

Gelfand models and Robinson–Schensted correspondence

Fabrizio Caselli · Roberta Fulci

Received: 24 January 2011 / Accepted: 29 October 2011 / Published online: 29 November 2011
© Springer Science+Business Media, LLC 2011

Abstract In F. Caselli (Involutory reflection groups and their models, J. Algebra 24:370–393, 2010), a uniform Gelfand model is constructed for all nonexceptional irreducible complex reflection groups which are involutory. Such models can be naturally decomposed into the direct sum of submodules indexed by S_n -conjugacy classes, and we present here a general result that relates the irreducible decomposition of these submodules with the projective Robinson–Schensted correspondence. This description also reflects, in a very explicit way, the existence of split representations for these groups.

Keywords Complex reflection groups · Characters and representations of finite groups · Clifford theory

1 Introduction

Given a finite-dimensional vector space V over the complex field, a reflection group is a subgroup $G < GL(V)$ that is generated by reflections, i.e., elements of finite order fixing a hyperplane of V pointwise. Finite irreducible complex reflection groups were completely classified in the 1950s [12] by Shephard and Todd. They consist of an infinite family of groups denoted $G(r, p, n)$, where $r, p, n \in \mathbb{N}$ and $p|r$, which are the main subject of this paper, and 34 more sporadic groups.

This work finds its roots in the introduction of a new family of groups, called *projective reflection groups* [4]. They can be roughly described as quotients—modulo a scalar group—of finite reflection groups. If we quotient a group $G(r, p, n)$ modulo the cyclic scalar subgroup C_q , we find a new group $G(r, p, q, n)$, so that in this

F. Caselli (✉) · R. Fulci
Dipartimento di Matematica, Università di Bologna, Bologna, Italy
e-mail: caselli@dm.unibo.it

R. Fulci
e-mail: roberta.fulci2@unibo.it

notation we have $G(r, p, n) = G(r, p, 1, n)$. We define the *dual group* $G(r, p, q, n)^*$ as the group $G(r, q, p, n)$ obtained by simply exchanging the parameters p and q . It turns out that many objects related to the algebraic structure of a projective reflection group G can be naturally described by means of the combinatorics of its dual G^* (see [4, 5]), for example, its representations.

A Gelfand model of a finite group G is a G -module isomorphic to the multiplicity-free sum of all the irreducible complex representations of G . The study of Gelfand models originated from [3] and has found a wide interest in the case of reflection groups and other related groups (see, e.g., [1, 2, 7–10]). In [5], a Gelfand model (M, ϱ) was constructed (relying on the concept of duality in an essential way) for every *involutory* projective reflection group $G(r, p, q, n)$ satisfying $\text{GCD}(p, n) = 1, 2$.

A finite subgroup of $GL(V)$ is involutory if the number of its absolute involutions, i.e., elements g such that $g\bar{g} = 1$, coincides with the dimension of its Gelfand model. A group $G(r, p, n)$ turns out to be involutory if and only if $\text{GCD}(p, n) = 1, 2$ [5, Theorem 4.5], so that, in particular, all infinite families of finite irreducible Coxeter groups are involutory.

The model (M, ϱ) provided in [5] is as follows:

- M is a formal vector space generated by all absolute involutions $I(r, p, n)^*$ of the dual group $G(r, p, n)^*$,

$$M \stackrel{\text{def}}{=} \bigoplus_{v \in I(r, p, n)^*} \mathbb{C}C_v;$$

- The group acts via $\varrho : G(r, p, n) \rightarrow GL(M)$ by means of an *absolute conjugation* of $G(r, p, n)$ on the elements indexing the basis of M :

$$\varrho(g)(C_v) \stackrel{\text{def}}{=} \psi(g, v)C_{|g|v|g|^{-1}}, \tag{1}$$

where $\psi(g, v)$ is a scalar, and $|g|$ is the natural projection of g in the symmetric group S_n .

If $g, h \in G(r, p, n)^*$, we say that g and h are S_n -conjugate if there exists $\sigma \in S_n$ such that $g = \sigma h \sigma^{-1}$, and we call S_n -conjugacy classes the corresponding equivalence classes. If c is an S_n -conjugacy class of absolute involutions in $I(r, p, n)^*$, we denote by $M(c)$ the subspace of M spanned by the basis elements C_v indexed by the absolute involutions v belonging to the class c . Then it is clear from (1) that we have a decomposition

$$M = \bigoplus_c M(c) \quad \text{as } G(r, p, n)\text{-modules,}$$

where the sum runs through all S_n -conjugacy classes of absolute involutions in $I(r, p, n)^*$. It is natural to ask if we can describe the irreducible decomposition of the submodules $M(c)$, and the main goal of this paper is to answer this question for every group $G(r, p, n)$ with $\text{GCD}(p, n) = 1, 2$. The special case of this result for the symmetric group $S_n = G(1, 1, n)$ was established in [7], while the corresponding result for wreath products $G(r, 1, n)$ has been recently proved by the authors in [6]. Though the main result of this paper is a generalization of [6], we should mention that the proof is not, in the sense that we will actually make use here of the main results of [6].

The decomposition of the submodules $M(c)$ in this wider setting is much more subtle. Indeed, when $\text{GCD}(p, n) = 2$, the Gelfand model M splits also in a different way as the direct sum of two distinguished modules: the symmetric submodule M_{Sym} , which is spanned by the elements C_v indexed by symmetric absolute involutions, and the antisymmetric submodule M_{Asym} , which is defined similarly. This decomposition is compatible with the one described above in the sense that every submodule $M(c)$ is contained either in the symmetric or in the antisymmetric submodule. The existence of the antisymmetric submodule and of the submodules $M(c)$ contained therein will reflect in a very precise way the existence of split representations for these groups. The study of the irreducible decomposition of $M(c)$ when c is made up of antisymmetric elements requires a particular machinery developed in Sects. 6, 7, and 8 that was not needed in the case of wreath products $G(r, n)$, where the antisymmetric submodule vanishes, and so the Gelfand model coincides with its symmetric submodule.

The final description of the irreducible decomposition of the modules $M(c)$ has a rather elegant formulation due to its compatibility with the projective Robinson–Schensted correspondence. Namely, the irreducible subrepresentations of $M(c)$ are indexed by the shapes which are obtained when performing this correspondence to the elements in c .

Here is a plan of this paper. In Sect. 2 we collect the background of preliminary results that are needed to afford the topic. Here an introduction to the groups $G(r, p, q, n)$ can be found, as well as the description of their irreducible representations and a brief account of the projective Robinson–Schensted correspondence. In Sect. 3, for the reader’s convenience, we recall the important definition of symmetric and antisymmetric elements given in [5] and the Gelfand model constructed therein. Also a brief account of the main result for the case of $G(r, n)$ can be found here. Section 4 consists of an outline of the proof of the main results of this work for the special case of Weyl groups of type D . Afterwards, the more general case of all involutory groups of the form $G(r, p, n)$ is treated in full detail. Section 5 is devoted to the description of the conjugacy classes of such groups. In Sect. 6 we study the discrete Fourier transform, a tool which will be used later in Sect. 7, where the irreducible decomposition of the antisymmetric submodule is treated. Section 8 then provides an explicit description of the irreducible decomposition of the modules $M(c)$ contained in the antisymmetric submodule. Section 9 describes the irreducible decomposition of the submodule $M(c)$, where c is any S_n -conjugacy class of symmetric absolute involutions, and Sect. 10 contains a general result, Theorem 10.1, that includes all partial results of the previous sections in a very concise way as well as a further generalization to all groups $G(r, p, q, n)$ satisfying $\text{GCD}(p, n) = 1, 2$.

