to x_1 are the vertices $y_{1,j}, y_{j,1}$.

Consequently, Γ contains no further long vertices besides the 12 vertices $y_{i,j}$, and hence, by Proposition 13, $|\Gamma| = 24$. Apply Proposition 17.   □

5 Group-theoretic applications

Our guiding example for the application of the local recognition of graphs in group theory is the characterization of the symmetric groups by means of the structure of its transposition centralizers; cf. [10, Theorem 27.1]. A detailed proof of [10, Theorem 27.1] using local recognition results for the Weyl graphs of type A_n is contained in the third author's thesis [28]; that proof runs along the lines of the proof of [7, Theorem 1.2]. An early example of such results can be found in [21] which along with [4] has been one of the original motivations for the second author to pursue the local recognition of Kneser graphs in [16]. Another fundamental example for this theme is [22].

Likewise, the recognition results for the Weyl graph of type F_4 discussed in the previous section give rise to the following local characterization of the Weyl group $W(F_4)$. Again, we refer to [28] for a proof inspired by [7].

Theorem 20 *Let G be a group with non-conjugate involutions x, y such that*

- $C_G(x) = \langle x \rangle \times J$ *with* $J \cong W(B_3)$,
- $C_G(y) = \langle y \rangle \times K$ *with* $K \cong W(C_3)$,
- x *(respectively y) is a short (respectively long) root reflection in K (respectively J),*
- $J \cap K$ *contains involutions x_1, y_1 such that x_1 (respectively y_1) is a short (respectively long) root reflection in K as well as in J, and*
- *there are a long root reflection $y_0 \neq y, y_1$ in J and a short root reflection $x_0 \neq x, x_1$ in K such that x_0 and y_0 commute.*

If $G = \langle J, K \rangle$ then $G \cong W(F_4)$.

The interest in group-theoretic local recognition results like Theorem 20 stems from the classification of finite simple groups (outlined in [10]) and the fact that the majority of finite simple groups arises from (twisted) Chevalley groups. These can be defined as groups generated by subgroups isomorphic to $SL(2, q)$ subject to certain relations by the Curtis–Tits theorem formulated as in [19, 23, 29], by Phan's theorems [24, 25], and by the Phan-type theorems [2, 12, 13]. Recently, see [11, Local recognition theorem 1], Kristina Altmann and the first author proved a local recognition result for Chevalley groups of (twisted) type A_7 and E_6 based on results

and techniques of [1]; this result makes serious use of the local recognition of graphs that are locally the Weyl graph of type A_5 and of the Curtis–Tits theorem and Phan's theorems. We hope that our analysis can help to approach a similar recognition result for Chevalley groups of type F_4 based on the Phan-type theorem of type F_4 proved by Hoffman, Mühlherr, Shpectorov and the first author and published in [13]. For more details we refer to the thesis [28] of the third author and the survey [11] of the first author.

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