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Lie expression for multi-parameter Klyachko idempotent

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Abstract An expression for the multi-parameter Klyachko idempotent as a linear combination of Lie base elements is given.

Keywords Free Lie algebras · Lie elements · Klyachko idempotent

1 Introduction

Let $\mathcal{F}$ be the free associative algebra with generators $a_1, a_2, \dots,$ over the field of complex numbers **C**. Its elements are associative polynomials in non-commuting variables $a_1, a_2, \dots,$ with coefficients from **C**. Let $\mathcal{F}^-$ be the Lie algebra of $\mathcal{F}$ under commutator $[a, b] = ab - ba$. Endow $\mathcal{F}$ with the structure of a Hopf algebra by defining a coproduct δ such that $\delta(a_i) = a_i \otimes 1 + 1 \otimes a_i$. Let $\mathcal{L}$ be the Lie subalgebra of $\mathcal{F}$ generated by $a_1, a_2, \dots$. So $\mathcal{L}$ is the free Lie algebra on generators $a_1, a_2, \dots$.

An element $X = f(a_1, \dots, a_n) \in \mathcal{F}$ is called a *Lie element* if $X \in \mathcal{L}$. There is a well-known criterion for X to be a Lie element (Friedrich's criterion): X is a Lie element if and only if X is primitive element of $\mathcal{F}$ (considered as a Hopf algebra),

$$\delta(X) = X \otimes 1 + 1 \otimes X.$$

If X is a homogeneous Lie element of degree m, the Dynkin–Specht–Wever theorem allows us to construct its Lie expression

$$\pi X = mX,$$

where πX denotes Lie element corresponding to X constructed by

$$\pi a_i = a_i, \qquad \pi(a_{i_1} \dots a_{i_m}) = \big[a_{i_1}, \big[\cdots [a_{i_{m-1}}, a_{i_m}] \cdots\big]\big].$$

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See [4] or [9] for details. For example, a_1^2 is not Lie element, but $X = -a_1^2 a_2 + 2a_1a_2a_1 - a_2a_1a_1$ is a Lie element with $X = -[a_1, [a_1, a_2]]$.

A.A. Klyachko in [5] has constructed the following idempotent in the group algebra of the permutation group:

$$k_n = \frac{1}{n} \sum_{\sigma \in S_n} q^{maj(\sigma)} \sigma.$$

Recall that the major-index $maj(\sigma)$ is defined as a sum of descent indices,

$$maj(\sigma) = \sum_{1 \le i < n, \sigma(i) > \sigma(i+1)} i.$$

An important property of the Klyachko idempotent is that the element

$$k_n(a_1, \dots, a_n) = \frac{1}{n} \sum_{\sigma} q^{maj(\sigma)} a_{\sigma(1)} \cdots a_{\sigma(n)}$$

is a Lie element, assuming q is a primitive n-th root of unity. For example,

$$2\, k_2(a_1, a_2) = a_1a_2 - a_2a_1 = [a_1, a_2] \in \mathcal{L}.$$

The case $n = 3$ is not so evident. If

$$q = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2},$$

then

$$3\, k_3(a_1, a_2, a_3) = a_1a_2a_3 + q^2 a_1a_3a_2 + q\, a_2a_1a_3 + q^2 a_2a_3a_1 + q\, a_3a_1a_2 + q^3 a_3a_2a_1.$$

By Friedrich's criterion

$$k_3(a_1, a_2, a_3) \in \mathcal{L},$$

and by the Dynkin–Specht–Weber theorem,

$$\begin{aligned} &3k_3(a_1, a_2, a_3) \\ &\quad = \big[[a_1, a_2], a_3\big] + q^2\big[[a_1, a_3], a_2\big] + q\big[[a_2, a_1], a_3\big] + q^2\big[[a_2, a_3], a_1\big] \\ &\qquad + q\big[[a_3, a_1], a_2\big] + q^3\big[[a_3, a_2], a_1\big]. \end{aligned}$$

But one can check that the following is a simpler expression for k_3 as a Lie element,

$$k_3(a_1, a_2, a_3) = \big[a_1, [a_2, a_3]\big] + q\big[a_2, [a_1, a_3]\big] \in \mathcal{L},$$

where in all summands the last element a_3 seats in the last place.