2 Notation and prerequisites

We let \mathbb{Z} and \mathbb{N} be the sets of integer numbers and nonnegative integer numbers. For $a, b \in \mathbb{Z}$ such that $a \leq b$, we denote $[a, b] = \{a, a + 1, \dots, b\}$, and, for $n \in \mathbb{N}$, we let $[n] \stackrel{\text{def}}{=} [1, n]$. For $r \in \mathbb{N}$, $r > 0$, we let $\mathbb{Z}_r \stackrel{\text{def}}{=} \mathbb{Z}/r\mathbb{Z}$ and we denote by ζ_r the primitive r th root of unity, $\zeta_r \stackrel{\text{def}}{=} e^{\frac{2\pi i}{r}}$.

The group $G(r, n)$ consists of all $n \times n$ complex matrices satisfying the following conditions:

- the nonzero entries are r th roots of unity;
- there is exactly one nonzero entry in every row and every column.

Let now $p|r$. The group $G(r, p, n)$ is the subgroup of $G(r, n)$ of the elements satisfying one extra condition:

- if we write every nonzero element as a power of ζ_r , the sum of all the exponents of ζ_r appearing in the matrix is a multiple of p .

We denote by $z_i(g) \in \mathbb{Z}_r$ the exponent of ζ_r appearing in the i th row of g . We say that $z_i(g)$ is the *color* of i in g , and the sum $z(g) \stackrel{\text{def}}{=} z_1(g) + \dots + z_n(g)$ will be called the *color* of g .

It is sometimes convenient to use an alternative notation to denote an element in $G(r, n)$, other than the matrix representation. We write $g = [\sigma_1^{z_1}, \dots, \sigma_n^{z_n}]$ meaning that, for all $j \in [n]$, the unique nonzero entry in the j th row appears in the σ_j th column and equals $\zeta_r^{z_j}$ (i.e., $z_j(g) = z_j$). We call this the *window* notation of g . In this case we also write $g_i = \sigma_i^{z_i}$. Observe that $[\sigma_1, \dots, \sigma_n]$ is actually a permutation in S_n , and we denote it by $|g|$. We also observe that the map $g \mapsto (|g|, (z_1(g), \dots, z_n(g)))$ gives an isomorphism of $G(r, n)$ with the semidirect product $S_n \ltimes \mathbb{Z}_r^n$ where S_n acts on \mathbb{Z}_r^n by permuting coordinates. Elements of $G(r, n)$ also have a cyclic decomposition which is analogous to the cyclic decomposition of permutations. A *cycle* c of $g \in G(r, n)$ is an object of the form $c = (a_1^{z_{a_1}}, \dots, a_k^{z_{a_k}})$, where (a_1, \dots, a_k) is a cycle of the permutation $|g|$, and $z_{a_i} = z_{a_i}(g)$ for all $i \in [k]$. We let k be the *length* of c , $z(c) \stackrel{\text{def}}{=} z_{a_1} + \dots + z_{a_k}$ be the *color* of c , and $\text{Supp}(c) \stackrel{\text{def}}{=} \{a_1, \dots, a_k\}$ be the *support* of c . We will sometimes write an element $g \in G(r, n)$ as the product of its cycles. For example, if $g \in G(3, 6)$ has window notation $g = [3^0, 4^1, 6^1, 2^0, 5^2, 1^2]$, we have that the cyclic decomposition of g is given by $g = (1^0, 3^1, 6^2)(2^1, 4^0)(5^2)$. Note that we use square brackets for the window notation and round brackets for the cyclic notation.

If $v = (n_0, \dots, n_k)$ is a composition of n , we let $G(r, v) \stackrel{\text{def}}{=} G(r, n_0) \times \dots \times G(r, n_k)$ be the (Young) subgroup of $G(r, n)$ given by

$$G(r, v) = \{[\sigma_1^{z_1}, \dots, \sigma_n^{z_n}] \in G(r, n) : \sigma_i \leq n_0 + \dots + n_j \text{ if and only if } i \leq n_0 + \dots + n_j\}.$$

If $S \subseteq [n]$, we also let

$$G(r, S) = \{[\sigma_1^{z_1}, \dots, \sigma_n^{z_n}] \in G(r, n) : \sigma_i^{z_i} = i^0 \text{ for all } i \notin S\}.$$

Consider a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of n . The *Ferrers diagram of shape* λ is a collection of boxes, arranged in left-justified rows, with λ_i boxes in row i . We denote by $\text{Fer}(r, n)$ the set of r -tuples $(\lambda^{(0)}, \dots, \lambda^{(r-1)})$ of Ferrers diagrams such that $\sum |\lambda^{(i)}| = n$.

The set of conjugacy classes of $G(r, n)$ is naturally parameterized by $\text{Fer}(r, n)$ in the following way. If $(\alpha^{(0)}, \dots, \alpha^{(r-1)}) \in \text{Fer}(r, n)$, we let $m_{i,j}$ be the number of parts

of $\alpha^{(i)}$ equal to j . Then the set

$$cl_{\alpha^{(0), \dots, \alpha^{(r-1)}}} = \{g \in G(r, n) : g \text{ has } m_{i,j} \text{ cycles of color } i \text{ and length } j\}$$

is a conjugacy class of $G(r, n)$, and all conjugacy classes are of this form.

The set of equivalence classes of irreducible complex representations of $G(r, n)$ is also parameterized by the elements of $\text{Fer}(r, n)$. These representations are described in the following result (where we use the symbol \otimes for the internal tensor product of representations and the symbol \odot for the external tensor product of representations).

Proposition 2.1 *Let $\lambda = (\lambda^{(0)}, \dots, \lambda^{(r-1)}) \in \text{Fer}(r, n)$, $n_i = |\lambda^{(i)}|$, and $v = (n_0, \dots, n_{r-1})$. Consider the $G(r, n)$ -representation ρ_λ given by*

$$\rho_\lambda \stackrel{\text{def}}{=} \text{Ind}_{G(r,v)}^{G(r,n)} \left(\bigodot_{i=0}^{r-1} (\gamma_{n_i}^{\otimes i} \otimes \tilde{\rho}_{\lambda^{(i)}}) \right),$$

where:

- $\tilde{\rho}_{\lambda^{(i)}}$ is the natural extension to $G(r, n_i)$ of the irreducible (Specht) representation $\rho_{\lambda^{(i)}}$ of S_{n_i} , i.e., $\tilde{\rho}_{\lambda^{(i)}}(g) \stackrel{\text{def}}{=} \rho_{\lambda^{(i)}}(|g|)$ for all $g \in G(r, n_i)$.
- γ_{n_i} is the one-dimensional representation of $G(r, n_i)$ given by

$$\begin{aligned} \gamma_{n_i} : G(r, n_i) &\rightarrow \mathbb{C}^* \\ g &\mapsto \zeta_r^{z(g)}. \end{aligned}$$

Then the set $\text{Irr}(r, n) \stackrel{\text{def}}{=} \{\rho_{(\lambda^{(0)}, \dots, \lambda^{(r-1)})} \text{ with } (\lambda^{(0)}, \dots, \lambda^{(r-1)}) \in \text{Fer}(r, n)\}$ is a set of representatives of the distinct equivalence classes of irreducible representations of $G(r, n)$.