In this paper we show that this kind of construction exists for any n. We give expression of the Klyachko element as a Lie element in a more general context. Let $[n] = \{1, 2, \dots, n\}$ and S_n be its permutation group. We will use one-line notation for permutations: instead of $\left(\begin{smallmatrix}1 & 2 & \cdots & n\\ \sigma(1) & \sigma(2) & \cdots & \sigma(n)\end{smallmatrix}\right)$ we will write $\sigma(1)\cdots\sigma(n)$. More generally we will consider words in the alphabet $[n]$ that have at most one occurrence of each letter $i \in [n]$. If every element of $[n]$ appears in such a word $u = i_1 \dots i_k$, then $k = n$ and we can consider w as a permutation. For a word $u = i_1 \dots i_k$, say that $k = |u|$ is its *length* and that $s \in [n-1]$ is *a descent index* if $i_s > i_{s+1}$. Denote by $Des(u)$ a set of descent indices of u. The sum of all descent indices, as we mentioned above, is called *major index* of u and is denoted $maj(u)$,

$$maj(u) = \sum_{j \in Des(u)} j.$$

Define the *multi-parametric* **q**-*major index* $maj_{\mathbf{q}}(u)$ of word u by

$$maj_{\mathbf{q}}(u) = \frac{\prod_{j \in Des(u)} q_{u(1)} \cdots q_{u(j)}}{\prod_{i=1}^{|u|-1} (1 - q_{u(1)} \cdots q_{u(i)})},$$

where $q_1, \dots, q_n$ are some variables.

Let $u = i_1 \cdots i_k$ and $i_s = \max\{i_1, \dots, i_k\}$ be maximum of the letters of u. Call the subwords $L(u) = i_1 \dots i_{s-1}$ the *left part* and $R(u) = i_{s+1} \cdots i_k$ the *right part* of u. For a word $u = i_1 \cdots i_{k-1} i_k$ denote by $rev(u) = i_k i_{k-1} \cdots i_1$ the word u written in reverse order. For example, if $u = 513942$, then

$$|u| = 6, \qquad Des(u) = \{1, 4, 5\}, \qquad maj(u) = 1 + 4 + 5 = 10,$$

$$maj_{\mathbf{q}}(u) = \frac{q_1^2 q_3^2 q_4 q_5^3 q_9^2}{(1 - q_5)(1 - q_1 q_5)(1 - q_1 q_3 q_5)(1 - q_1 q_3 q_5 q_9)(1 - q_1 q_3 q_4 q_5 q_9)},$$

$$L(u) = 513, \qquad R(u) = 42, \qquad rev(u) = 249315.$$

Let $K(\mathbf{q})$ be the field of rational functions in $\mathbf{q}$ over the field K and $K(\mathbf{q})S_n$ be group algebra of symmetric group with coefficients in $K(\mathbf{q})$. There is a natural left action of S_n on $K(\mathbf{q})$: if $f(\mathbf{q}) = f(q_1, \dots, q_n) \in K(\mathbf{q})$, then we define

$$\sigma\big[f(\mathbf{q})\big] = f(q_{\sigma(1)}, \dots, q_{\sigma(n)}).$$

The "correct" product for $K(\mathbf{q})S_n$ is not the straightforward analogue of the product for KS_n. Instead we need the *twisted product* of $f(\mathbf{q})\sigma$ and $g(\mathbf{q})\tau$, denoted $f(\mathbf{q})\sigma \ltimes g(\mathbf{q})\tau$, defined by

$$f(\mathbf{q})\sigma \ltimes g(\mathbf{q})\tau = (f(\mathbf{q})\sigma\big[g(\mathbf{q})\big]\sigma\tau.$$

For example,

$$213 \ltimes \frac{q_2 q_3}{(1 - q_2)(1 - q_2 q_3)} 231 = \frac{q_1 q_3}{(1 - q_1)(1 - q_1 q_3)} 132.$$

We recall the following multi-parameter generalization of Klyachko idempotent introduced in [7]. Namely, an element $k_n(\mathbf{q}) \in K(\mathbf{q})S_n$ defined by

$$k_n(\mathbf{q}) = \sum_{\sigma \in S_n} maj_{\mathbf{q}}(\sigma)\sigma$$

is an idempotent in $K(\mathbf{q})S_n$ under multiplication $\ltimes$. Moreover, $k_n(\mathbf{q})(a_1,\dots,a_n)$ is a Lie element, if $q_1q_2\cdots q_n=1$, but $q_{i_1}q_{i_2}\cdots q_{i_r}\neq 1$, for any proper subset $\{i_1,\dots,i_r\}\subset[n]$.

The multilinear part of the free Lie algebra of degree n is $(n-1)!$-dimensional and any multilinear Lie element should be a linear combination of *base* elements of a form $[a_{\sigma(1)},[\cdots[a_{\sigma(n-1)},a_n]\cdots]]$. The existence of a Lie expression for the Klyachko element as such a linear combination was established in many papers (see for example [1–3, 8]).

In our paper we give an explicit such expression of the multi-parameter Klyachko element as a Lie word.