Let us now consider a group $G(r, p, n)$. Given $q \in \mathbb{N}$ such that $q|r, pq|rn$, $G(r, p, n)$ contains a unique cyclic scalar subgroup C_q of order q , generated by $[1^{r/q}, 2^{r/q}, \dots, n^{r/q}]$. In this case, we can consider the quotient group (see [4, Sect. 4])

$$G(r, p, q, n) \stackrel{\text{def}}{=} \frac{G(r, p, n)}{C_q}.$$

A group of this form is called a *projective reflection group*. Since the conditions of existence of $G(r, p, q, n)$ are symmetric with respect to p and q , we can give the following

Definition Let $G = G(r, p, q, n)$ as above. Its *dual group* G^* is the group obtained from G by simply exchanging the roles of p and q :

$$G^* \stackrel{\text{def}}{=} G(r, q, p, n).$$

If $\lambda = (\lambda^{(0)}, \dots, \lambda^{(r-1)}) \in \text{Fer}(r, n)$, we define the *color* of λ by $z(\lambda) = \sum_i i|\lambda^{(i)}|$, and if $p|r$, we let $\text{Fer}(r, p, 1, n) \stackrel{\text{def}}{=} \{\lambda \in \text{Fer}(r, n) : z(\lambda) \equiv 0 \pmod p\}$. The irreducible representations of the group $G(r, 1, q, n) = G(r, q, n)^*$ are given by those representations of $G(r, n)$ whose kernels contain the scalar cyclic subgroup C_q . It follows from this observation and the description in Proposition 2.1 that the set

$$\text{Irr}(r, 1, q, n) = \{\rho_\lambda : \lambda \in \text{Fer}(r, q, 1, n)\}$$

is a set of representatives of the distinct equivalence classes of irreducible representations of $G(r, 1, q, n)$.

The irreducible representations of $G(r, p, q, n)$ can now be deduced essentially by Clifford theory from those of $G(r, 1, q, n)$. We apply this theory in this case as the final description will be very explicit.

The irreducible representations of $G(r, 1, q, n)$ may restrict to reducible representations of $G(r, p, q, n)$. Let us see which of them split into more than one $G(r, p, q, n)$ -module. Consider the *color representation* $\gamma_n : G(r, n) \rightarrow \mathbb{C}^*$ given by $g \mapsto \zeta_r^{z(g)}$. We note that $\gamma_n^{r/p}$ is a well-defined representation of $G(r, 1, q, n)$ of order p and that the kernel of the cyclic group $\Gamma_p = \langle \gamma_n^{r/p} \rangle$ of representations of $G(r, 1, q, n)$ is $G(r, p, q, n)$. The group Γ_p acts on the set of the irreducible representations of $G(r, 1, q, n)$ by internal tensor product. If we let $n_i = |\lambda^{(i)}|$ and $v = (n_0, \dots, n_{r-1})$, we have that this action is given by

$$\begin{aligned} \gamma_n^{r/p} \otimes \rho_{\lambda^{(0)}, \dots, \lambda^{(r-1)}} &= \text{Ind}_{G(r,v)}^{G(r,n)} \left(\left(\gamma_n^{r/p} |_{G(r,v)} \right) \otimes \bigotimes_{i=0}^{r-1} (\gamma_{n_i}^{\otimes i} \otimes \tilde{\rho}_{\lambda^{(i)}}) \right) \\ &= \rho_{\lambda^{(r-r/p)}, \dots, \lambda^{(r-1)}, \lambda^{(0)}, \dots, \lambda^{(r-1-r/p)}}, \end{aligned} \tag{2}$$

and so it simply corresponds to a shift of r/p of the indexing partitions.

It is now natural to let $\text{Fer}(r, q, p, n)$ be the set of orbits in $\text{Fer}(r, q, 1, n)$ with respect to the action of Γ_p described in (2). If $\lambda = (\lambda^{(0)}, \dots, \lambda^{(r-1)}) \in \text{Fer}(r, q, 1, n)$, we denote by $[\lambda]$ or $[\lambda^{(0)}, \dots, \lambda^{(r-1)}] \in \text{Fer}(r, q, p, n)$ the corresponding orbit. Moreover, if $[\lambda^{(0)}, \dots, \lambda^{(r-1)}] \in \text{Fer}(r, q, p, n)$, we let $\mathcal{ST}_{[\lambda^{(0)}, \dots, \lambda^{(r-1)}]}$ be the set of *standard multitableaux* obtained by filling the boxes of any element in $[\lambda^{(0)}, \dots, \lambda^{(r-1)}]$ with all the numbers from 1 to n appearing once, in such a way that rows are increasing from left to right and columns are increasing from top to bottom (see [4, Sect. 6]).

We will now state a theorem which applies in full generality to every group $G(r, p, q, n)$ and fully clarifies the nature of its irreducible representations. Here and in what follows, if $\lambda \in \text{Fer}(r, n)$, we let $m_p(\lambda) = |\text{Stab}_{\Gamma_p}(\lambda)|$, and we observe that if $[\lambda] = [\mu] \in \text{Fer}(r, q, p, n)$, then $m_p(\lambda) = m_p(\mu)$.

Theorem 2.2 *For every $\lambda \in \text{Fer}(r, q, 1, n)$, we have that the equivalence class of the restriction $\text{Res}_{G(r,p,q,n)}^{G(r,1,q,n)}(\rho_\lambda)$ depends only on the class $[\lambda] \in \text{Fer}(r, p, q, n)$, and it is the direct sum of $m_p(\lambda)$ irreducible nonequivalent $G(r, p, q, n)$ representations that*

we denote by $\rho_{[\lambda]}^0, \dots, \rho_{[\lambda]}^{m_p(\lambda)-1}$:

$$\text{Res}_{G(r,p,q,n)}^{G(r,1,q,n)}(\rho_\lambda) = \bigoplus_{j=0}^{m_p(\lambda)-1} \rho_{[\lambda]}^j.$$

The set

$$\text{Irr}(r, p, q, n) \stackrel{\text{def}}{=} \{ \rho_{[\lambda]}^j : [\lambda] \in \text{Fer}(r, q, p, n) \text{ and } j \in [0, m_p(\lambda) - 1] \}$$

represents the distinct equivalence classes of irreducible representations of $G(r, p, q, n)$. Moreover, we have that $\dim(\rho_{[\lambda]}^j) = |\mathcal{ST}_{[\lambda]}|$ for all $[\lambda] \in \text{Fer}(r, q, p, n)$ and $j \in [0, m_p(\lambda) - 1]$.

If $m_p(\lambda) = 1$, we sometimes write $\rho_{[\lambda]}$ instead of $\rho_{[\lambda]}^0$, and we say that this is an *unsplit* representation. On the other hand, whenever $m_p(\lambda) > 1$, we say that all representations of the form $\rho_{[\lambda]}^j$ are *split* representations. We will come back to this description of the irreducible representations in Sect. 6 with more details.