Theorem 1 *If* $q_1q_2\cdots q_n=1$, *but* $q_{i_1}q_{i_2}\cdots q_{i_r}\neq 1$, *for any proper subset* $\{i_1,\dots,i_r\}$ *of* $\{1,2,\dots,n\}$, *then*

$$k_n(\mathbf{q})=\sum_{\sigma\in S_n,\sigma(n)=n} maj_{\mathbf{q}}(\sigma)\big[\sigma(1),\big[\sigma(2),\cdots,\big[\sigma(n-1),n\big]\cdots\big]\big].$$

Equivalently,

$$\sum_{\sigma\in S_n} maj_{\mathbf{q}}(\sigma)a_{\sigma(1)}\cdots a_{\sigma(n)}=\sum_{\tau\in S_{n-1}}\frac{maj_{\mathbf{q}}(\tau)[a_{\tau(1)},[a_{\tau(2)},\cdots[a_{\tau(n-1)},a_n]\cdots]]}{(1-q_1\cdots q_{n-1})}. \tag{1}$$

Example Let us demonstrate calculations for the case $n=3$. Denote by H the right-hand side of equality (1). We have

$$maj_{\mathbf{q}}(123)=\frac{1}{(1-q_1)(1-q_1q_2)},\qquad maj_{\mathbf{q}}(132)=\frac{q_1q_3}{(1-q_1)(1-q_1q_3)},$$

$$maj_{\mathbf{q}}(213)=\frac{q_2}{(1-q_2)(1-q_1q_2)},\qquad maj_{\mathbf{q}}(231)=\frac{q_2q_3}{(1-q_2)(1-q_2q_3)},$$

$$maj_{\mathbf{q}}(312)=\frac{q_3}{(1-q_3)(1-q_1q_3)},\qquad maj_{\mathbf{q}}(321)=\frac{q_2q_3^2}{(1-q_3)(1-q_2q_3)},$$

and

$$\begin{aligned}k_3(\mathbf{q})(a,b,c)&=\frac{abc}{(1-q_1)(1-q_1q_2)}+\frac{q_1q_3\,acb}{(1-q_1)(1-q_1q_3)}\\&\quad+\frac{q_2\,bac}{(1-q_2)(1-q_1q_2)}+\frac{q_2q_3\,bca}{(1-q_2)(1-q_2q_3)}\\&\quad+\frac{q_3\,cab}{(1-q_3)(1-q_1q_3)}+\frac{q_2q_3^2\,cba}{(1-q_3)(1-q_2q_3)}.\end{aligned}$$

On the other hand,

$$maj_{\mathbf{q}}(12)=\frac{1}{1-q_1},\qquad maj_{\mathbf{q}}(21)=\frac{q_2}{1-q_2},$$

and

$$\begin{aligned}
H &= \frac{1}{1-q_1q_2}\left(\frac{[a,[b,c]]}{1-q_1} + \frac{q_2[b,[a,c]]}{1-q_2}\right)\\
&= \frac{1}{1-q_1q_2}\left(\frac{abc-acb-bca+cba}{1-q_1} + \frac{q_2(bac-bca-acb+cab)}{1-q_2}\right)\\
&= \frac{((1-q_2)abc-(1-q_1q_2)acb+(q_2-q_1q_2)bac-(1-q_1q_2)bca+(q_2-q_1q_2)cab+(1-q_2)cba)}{(1-q_1)(1-q_2)(1-q_1q_2)}\\
&= \frac{abc}{(1-q_1)(1-q_1q_2)} - \frac{acb}{(1-q_1)(1-q_2)} + \frac{q_2\,bac}{(1-q_2)(1-q_1q_2)}\\
&\quad - \frac{bca}{(1-q_1)(1-q_2)} + \frac{q_2\,cab}{(1-q_2)(1-q_1q_2)} + \frac{cba}{(1-q_1)(1-q_1q_2)}.
\end{aligned}$$

Since $q_1q_2q_3 = 1$, this means that

$$k_3(\mathbf{q})(a,b,c) = H.$$

If q is a primitive root of 1 of degree 3 and $q_i = q$, then $(1-q)(1-q^2) = 1-q-q^2+q^3 = 3$. Therefore in this case we obtain the Klyachko element

$$k_3(\mathbf{q}) = k_3.$$

These kind of calculations can be done for any n.

As a corollary of Theorem 1 we obtain an exact expression of $k_n(a_1,\dots,a_n)$ as a Lie element.

Corollary 2 *If $q^n = 1, q^m \neq 1, 0 < m < n$, then*

$$\sum_{\sigma\in S_n} q^{maj(\sigma)} a_{\sigma(1)}\cdots a_{\sigma(n)} = \sum_{\tau\in S_{n-1}} q^{maj(\tau)}\big[a_{\tau(1)},\big[a_{\tau(2)},\big[\cdots,[a_{\tau(n-1)},a_n]\cdots\big]\big]$$

Let $⧢$ be the shuffle product on a space of words A. For elements $u = x_1\dots x_k, v = y_1\dots y_l \in A$ recall that $u⧢v$ is defined as the sum of elements of the form $w = z_1\dots z_{k+l}$ such that $\{z_1,\dots,z_{k+l}\} = \{x_1,\dots,x_k,y_1,\dots,y_l\}$ and if $z_{i_1} = x_1,\dots,z_{i_k} = x_k,\ z_{j_1} = y_1,\dots z_{j_l} = y_l$, then $i_1 < i_2 < \cdots < i_k,\ j_1 < j_2 < \cdots < j_l$. For $w \in A$ we say that w is a part of the shuffle product $u⧢v$ and denote $w \in u⧢v$ if w appears as a summand in $u⧢v$. For example,

$$ab⧢cd = abcd + acbd + acdb + cabd + cadb + cdab$$

and $acdb$ is a part of $ab⧢cd$, but $adcb$ is not.

By Lemma 5 (see below) one can reformulate the above results in terms of major-indices and shuffle products.

Corollary 3 *If* $q_1 \cdots q_n = 1$, *but* $q_{i_1} \cdots q_{i_r} \neq 1$, *for any proper subset* $\{i_1, \dots, i_r\} \subset [n]$, *then for any permutation* $\sigma \in S_n$,

$$maj_{\mathbf{q}}(\sigma) = \frac{(-1)^{|R(\sigma)|} q_n}{(q_n - 1)} \sum_{w \in L(\sigma) \sqcup\sqcup rev(R(\sigma))} maj_{\mathbf{q}}(w),$$

where $L(\sigma)$ *and* $R(\sigma)$ *are the left and right parts of* σ.

Corollary 4 *If* $q^n = 1, q^m \neq 1, 0 < m < n$, *then for any permutation* $\sigma \in S_n$,

$$q^{maj(\sigma)} = (-1)^{|R(\sigma)|} \sum_{w \in L(\sigma) \sqcup\sqcup rev(R(\sigma))} q^{maj(w)},$$

where $L(\sigma)$ *and* $R(\sigma)$ *are the left and right parts of* σ.

2 Proof of Theorem 1

Let $S_{n,r}$ be the set of shuffle permutations,

$$S_{n,r} = \big\{\sigma \in S_n | \sigma(1) < \cdots \sigma(r),\ \sigma(r+1) < \cdots < \sigma(n)\big\}.$$

Lemma 5 *For any* $a_1, \dots, a_n \in A$,

$$\begin{aligned}&[a_1, \big[a_2, \big[\cdots [a_{n-1}, a_n] \cdots \big]\big] \\ &= \sum_{r=0}^{n-1} \sum_{\sigma \in S_{n-1,r}} (-1)^r a_{\sigma(1)} \cdots a_{\sigma(r)} a_n \, rev(a_{\sigma(r+1)} \cdots a_{\sigma(n-1)}).\end{aligned}$$

Proof Follows from Theorem 8.16 [9]. □

Lemma 6 *Let* u *and* v *be complementary words on* $[n]$ *with* $|u| = k, |v| = l$. *Then*

$$\sum_{w \in u \sqcup\sqcup v} maj_{\mathbf{q}}(w) = \frac{(1 - q_1 \dots q_n)}{(1 - q_{u(1)} \cdots q_{u(k)})(1 - q_{v(1)} \cdots q_{v(l)})} maj_{\mathbf{q}}(u) maj_{\mathbf{q}}(v).$$

Proof Let $\langle\ ,\ \rangle$ be inner (scalar) product on $\mathcal{F}$ defined by $\langle u, v\rangle = \delta_{u,v}$ for any words u and v and extended to $\mathcal{F}$ by linearity. By the following statement (see end of Sect. 2 of the paper [7]),

$$\begin{aligned}&\big\langle k_n(\mathbf{q}), u \sqcup\sqcup v\big\rangle \\ &= (1 - q_1 \cdots q_n) \frac{(\prod_{j \in D(u)} q_{u(1)} \cdots q_{u(n)})(\prod_{j \in D(v)} q_{v(1)} \cdots q_{v(n)})}{(\prod_{i=1}^{k} q_{u(1)} \cdots q_{u(i)})(\prod_{j=1}^{l} q_{v(1)} \cdots q_{v(j)})}\end{aligned}$$

we have

$$\big\langle k_n(\mathbf{q}), u \sqcup\sqcup v\big\rangle = \frac{(1 - q_1 \cdots q_n) maj_{\mathbf{q}}(u) maj_{\mathbf{q}}(v)}{(1 - q_{u(1)} \cdots q_{u(k)})(1 - q_{v(1)} \cdots q_{v(l)})}.$$

It remains to note that

$$\langle k_n(\mathbf{q}), u \sqcup\!\sqcup v\rangle = \sum_{w \in u \sqcup\!\sqcup v} maj_{\mathbf{q}}(w).$$ □