Let us now turn to give a brief account of the *projective Robinson–Schensted correspondence*, which is an extension of the Robinson–Schensted correspondence for the symmetric group [14, Sect. 7.11] and wreath products $G(r, n)$ [15] to all groups of the form $G(r, p, q, n)$. This is a surjective map

$$G(r, p, q, n) \longrightarrow \bigcup_{[\lambda] \in \text{Fer}(r,p,q,n)} \mathcal{ST}_{[\lambda]} \times \mathcal{ST}_{[\lambda]}$$

such that, if $P, Q \in \mathcal{ST}_{[\lambda]}$, then the cardinality of the inverse image of (P, Q) is equal to $m_q(\lambda)$. In particular, we have that this correspondence is a bijection if and only if $\text{GCD}(q, n) = 1$. We refer the reader to [4, Sect. 10] for the precise definition and further properties of this correspondence.

Note that while in Theorem 2.2 we use elements $[\lambda] \in \text{Fer}(r, q, p, n)$, in the projective Robinson–Schensted correspondence the elements $[\lambda]$ involved belong to $\text{Fer}(r, p, q, n)$. This is one of the reasons why it is natural to look at the dual groups when studying the combinatorial representation theory of any projective reflection group of the form $G(r, p, q, n)$.

3 The model and its natural decomposition

A Gelfand model for a group G is a G -module affording each irreducible representation of G exactly once. A Gelfand model was constructed in [5] for every $G(r, p, q, n)$ such that $\text{GCD}(p, n) = 1, 2$. In order to illustrate it, we need to introduce some new concepts and definitions.

An element $g \in G(r, p, q, n)$ is an *absolute involution* if $g\bar{g} = 1$, \bar{g} being the complex conjugate of g (note that this is well defined since complex conjugation stabilizes the cyclic scalar group C_q). We denote by $I(r, n)$ the set of the absolute involutions of

the group $G(r, n)$, and we similarly define $I(r, p, n)$ and $I(r, p, q, n)$. Moreover, we let $I(r, p, n)^*$ stand for the set of the absolute involutions of the group $G(r, p, n)^*$.

The absolute involutions in $I(r, p, q, n)$ can be either symmetric or antisymmetric, according to the following definition:

Definition Let $v \in G(r, p, q, n)$. We say that it is:

- symmetric if every lift of v in $G(r, n)$ is a symmetric matrix;
- antisymmetric if every lift of v in $G(r, n)$ is an antisymmetric matrix.

We observe that while a symmetric element is always an absolute involution, an antisymmetric element of $G(r, p, q, n)$ is an absolute involution if and only if q is even (see [5, Lemma 4.2]). Antisymmetric elements can also be characterized in terms of the projective Robinson–Schensted correspondence (see [5, Lemma 4.3]):

Lemma 3.1 *Let $v \in G(r, n)$. Then the following are equivalent:*

- (1) v is antisymmetric;
- (2) r is even, and $v \mapsto ((P_0, \dots, P_{r-1}), (P_{\frac{r}{2}}, \dots, P_{r-1}, P_0, \dots, P_{\frac{r}{2}-1}))$ for some $(P_0, \dots, P_{r-1}) \in \mathcal{ST}_\lambda$ and $\lambda \in \text{Fer}(r, n)$ by the Robinson–Schensted correspondence.

Now we can deduce the following combinatorial interpretation for the number of antisymmetric elements in a projective reflection group. Since we often deal with even integers, here and in the rest of this paper we let $k' \stackrel{\text{def}}{=} \frac{k}{2}$ whenever k is an even integer.

Proposition 3.2 *Let $\text{asym}(r, q, p, n)$ be the number of antisymmetric elements in $G(r, q, p, n)$. Then*

$$\text{asym}(r, q, p, n) = \sum_{[\mu, \mu] \in \text{Fer}(r, q, p, n)} |\mathcal{ST}_{[\mu, \mu]}|,$$

where $[\mu, \mu] \in \text{Fer}(r, q, p, n)$ means that $[\mu, \mu]$ varies among all elements in $\text{Fer}(r, q, p, n)$ of the form $[\mu^{(0)}, \dots, \mu^{(r'-1)}, \mu^{(0)}, \dots, \mu^{(r'-1)}]$ for some $\mu = (\mu^{(0)}, \dots, \mu^{(r'-1)}) \in \text{Fer}(r', n')$.

Proof Observe that if $v \in G(r, q, n)$ is antisymmetric and if (P_0, \dots, P_{r-1}) and λ are as in Lemma 3.1, then necessarily $\lambda \in \text{Fer}(r, q, 1, n)$ is of the form $\lambda = (\mu, \mu)$, for some $\mu \in \text{Fer}(r', n')$. So, if $v \mapsto (P, Q)$ is antisymmetric, we have that P is an element in $\mathcal{ST}_{(\mu, \mu)}$ for some $\mu \in \text{Fer}(r', n')$ whilst Q is uniquely determined by P . So we deduce that

$$\text{asym}(r, q, 1, n) = \sum_{(\mu, \mu) \in \text{Fer}(r, q, 1, n)} |\mathcal{ST}_{(\mu, \mu)}|.$$

The result now follows since every antisymmetric element in $G(r, q, p, n)$ has p distinct lifts in $G(r, q, n)$ and any element in $\mathcal{ST}_{[\mu, \mu]}$ has p distinct lifts in $\cup_{(v, v) \in [\mu, \mu]} \mathcal{ST}_{(v, v)}$. □

Before describing the Gelfand model for the involutory reflection groups, we need to recall some further notation from [5]. If $\sigma, \tau \in S_n$ with $\tau^2 = 1$, we let

- $\text{Inv}(\sigma) = \{\{i, j\} : (j - i)(\sigma(j) - \sigma(i)) < 0\}$;
- $\text{Pair}(\tau) = \{\{i, j\} : \tau(i) = j \neq i\}$;
- $\text{inv}_\tau(\sigma) = |\{\text{Inv}(\sigma) \cap \text{Pair}(\tau)\}|$.

If $g \in G(r, p, q, n)$, $v \in I(r, q, p, n)$, \tilde{g} any lift of g in $G(r, p, n)$, and \tilde{v} any lift of v in $G(r, q, n)$, we let

- $\text{inv}_v(g) = \text{inv}_{|v|}(|g|)$;
- $\langle g, v \rangle = \sum_{i=1}^n z_i(\tilde{g})z_i(\tilde{v}) \in \mathbb{Z}_r$;
- $a(g, v) = z_1(\tilde{v}) - z_{|g|^{-1}(1)}(\tilde{v}) \in \mathbb{Z}_r$.

The verification that $\langle g, v \rangle$ and $a(g, v)$ are well defined is straightforward.

We are now ready to present the Gelfand model constructed in [5].

Theorem 3.3 *Let $\text{GCD}(p, n) = 1, 2$, and let*

$$M(r, q, p, n) \stackrel{\text{def}}{=} \bigoplus_{v \in I(r, q, p, n)} \mathbb{C}C_v.$$

Define $\varrho : G(r, p, q, n) \rightarrow GL(M(r, q, p, n))$ by

$$\varrho(g)(C_v) \stackrel{\text{def}}{=} \begin{cases} \zeta_r^{\langle g, v \rangle} (-1)^{\text{inv}_v(g)} C_{|g|v|g|^{-1}} & \text{if } v \text{ is symmetric,} \\ \zeta_r^{\langle g, v \rangle} \zeta_r^{a(g, v)} C_{|g|v|g|^{-1}} & \text{if } v \text{ is antisymmetric.} \end{cases} \tag{3}$$

Then $(M(r, q, p, n), \varrho)$ is a Gelfand model for $G(r, p, q, n)$.