Lemma 7 *For any $\tau \in S_n$,*

$$maj_{\mathbf{q}}(\tau) = \frac{(-1)^{|\tau|+1}(\prod_{j \in Des(rev(\tau))} q_{\tau(1)} \cdots q_{\tau(n-j)})^{-1}}{\prod_{i=1}^{n-1}(1-(q_{\tau(1)} \cdots q_{\tau(i)})^{-1})}$$

Proof For $\tau \in S_n$ let

$$Des(\tau) = \{i \in [n-1] | \tau(i) > \tau(i+1)\},$$

$$Rise(\tau) = \{i \in [n-1] | \tau(i) < \tau(i+1)\}$$

be sets of descent and rise indices. Then

$$Des(\tau) \cup Rise(\tau) = \{1, 2, \dots, n-1\},$$

$$Des(\tau) \cap Rise(\tau) = \emptyset,$$

$$|Rise(\tau)| = n - 1 - |Des(\tau)|,$$

$$Des(rev(\tau)) = \{n, \dots, n\} - rev(Rise(\tau))$$

(here $(n, \dots, n)$ is a sequence with components n and length $|Rise(\tau)|$). Thus,

$$Rise(\tau) = \{n - j \,|\, j \in Des(rev(\tau))\}.$$

Therefore,

$$\begin{aligned}
&\frac{(-1)^{n+1}}{\prod_{j \in Des(rev(\tau))} q_{\tau(1)} \cdots q_{\tau(n-j)} \prod_{i=1}^{n-1}(1-(q_{\tau(1)} \cdots q_{\tau(i)})^{-1})} \\
&\quad = \frac{(-1)^{n+1}\prod_{i=1}^{n-1}(q_{\tau(1)} \cdots q_{\tau(i)})}{\prod_{j \in Des(rev(\tau))} q_{\tau(1)} \cdots q_{\tau(n-j)} \prod_{i=1}^{n-1}(-1+(q_{\tau(1)} \cdots q_{\tau(i)}))} \\
&\quad = \frac{\prod_{i=1}^{n-1}(q_{\tau(1)} \cdots q_{\tau(i)})}{\prod_{j \in Des(rev(\tau))} q_{\tau(1)} \cdots q_{\tau(n-j)} \prod_{i=1}^{n-1}(1-q_{\tau(1)} \cdots q_{\tau(i)})} \\
&\quad = \frac{\prod_{i \in Des(\tau)} q_{\tau(1)} \cdots q_{\tau(i)}}{\prod_{i=1}^{n-1}(1-q_{\tau(1)} \cdots q_{\tau(i)})} \\
&\quad = maj_{\mathbf{q}}(\tau).
\end{aligned}$$ □

Lemma 8 *Let u and v be complementary nonempty words on $[n-1]$. Let $q_1\cdots q_n=1$, but $q_{i_1}\cdots q_{i_r}\neq 1$, for any proper subset $\{i_1,\dots,i_r\}\subset\{1,\dots,n\}$. Then*

$$maj_{\mathbf{q}}\big(u\,n\,rev(v)\big)=\frac{(-1)^{|v|}maj_{\mathbf{q}}(u)maj_{\mathbf{q}}(v)}{(1-\prod_{i=1}^{|u|}q_{u(i)})(1-\prod_{j=1}^{|v|}q_{v(j)})}.$$

Proof Let $\sigma=u\,n\,rev(v)$. Note that $\sigma^{-1}(n)=|u|+1$ and $|u|\notin Des(\sigma)$. Therefore,

$$\begin{aligned} &maj_{\mathbf{q}}(\sigma)\\ &\quad=\frac{\prod_{j\in Des(\sigma)}q_{\sigma(1)}\cdots q_{\sigma(j)}}{\prod_{i=1}^{n-1}(1-q_{\sigma(1)}\cdots q_{\sigma(i)})}\\ &\quad=\frac{\prod_{j\in Des(\sigma),j<|u|}q_{u(1)}\cdots q_{u(j)}}{\prod_{i=1}^{|u|-1}(1-q_{u(1)}\cdots q_{u(i)})}\\ &\qquad\times\frac{(q_n\prod_{j=1}^{|u|}q_{\sigma(j)})}{(1-q_{u(1)}\cdots q_{u|u|)})(1-q_{u(1)}\cdots q_{u(|u|)}q_n)}\\ &\qquad\times\frac{\prod_{j\in Des(rev(v))}q_{u(1)}\cdots q_{u(|u|)}q_nq_{v(|v|-j+1)}\cdots q_{v(|v|)}}{\prod_{i=1}^{|v|-1}(1-q_{u(1)}\cdots q_{u(|u|)}q_nq_{v(1+i)}\cdots q_{v(|v|)})}. \end{aligned}$$

Now use the condition

$$q_{u(1)}\cdots q_{u(|u|)}q_nq_{v(1)}\cdots q_{v(|v|)}=q_1\cdots q_n=1.$$

We have

$$\begin{aligned} &maj_{\mathbf{q}}(\sigma)\\ &\quad=maj_{\mathbf{q}}(u)\times\frac{1}{(1-q_{u(1)}\cdots q_{u(|u|)})(-1+q_{v(1)}\cdots q_{v(|v|)})}\\ &\qquad\times\frac{\prod_{j\in Des(rev(v))}(q_{v(1)}\cdots q_{v(|v|-j)})^{-1}}{\prod_{i=1}^{|v|-1}(1-(q_{v(1)}\cdots q_{v(i)})^{-1})}. \end{aligned}$$

It remains to use Lemma 7. □

Proof of Theorem 1 For any $\sigma\in S_n$ let us calculate the coefficient of $a_{\sigma(1)}\cdots a_{\sigma(n)}$. On the left hand side of (1) it is $maj_{\mathbf{q}}(\sigma)$.

Now consider the coefficient on the right hand side of (1). Denote it as λ_σ. If $r=\sigma^{-1}(n)$, then $u=L(\sigma)=\sigma(1)\cdots\sigma(r-1)$ and $v=R(\sigma)=\sigma(r+1)\cdots\sigma(n)$. Then $a_{\sigma(1)}\cdots a_{\sigma(n)}$ appears only in terms of a form

$$\frac{maj_{\mathbf{q}}(w)[a_{w(1)},[\cdots[a_{w(n-1)},a_n]\cdots]]}{(1-q_1\cdots q_{n-1})},$$

where $w \in u \sqcup\!\sqcup rev(v)$, with coefficient $(-1)^{|v|}$. Hence, by Lemma 5,

$$\lambda_\sigma = \sum_{w \in u \sqcup\!\sqcup rev(v)} \frac{(-1)^{|v|}}{(1 - q_1 \cdots q_{n-1})} maj_{\mathbf{q}} w.$$

Since u and v are complementary words on $[n-1]$, by Lemma 6

$$\lambda_\sigma = \frac{(-1)^{|v|}}{(1 - q_{u(1)} \cdots q_{u(k)})(1 - q_{v(1)} \cdots q_{v(l)})} maj_{\mathbf{q}}(u) maj_{\mathbf{q}}\big(rev(v)\big).$$

By Lemma 8 we see that

$$\lambda_\sigma = maj_{\mathbf{q}}(u\, n\, v)$$

So, coefficients at $a_{\sigma(1)} \cdots a_{\sigma(n)}$ on the left and right hand sides of (1) are equal for any $\sigma \in S_n$. Theorem 1 is proved. □

3 Additional remarks

Remark 1 It will be interesting to study associative algebra as an n-algebra under k_n as an n-ary multiplication. For example, for $n = 3$ the 3-multiplication $[a, b, c]$ defined by

$$[a_1, a_2, a_3] = \sum_{\sigma \in S_3} q^{maj(\sigma)} a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)}, \qquad q = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2},$$

can be re-written as

$$[a, b, c] = (ab)c + (cb)a + q\big((ba)c + (ca)b\big) + q^2\big((ac)b + (bc)a\big)$$

It satisfies the cyclic q-identity of 3-degree 1,

$$[a, b, c] = q[c, a, b],$$

and the following 25 terms identity of 3-degree 2

$$\begin{aligned}
&- q^2\big[a, b, [c, d, e]\big] - q\big[a, b, [d, c, e]\big] + \big[a, c, [b, d, e]\big] + q\big[a, c, [d, b, e]\big] \\
&\quad - \big[a, d, [b, c, e]\big] + q^2\big[a, d, [c, b, e]\big] - \big[a, e, [b, c, d]\big] + q^2\big[a, e, [c, b, d]\big] \\
&\quad + q^2\big[b, a, [c, d, e]\big] + \big[b, a, [d, c, e]\big] - q\big[b, c, [a, d, e]\big] - \big[b, c, [d, a, e]\big] \\
&\quad + q\big[b, d, [a, c, e]\big] - q^2\big[b, d, [c, a, e]\big] + q\big[b, e, [a, c, d]\big] - q^2\big[b, e, [c, a, d]\big] \\
&\quad - \big[c, a, [b, d, e]\big] - q^2\big[c, a, [d, b, e]\big] + q\big[c, b, [a, d, e]\big] + q^2\big[c, b, [d, a, e]\big] \\
&\quad - q\big[c, d, [a, b, e]\big] + \big[c, d, [b, a, e]\big] - q\big[c, e, [a, b, d]\big] + \big[c, e, [b, a, d]\big] \\
&\quad + \big(q - q^2\big)\big[d, e, [a, b, c]\big] = 0.
\end{aligned}$$