Let us have a short digression to recall what happens for the wreath products $G(r, n)$. In this case the setting is much simpler since the group $G(r, n)$ coincides with its dual, there are no split representations, and no antisymmetric absolute involutions. Moreover, the absolute involutions are characterized as those elements v satisfying

$$v \mapsto (P, P)$$

for some $P \in \mathcal{ST}_\lambda$, $\lambda \in \text{Fer}(r, n)$, via the Robinson–Schensted correspondence. We write in this case $\text{Sh}(v) \stackrel{\text{def}}{=} \lambda$. If c is an S_n -conjugacy class of absolute involutions in $G(r, n)$, we also let $\text{Sh}(c) = \cup_{v \in c} \text{Sh}(v) \subset \text{Fer}(r, n)$. The main result in [6] is the following theorem of compatibility with respect to the Robinson–Schensted correspondence of the irreducible decomposition of the submodules $M(c)$ of $M(r, 1, 1, n)$ defined in the introduction.

Theorem 3.4 *Let c be an S_n -conjugacy class of absolute involutions in $G(r, n)$. Then the following decomposition holds:*

$$M(c) \cong \bigoplus_{\lambda \in \text{Sh}(c)} \rho_\lambda.$$

An analogous result about the model for $G(r, n)$ constructed by Adin, Postnikov, and Roichman [1] was conjectured in [1, Conjecture 7.1] and proved by Marberg in [10].

The main target of this paper is to establish a result analogous to Theorem 3.4 for all groups $G(r, p, q, n)$ satisfying $\text{GCD}(p, n) = 1, 2$ (this will be given as Theorem 10.1). In this general context we also have the decomposition

$$M = M_{\text{Sym}} \oplus M_{\text{Asym}},$$

where M_{Sym} is the symmetric submodule of M , i.e., the submodule spanned by all elements C_v indexed by symmetric absolute involutions, and M_{Asym} is the antisymmetric submodule defined similarly. In fact, the main step in the description of the irreducible decomposition of the modules $M(c)$ will be an intermediate result that provides the irreducible decomposition of the symmetric and the antisymmetric submodules.

4 An outline: the irreducible decomposition of $M(c)$ in type D

In this section we give an outline of the proofs of the main results in the special case $D_n = G(2, 2, n)$. Here we may take advantage of some results which are already known in the literature, such as the description of the split conjugacy classes and of the split representations and their characters. Recall that D_n is a subgroup of index 2 of the group $B_n = G(2, n)$ of signed permutations and observe that its dual group is given by $D_n^* = G(2, 1, 2, n) = B_n / \pm I$.

The irreducible representations of B_n are indexed by elements $(\lambda, \mu) \in \text{Fer}(2, n)$. If $(\lambda, \mu) \in \text{Fer}(2, n)$ is such that $\lambda \neq \mu$, then the two representations $\rho_{(\lambda, \mu)}$ and $\rho_{(\mu, \lambda)}$, when restricted to D_n , are irreducible and isomorphic by Theorem 2.2, and we denote this representation by $\rho_{[\lambda, \mu]}$. If $n = 2m$ is even, Theorem 2.2 also implies that the irreducible representations of B_{2m} of the form $\rho_{(\mu, \mu)}$, when restricted to D_n , split into two irreducible representations that we denote by $\rho_{[\mu, \mu]}^0$ and $\rho_{[\mu, \mu]}^1$.

The conjugacy classes of B_n contained in D_n are those indexed by ordered pairs of partitions (α, β) , with $|\ell(\beta)| \equiv 0 \pmod 2$. They all do not split as D_n -conjugacy classes with the exception of those indexed by $(2\alpha, \emptyset)$, which split into two D_n -conjugacy classes that we denote by $\text{cl}_{2\alpha}^0$ and $\text{cl}_{2\alpha}^1$ (and we make the convention that $\text{cl}_{2\alpha}^0$ is the class containing all the elements g belonging to the B_n -class labeled by $(2\alpha, \emptyset)$ and satisfying $z_i(g) = 0$ for all $i \in [n]$).

The characters of the unsplit representations are clearly the same as those of the corresponding representations of the groups B_n (being the corresponding restrictions). The characters of the split representations, denoted $\chi_{[\mu, \mu]}^\epsilon$, are given by the following result (see [11, 13]).

Lemma 4.1 *Let $g \in D_{2m}$, and let $\mu \vdash m$. Then*

$$\chi_{[\mu, \mu]}^\epsilon(g) = \begin{cases} \frac{1}{2} \chi_{(\mu, \mu)}(2\alpha, \emptyset) + (-1)^{\epsilon+\eta} 2^{\ell(\alpha)-1} \chi_\mu(\alpha) & \text{if } g \in \text{cl}_{2\alpha}^\eta, \\ \frac{1}{2} \chi_{(\mu, \mu)}(g) & \text{otherwise,} \end{cases}$$

where $\epsilon, \eta = 0, 1$, $\chi_{(\mu, \mu)}$ is the character of the B_{2m} -representation $\rho_{(\mu, \mu)}$, and χ_μ is the character of S_m indexed by μ .

In Sect. 5 we prove a generalization of this result for all groups $G(r, p, n)$ such that $\text{GCD}(p, n) = 2$.

An antisymmetric element in B_{2m} is necessarily the product of cycles of length 2 and color 1, i.e., cycles of the form (a^0, b^1) . It follows that the antisymmetric elements of B_{2m} , and hence also those of $B_{2m}/\pm I$, are all S_n -conjugate. This is a special feature of this case and is not true for generic involutory reflection groups (see Sect. 8). We denote by c^1 the unique S_n -conjugacy class of antisymmetric absolute involutions in $B_{2m}/\pm I$, and we will now find out which of the irreducible representations of D_{2m} are afforded by the antisymmetric submodule M_{Asym} , which coincides in this case with $M(c^1)$. The crucial observation is the following result, which is a straightforward consequence of Lemma 4.1.

Remark Let $g \in D_{2m}$. Then

$$\sum_{\mu \vdash m} (\chi_{[\mu, \mu]}^0 - \chi_{[\mu, \mu]}^1)(g) = \begin{cases} (-1)^{\eta} 2^{\ell(\alpha)} \sum_{\mu \vdash m} \chi_\mu(\alpha) & \text{if } g \in \text{cl}_{2\alpha}^\eta, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The main result here is the following.

Theorem 4.2 *Let c^1 be the S_n -conjugacy class consisting of the antisymmetric involutions in $D_n^* = B_n/\pm I$. Then*

$$M(c^1) \cong \bigoplus_{\mu \vdash m} \rho_{[\mu, \mu]}^1.$$

Proof We present here a sketch of the proof only since this result will be generalized and proved in full detail in Sect. 7.