Remark 2 The Dynkin–Specht–Wever theorem allows us to check whether an element $X \in \mathcal{F}$ can be presented as a linear combination of Lie commutators. Let us give some modification of this theorem that allows to write a multilinear Lie element as a linear combination of Lie *base* elements. The algorithm is the following:

Step 1 Write multilinear element X as a linear combination of Lie commutators. One can use here the Dynkin–Specht–Wever theorem.
Step 2 Write all Lie commutators in a right-bracketed form by the rule (Jacobi identity) $[[a,b]c] := [a,[b,c]] - [b,[a,c]]$.
Step 3 Write any right-bracketed Lie commutator $[a_{i_1}, [\cdots [a_{i_{n-1}}, a_{i_n}] \cdots]$ as a linear combination of multilinear base elements $[a_{\sigma(1)}, [\cdots [a_{\sigma(n-1)}, a_n] \cdots]]$, where $\sigma \in S_{n-1}$. Here one can use the following formula:

$$\begin{aligned}&\big[a_n, \big[a_1, \big[\cdots, [a_{n-2}, a_{n-1}]\big]\big]\big]\\&\quad= \sum_{r=0}^{n-2} \sum_{\sigma \in S_{n-1,r+1}, \sigma(r+1)=n-1} \big[a_{\sigma(1)}, \big[\cdots \big[a_{\sigma(r)}, \big[a_{n-1}, \big[a_{\sigma(n-2)},\\&\qquad \big[\cdots [a_{\sigma(r+1)}, a_n]\big]\big]\big]\big]\big]. \qquad (2)\end{aligned}$$

Example Let us present an element $X = [[a_2, a_5], [[a_3, a_4], a_1]]$ as a linear combination of Lie base elements. We can begin from step 2,

$$\begin{aligned}X &= \big[a_2, \big[a_5, \big[a_3, [a_4, a_1]\big]\big]\big] - \big[a_2, \big[a_5, \big[a_4, [a_3, a_1]\big]\big]\big] - \big[a_5, \big[a_2, \big[a_3, [a_4, a_1]\big]\big]\big]\\&\quad + \big[a_5, \big[a_2, \big[a_4, [a_3, a_1]\big]\big]\big].\end{aligned}$$

By (2) we have

$$\begin{aligned}\big[a_5, \big[a_3, [a_4, a_1]\big]\big] &= -\big[a_3, \big[a_4, [a_1, a_5]\big]\big] + \big[a_3, \big[a_1, [a_4, a_5]\big]\big]\\&\quad + \big[a_4, \big[a_1, [a_3, a_5]\big]\big] - \big[a_1, \big[a_4, [a_3, a_5]\big]\big],\end{aligned}$$

and,

$$\begin{aligned}&\big[a_2, \big[a_5, \big[a_3, [a_4, a_1]\big]\big]\big]\\&\quad = -\big[a_2, \big[a_3, \big[a_4, [a_1, a_5]\big]\big]\big] + \big[a_2, \big[a_3, \big[a_1, [a_4, a_5]\big]\big]\big] + \big[a_2, \big[a_4, \big[a_1, [a_3, a_5]\big]\big]\big]\\&\qquad - \big[a_2, \big[a_1, \big[a_4, [a_3, a_5]\big]\big]\big].\end{aligned}$$

Similarly,

$$\begin{aligned}&\big[a_2, \big[a_5, \big[a_4, [a_3, a_1]\big]\big]\big]\\&\quad = -\big[a_2, \big[a_4, \big[a_3, [a_1, a_5]\big]\big]\big] + \big[a_2, \big[a_4, \big[a_1, [a_3, a_5]\big]\big]\big] + \big[a_2, \big[a_3, \big[a_1, [a_4, a_5]\big]\big]\big]\\&\qquad - \big[a_2, \big[a_1, \big[a_3, [a_4, a_5]\big]\big]\big].\end{aligned}$$

By (2),

$$\big[a_5, \big[a_2, \big[a_3, [a_4, a_1]\big]\big]\big]$$

$$
\begin{aligned}
&= \big[a_1, \big[a_4, \big[a_3, [a_2, a_5]\big]\big]\big] - \big[a_2, \big[a_1, \big[a_4, [a_3, a_5]\big]\big]\big] + \big[a_2, \big[a_3, \big[a_1, [a_4, a_5]\big]\big]\big] \\
&\quad - \big[a_2, \big[a_3, \big[a_4, [a_1, a_5]\big]\big]\big] + \big[a_2, \big[a_4, \big[a_1, [a_3, a_5]\big]\big]\big] \\
&\quad - \big[a_3, \big[a_1, \big[a_4, [a_2, a_5]\big]\big]\big] + \big[a_3, \big[a_4, \big[a_1, [a_2, a_5]\big]\big]\big] \\
&\quad - \big[a_4, \big[a_1, \big[a_3, [a_2, a_5]\big]\big]\big],
\end{aligned}
$$