Consider the two representations ϕ^0 and ϕ^1 of D_{2m} on the vector space $M(c^1)$ given by

$$\phi^0(g)(C_v) \stackrel{\text{def}}{=} (-1)^{\langle g, v \rangle} C_{|g|v||g|^{-1}}, \quad \phi^1(g)(C_v) \stackrel{\text{def}}{=} (-1)^{\langle g, v \rangle} (-1)^{a(g, v)} C_{|g|v||g|^{-1}}$$

(notice that $\phi^1(g) = \varrho(g)|_{M(c^1)}$). We will simultaneously prove that

$$\chi_{\phi^0} = \sum_{\mu \vdash m} \chi_{[\mu, \mu]}^0 \quad \text{and} \quad \chi_{\phi^1} = \sum_{\mu \vdash m} \chi_{[\mu, \mu]}^1,$$

the latter equality being equivalent to the statement that we have to prove. To this end, we observe that it will be enough to show that

$$\chi_{\phi^0} - \chi_{\phi^1} = \sum_{\mu \vdash m} (\chi_{[\mu, \mu]}^0 - \chi_{[\mu, \mu]}^1). \tag{5}$$

In fact if (5) is satisfied, we have

$$\chi_{\phi^1} + \sum_{\mu \vdash m} \chi_{[\mu, \mu]}^0 = \chi_{\phi^0} + \sum_{\mu \vdash m} \chi_{[\mu, \mu]}^1. \tag{6}$$

Now, since the irreducible characters are linearly independent, it follows that ϕ^0 has a subrepresentation isomorphic to $\oplus \rho_{[\mu, \mu]}^0$, and similarly for ϕ^1 . By Theorem 2.2 and Proposition 3.2 we also have that

$$\sum_{\mu \vdash m} \dim(\rho_{[\mu, \mu]}^0) = \sum_{[\mu, \mu] \in \text{Fer}(2, 1, 2, 2m)} |\mathcal{ST}_{[\mu, \mu]}| = |c^1| = \dim(\phi^0)$$

and, analogously, $\sum_{\mu \vdash m} \dim(\rho_{[\mu, \mu]}^1) = \dim(\phi^1)$, and we are done.

To prove (5), one has to compute explicitly the difference $\chi_{\phi^0} - \chi_{\phi^1}$ and show that this agrees with the right-hand side of (4). To this end, we will need to observe that $\sum \chi_{\mu}$ is actually the character of a Gelfand model of the symmetric group S_m , which has an already known combinatorial interpretation (see, e.g., [5, Proposition 3.6]). □

Let us now consider a class c of symmetric involutions in $D_n^* = B_n / \pm I$ (note that in this case an absolute involution is actually an involution since all the involved matrices are real). The lift of c to B_n is the union of two S_n -conjugacy classes c_1 and c_2 of B_n that may coincide. Since we already know the irreducible decompositions of $M(c_1)$ and of $M(c_2)$ as B_n -modules (by Theorem 3.4) and hence also as D_n -modules (by Theorem 2.2), the main point in the proof of the following result will be that $M(c)$ is actually isomorphic to a subrepresentation of the restriction of $M(c_1) \oplus M(c_2)$ to D_n , together with straightforward applications of Theorems 3.3 and 4.2.

Theorem 4.3 *Let c be an S_n -conjugacy class of symmetric involutions in $D_n^* = B_n / \pm I$. Then*

$$M(c) \cong \bigoplus_{[\lambda, \mu] \in \text{Sh}(c)} \rho_{[\lambda, \mu]}^0.$$

In Sect. 9 one can find the formal definition and an explicit simple combinatorial description of the sets $\text{Sh}(c)$ for any symmetric S_n -conjugacy class c of absolute involutions in $G(r, p, n)^*$. This is illustrated in the following example.

Example 4.4 Let $v \in B_6 / \pm I$ be given by $v = [6^1, 4^0, 3^0, 2^0, 5^1, 1^1]$. Then the S_n -conjugacy class c of v has 90 elements, and the decomposition of the D_n -module $M(c)$ is given by all representations $\rho_{[\lambda, \mu]}^0$, $[\lambda, \mu] \in \text{Fer}(2, 1, 2, 6)$, where both λ and μ are partitions of 3 and have exactly one column of odd length. Therefore,

$$M(c) \cong \rho_{\left[\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right]} \oplus \rho_{\left[\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \right]} \oplus \rho_{\left[\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right]}.$$

Note in particular that in this case we obtain both unsplit and split representations.

5 On the split conjugacy classes

In the more general case of any involutory reflection group $G(r, p, n)$, we have not been able to find the nature of the conjugacy classes that split from $G(r, n)$ to $G(r, p, n)$ in the literature. This is the content of the present section.

Let r be even, so that it makes sense to talk about even and odd elements in \mathbb{Z}_r . Let c be a cycle in $G(r, n)$ of even length and even color. If $c = (i_1^{z_{i_1}}, i_2^{z_{i_2}}, \dots, i_{2d}^{z_{i_{2d}}})$, we define the *signature* of c to be

$$\text{sign}(c) = z_{i_1} + z_{i_3} + \dots + z_{i_{2d-1}} = z_{i_2} + z_{i_4} + \dots + z_{i_{2d}} \in \mathbb{Z}_2,$$

so that the signature can be either 0 or 1. If g is a product of disjoint cycles of even length and even color, we define the *signature* $\text{sign}(g)$ of g as the sum of the signatures of its cycles.

Lemma 5.1 *Let r be even, and let c be a cycle in $G(r, n)$ of even length and even color. Let $h \in G(r, n)$. Then*

$$\text{sign}(h^{-1}ch) = \text{sign}(c) + \sum_{j \in |h|^{-1}(\text{Supp}(c))} z_j(h) \in \mathbb{Z}_2.$$

In particular, if $g \in G(r, n)$ is a product of cycles of even length and even color, then

$$\text{sign}(h^{-1}gh) = \text{sign}(g) + z(h) \in \mathbb{Z}_2.$$

Proof Let $|c| = (i_1, i_2, \dots, i_{2d})$. We have that $h^{-1}ch$ is a cycle and $|h^{-1}ch| = (\tau^{-1}(i_1), \dots, \tau^{-1}(i_{2d}))$, where $\tau = |h|$. Therefore,

$$\begin{aligned} \text{sign}(h^{-1}ch) &= \sum_{j \text{ odd}} z_{\tau^{-1}(i_j)}(h^{-1}ch) \\ &= \sum_{j \text{ odd}} z_{\tau^{-1}(i_j)}(h) + z_{i_j}(c) - z_{\tau^{-1}(i_{j+1})}(h) \\ &= \text{sign}(c) + \sum_{j \in |h|^{-1}(\text{Supp}(c))} z_j(h), \end{aligned}$$

where the sums in the first two lines are meant to be over all odd integers $j \in [2d]$. \square

It follows from Lemma 5.1 that the conjugacy classes cl_α of $G(r, n)$ contained in $G(r, p, n)$, where α has the special form $\alpha = (2\alpha^{(0)}, \emptyset, 2\alpha^{(2)}, \emptyset, \dots, 2\alpha^{(r-2)}, \emptyset)$, split in $G(r, p, n)$ into (at least) two conjugacy classes, according to the signature. How about the $G(r, n)$ -conjugacy classes of a different form? Do they split as $G(r, p, n)$ -classes?