$$
\begin{aligned}
&\big[a_5, \big[a_2, \big[a_4, [a_3, a_1]\big]\big]\big] \\
&= \big[a_1, \big[a_3, \big[a_4, [a_2, a_5]\big]\big]\big] - \big[a_2, \big[a_1, \big[a_3, [a_4, a_5]\big]\big]\big] + \big[a_2, \big[a_3, \big[a_1, [a_4, a_5]\big]\big]\big] \\
&\quad + \big[a_2, \big[a_4, \big[a_1, [a_3, a_5]\big]\big]\big] - \big[a_2, \big[a_4, \big[a_3, [a_1, a_5]\big]\big]\big] \\
&\quad - \big[a_3, \big[a_1, \big[a_4, [a_2, a_5]\big]\big]\big] - \big[a_4, \big[a_1, \big[a_3, [a_2, a_5]\big]\big]\big] \\
&\quad + \big[a_4, \big[a_3, \big[a_1, [a_2, a_5]\big]\big]\big].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
X &= \big[a_1, \big[a_3, \big[a_4, [a_2, a_5]\big]\big]\big] - \big[a_1, \big[a_4, \big[a_3, [a_2, a_5]\big]\big]\big] - \big[a_3, \big[a_4, \big[a_1, [a_2, a_5]\big]\big]\big] \\
&\quad + \big[a_4, \big[a_3, \big[a_1, [a_2, a_5]\big]\big]\big].
\end{aligned}
$$

Remark 3 Klyachko element has one more generalization. In [6] a Lie idempotent was constructed that generalises three other well-known idempotents. This generalization is the q-Solomon idempotent

$$
\phi_n(q) = \frac{1}{n} \sum_{\sigma \in S_n} \frac{(-1)^{des(\sigma)} q^{maj(\sigma) - \binom{d(\sigma)+1}{2}}}{\begin{bmatrix} n-1 \\ d(\sigma) \end{bmatrix}_q} \sigma.
$$

It has the following properties:

$$
\phi_n(\omega) = k_n(\omega),
$$

is the Klyachko element if ω is a primitive root of degree n,

$$
\phi_n(0) = \big[\cdots [1, 2], \ldots, n\big]
$$

is the Dynkin idempotent in case of $q = 0$ and

$$
\phi_n(1) = \sum_{\sigma \in S_n} \frac{(-1)^{des(\sigma)}}{\binom{n-1}{des(\sigma)}} \sigma
$$

gives us the (first) Euler idempotent if $q = 1$. Here $\begin{bmatrix} n-1 \\ p \end{bmatrix}_q$ denotes a q-binomial coefficient.

One can establish that the q-Solomon idempotent has a similar Lie presentation by base elements,

$$\phi_n(q) = \frac{1}{n} \sum_{\sigma \in S_n, \sigma(n)=n} \frac{(-1)^{des(\sigma)} q^{maj(\sigma) - \binom{d(\sigma)+1}{2}}}{\begin{bmatrix} n-1 \\ d(\sigma) \end{bmatrix}_q} [\sigma],$$

where $[\sigma]$ denotes the Lie commutator $[\sigma(1), [\cdots[\sigma(n-1), n]\cdots]]$. It follows from the following property of major indices

$$\sum_{w \in u \sqcup\!\sqcup rev(v)} \frac{(-1)^{|v|} q^{maj(w) - \binom{des(w)+1}{2}}}{\begin{bmatrix} |w| \\ des(w) \end{bmatrix}_q}$$
$$= - \frac{(-1)^{des(u)+des(v)} q^{maj(u)+maj(v)+(|u|+1)(des(v)+1) - \binom{des(u)+des(v)+2}{2}}}{\begin{bmatrix} |u|+|v| \\ des(u)+des(v)+1 \end{bmatrix}_q}.$$

In particular, the Lie presentation for Dynkin element is given by

$$\bigl[\bigl[\cdots[a_1, a_2], \cdots\bigr], a_n\bigr]$$
$$= \sum_{\sigma \in S_n, maj(\sigma) = \binom{des(\sigma)+1}{2}, \sigma(n)=n} (-1)^{des(\sigma)} \bigl[a_{\sigma(1)}, \bigl[\cdots[a_{\sigma(n-1)}, a_n]\cdots\bigr]\bigr]$$

and the Euler element has the following Lie presentation:

$$\sum_{\sigma \in S_n} \frac{(-1)^{des(\sigma)}}{\binom{n-1}{des(\sigma)}} a_{\sigma(1)} \cdots a_{\sigma(n)} = \sum_{\sigma \in S_n, \sigma(n)=n} \frac{(-1)^{des(\sigma)}}{\binom{n-1}{des(\sigma)}} \bigl[a_{\sigma(1)}, \bigl[\cdots[a_{\sigma(n-1)}, a_n]\cdots\bigr]\bigr].$$

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