If G is a group and $g \in G$, we denote by $\text{cl}_G(g)$ the conjugacy class of g and by $C_G(g)$ the centralizer of g in G . If $g \in G(r, p, n)$, then the $G(r, n)$ -conjugacy class

$\text{cl}_{G(r,n)}(g)$ of g splits into more than one $G(r, p, n)$ -conjugacy class if and only if

$$\begin{aligned} \frac{|\text{cl}_{G(r,n)}(g)|}{|\text{cl}_{G(r,p,n)}(g)|} &= \frac{\frac{|G(r,n)|}{|C_{G(r,n)}(g)|}}{\frac{|G(r,p,n)|}{|C_{G(r,p,n)}(g)|}} = \frac{[G(r, n) : G(r, p, n)]}{[C_{G(r,n)}(g) : C_{G(r,p,n)}(g)]} \\ &= \frac{p}{[C_{G(r,n)}(g) : C_{G(r,p,n)}(g)]} > 1, \end{aligned}$$

i.e.,

$$\text{cl}_{G(r,n)}(g) \text{ splits if and only if } [C_{G(r,n)}(g) : C_{G(r,p,n)}(g)] < p.$$

The following proposition clarifies which conjugacy classes of $G(r, n)$ split in $G(r, p, n)$.

Proposition 5.2 *Let $g \in G(r, p, n)$, and let $\text{cl}(g)$ be its conjugacy class in the group $G(r, n)$. Then the following holds:*

- (1) if $\text{GCD}(p, n) = 1$, $\text{cl}(g)$ does not split as a class of $G(r, p, n)$;
- (2) if $\text{GCD}(p, n) = 2$, $\text{cl}(g)$ splits into two different classes of $G(r, p, n)$ if and only if all the cycles of g have:
 - even length,
 - even color,

i.e., if $\text{cl}(g) = \text{cl}_{(2\alpha^{(0)}, \emptyset, 2\alpha^{(2)}, \emptyset, \dots, 2\alpha^{(r-2)}, \emptyset)}$.

Proof Let $G = G(r, n)$ and $H = G(r, p, n)$. We first make a general observation. If $C_G(g)$ contains an element x such that $z(x) \equiv 1 \pmod p$, we can split the group $C_G(g)$ into cosets modulo the subgroup $\langle x \rangle$: in each coset there is exactly one element having color $0 \pmod p$ every p elements. Thus,

$$[C_G(g) : C_H(g)] = p,$$

and $\text{cl}(g)$ does not split in H .

Now let $\text{GCD}(p, n) = 1$. By Bézout’s identity, there exist a, b such that $an + bp = 1$, i.e., there exists a such that the scalar matrix $\zeta_r^a \text{Id}$ has color $1 \pmod p$, so that $\text{cl}(g)$ does not split thanks to the observation above.

Assume now that $\text{GCD}(p, n) = 2$. Arguing as above, there exist a, b such that $ap + bn = 2$, so we know that $C_G(g)$ contains at least an element $\zeta_r^a \text{Id}$ with color $2 \pmod p$.

If there exists an element x of odd color in $C_G(g)$, the matrix $(\zeta_r^a \text{Id})^i \cdot x$ has color 1 for some i , so again $\text{cl}(g)$ does not split in H .

On the other hand, if there are no elements of odd color in $C_G(g)$, every coset of $\langle \zeta_r^a \text{Id} \rangle$ has exactly 1 element belonging to $G(r, p, n)$ out of p' elements. Thus,

$$[C_G(g) : C_H(g)] = p',$$

and $\text{cl}(g)$ splits into $p/p' = 2$ classes.

Let us see when this happens according the cyclic structure of g .

- (1) If g has at least a cycle of odd color, say c , c is in $C_G(g)$, and $\text{cl}(g)$ does not split.
- (2) If g has a cycle of odd length, say $(a_1^{z_1}, \dots, a_{2d+1}^{z_{2d+1}})$, then $(a_1^1, \dots, a_{2d+1}^1)$ has odd color and is in $C_G(g)$, so $\text{cl}(g)$ does not split.
- (3) We are left to study the case of g being a product of cycles all having even length and even color. Thanks to Lemma 5.1, every element in $C_G(g)$ has even color, so by the above argument $\text{cl}(g)$ splits into exactly two classes, and we are done. \square

If $2\alpha = (2\alpha^{(0)}, \emptyset, 2\alpha^{(2)}, \emptyset, \dots, 2\alpha^{(r-2)}, \emptyset)$ is such that $\text{cl}_{2\alpha} \subset G(r, p, n)$ (i.e., if $\sum 2i \ell(\alpha^{(2i)}) \equiv 0 \pmod p$), we denote by $\text{cl}_{2\alpha}^0$ the $G(r, p, n)$ -conjugacy class consisting of all elements in $\text{cl}_{2\alpha}$ having signature 0, and we similarly define $\text{cl}_{2\alpha}^1$.

6 The discrete Fourier transform

Recall from (2) that there is an action of the cyclic group Γ_p generated by $\gamma_n^{r/p}$ on the set of irreducible representations of $G(r, n)$. This action allows us to introduce the discrete Fourier transform, which will be essential in what follows for the case $\text{GCD}(p, n) = 2$. We will parallel and generalize in this section an argument due to Stembridge ([13], Sects. 6 and 7B).

We recall the following definition from [13].

Definition Let $\lambda \in \text{Fer}(r, n)$, (V, ρ_λ) be a concrete realization of the irreducible $G(r, n)$ -representation ρ_λ on the vector space V , and γ be a generator for $\text{stab}_{\Gamma_p}(\rho_\lambda)$. An *associator* for the pair (V, γ) is an element $S \in GL(V)$ exhibiting an explicit isomorphism of $G(r, n)$ -modules between

$$(V, \rho_\lambda) \quad \text{and} \quad (V, \gamma \otimes \rho_\lambda).$$

By Schur’s lemma, $S^{m_p(\lambda)}$ is a scalar, and therefore S can be normalized in such a way that $S^{m_p(\lambda)} = 1$.

Recall from Theorem 2.2 that a representation ρ_λ of $G(r, n)$ splits into exactly $m_p(\lambda)$ irreducible representations of $G(r, p, n)$.

Definition Let $\lambda \in \text{Fer}(r, n)$, and let S be an associator for the $G(r, n)$ -module (V, ρ_λ) . Then the *discrete Fourier transform* with respect to S is the family of $G(r, p, n)$ -class functions $\Delta_\lambda^i : G(r, p, n) \rightarrow \mathbb{C}^*$ given by

$$\Delta_\lambda^i(h) := \text{tr}(S^i \circ h), \quad i \in [0, m_p(\lambda) - 1].$$

A more detailed analysis of the associator shows that the irreducible representations $\rho_{[\lambda]}^i$ are exactly the eigenspaces of the associator S , and we make the convention that, once an associator S has been fixed, the representation $\rho_{[\lambda]}^i$ is the one afforded by the eigenspace of S of eigenvalue $\zeta_{m_p(\lambda)}^i$. Therefore,

$$\Delta_\lambda^i(h) = \sum_{j=0}^{m_p(\lambda)-1} \zeta_{m_p(\lambda)}^{ij} \chi_{[\lambda]}^j(h) \tag{7}$$

for all $h \in G(r, p, n)$, $\chi_{[\lambda]}^j$ being the character of the split representation $\rho_{[\lambda]}^j$ of $G(r, p, n)$.

Now let us consider a representation ρ_λ . Looking at the action described in (2), we see that $m_p(\lambda) = |\text{Stab}_{\Gamma_p}(\rho_\lambda)| = s$ only if $\lambda = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ consists of a smaller pattern repeated s times. It follows that $m_p(\lambda)$ is necessarily a divisor of both n and p .

In particular, if $\text{GCD}(p, n) = 2$, then $m_p(\lambda) = 1, 2$, and so the stabilizer of a representation with respect to Γ_p can either be trivial or be $\{\text{Id}, \gamma_n^{r'}\}$.

Notation From now on, when $\text{GCD}(p, n) = 2$, we use for the representation $\gamma_n^{r'}(g) = (-1)^{z(g)}$ the notation $\delta(g)$.

When $\text{Stab}_{\Gamma_p}(\rho_\lambda) = \{\text{Id}, \delta\}$, the representation ρ_λ has the form $\rho_{(\mu, \mu)}$ with $\mu \in \text{Fer}(r', n')$. Notice that μ may be considered as belonging to $\text{Fer}(r', 1, p', n')$: acting on μ with an element of $C_{p'}$ corresponds to acting on (μ, μ) with an element of C_p , and we know that elements of $\text{Fer}(r, n)$ in the same class modulo C_p parameterize the same irreducible representation of $G(r, p, n)$. These $\rho_{(\mu, \mu)}$ are the representations of $G(r, n)$ that split as $G(r, p, n)$ -modules. As in the case of D_n , they split into two different representations that we denote by $\rho_{[\mu, \mu]}^0$ and $\rho_{[\mu, \mu]}^1$. We also denote by $\chi_{[\mu, \mu]}^0$ and $\chi_{[\mu, \mu]}^1$ the corresponding characters. Then the discrete Fourier transform of $\rho_{\mu, \mu}$ is given by the two functions

$$\Delta_{\mu, \mu}^0(h) = \chi_{[\mu, \mu]}^0(h) + \chi_{[\mu, \mu]}^1(h); \quad \Delta_{\mu, \mu}^1(h) = \chi_{[\mu, \mu]}^0(h) - \chi_{[\mu, \mu]}^1(h).$$

In the rest of this section, we exploit the definition of $\Delta_{\mu, \mu}^1$ to provide an explicit computation of the difference character $\chi_{[\mu, \mu]}^0 - \chi_{[\mu, \mu]}^1$ for every $G(r, p, n)$ with $\text{GCD}(p, n) = 2$. This computation will turn out to be of crucial importance in the proof of Theorem 7.1.

Recall that when $\text{GCD}(p, n) = 2$, the conjugacy classes of $G(r, n)$ of the form $\text{cl}_{2\alpha}$ split into two distinct $G(r, p, n)$ -classes $\text{cl}_{2\alpha}^0$ and $\text{cl}_{2\alpha}^1$, where $2\alpha = (2\alpha^{(0)}, \emptyset, 2\alpha^{(2)}, \emptyset, \dots, 2\alpha^{(r-2)}, \emptyset)$.

Notation In what follows, we often need to compute class functions on $G(r, p, n)$. For this reason, it will be useful to fix one special element, that we call *normal*, for each $G(r, p, n)$ -conjugacy class. If the conjugacy class is not of the form $\text{cl}_{2\alpha}^1$, the normal element h is defined as follows:

- the elements of each cycle of h are chosen in increasing order, from the cycles of smallest color to the cycles of biggest color;
- in every cycle of color i , all the elements have color 0 but the biggest one whose color is i .

If the class is of the form $\text{cl}_{2\alpha}^1$ the normal element h is defined similarly with the unique difference that if the cycle containing n has color $2j$, then the color of n is $2j - 1$, and the color of $n - 1$ is 1. For example, if

$$2\alpha = \left(\begin{array}{|c|} \hline \square \\ \hline \end{array}, \emptyset, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}, \emptyset, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}, \emptyset \right),$$

then the normal element in $\text{cl}_{2\alpha}^0$ is $(1, 2)(3, 4, 5, 6^2)(7, 8, 9, 10^4)(11, 12^4)$, and the normal element in $\text{cl}_{2\alpha}^1$ is $(1, 2)(3, 4, 5, 6^2)(7, 8, 9, 10^4)(11^1, 12^3)$, where we have omitted all the exponents equal to 0.

Proposition 6.1 *Let $g \in G(r, p, n)$ and $\mu \in \text{Fer}(r', n')$. Let χ_μ denote the character of the representation of $G(r', n')$ indexed by μ . Then*

$$\Delta_{\mu, \mu}^1(g) = \begin{cases} (-1)^{\eta} 2^{\ell(\alpha)} \chi_\mu(\alpha^{(0)}, \alpha^{(2)}, \dots, \alpha^{(r-2)}) & \text{if } g \in \text{cl}_{2\alpha}^\eta, \\ 0 & \text{if } g \text{ belongs to an unsplit conjugacy class,} \end{cases}$$

where, if $\alpha = (\alpha^{(0)}, \emptyset, \alpha^{(2)}, \emptyset, \dots, \alpha^{(r-2)}, \emptyset)$, $\ell(\alpha) = \sum \ell(\alpha^{(i)})$.

Proof When g does not belong to a split conjugacy class, $\Delta_{\mu, \mu}^1(g) = 0$. In fact, $\chi_{[\mu, \mu]}^0$ and $\chi_{[\mu, \mu]}^1$ are conjugate characters, so they must coincide on every element belonging to an unsplit class.

When g does belong to a split conjugacy class, this proof consists of three steps:

- (1) Provide an explicit description for the $G(r, n)$ -module $\rho_{\mu, \mu}$.
- (2) Build an associator S for the $G(r, n)$ -module $\rho_{\mu, \mu}$.
- (3) Compute the trace $\text{tr}(S(g)) = \Delta_{\mu, \mu}^1(g)$.

Let us start with the first step. For brevity, we set $\tau = (t_0, \dots, t_{r'-1})$, where $t_i = |\mu^{(i)}|$, and $G(r, (\tau, \tau)) \stackrel{\text{def}}{=} G(r, t_0) \times \dots \times G(r, t_{r'-1}) \times G(r, t_0) \times \dots \times G(r, t_{r'-1}) < G(r, n)$.

Our representation $\rho_{\mu, \mu}$ looks like this (see Theorem 2.1):

$$\begin{aligned} \rho_{\mu, \mu} &= \text{Ind}_{G(r, (\tau, \tau))}^{G(r, n)} (\tilde{\rho}_{\mu^{(0)}} \odot (\gamma_{n_1} \otimes \tilde{\rho}_{\mu^{(1)}}) \odot \dots \odot (\gamma_{n_{r'-1}}^{\otimes(r'-1)} \otimes \tilde{\rho}_{\mu^{(r'-1)}}) \\ &\quad \odot (\gamma_{n_0}^{\otimes r'} \otimes \tilde{\rho}_{\mu^{(0)}}) \odot (\gamma_{n_1}^{\otimes(r'+1)} \otimes \tilde{\rho}_{\mu^{(1)}}) \odot \dots \odot (\gamma_{n_{r'-1}}^{\otimes(r-1)} \otimes \tilde{\rho}_{\mu^{(r'-1)}})) \\ &= \text{Ind}_{G(r, n') \times G(r, n')}^{G(r, n)} \left(\text{Ind}_{G(r, (\tau, \tau))}^{G(r, n') \times G(r, n')} (\tilde{\rho}_{\mu^{(0)}} \odot (\gamma_{n_1} \otimes \tilde{\rho}_{\mu^{(1)}}) \odot \dots \right. \\ &\quad \odot (\gamma_{n_{r'-1}}^{\otimes(r'-1)} \otimes \tilde{\rho}_{\mu^{(r'-1)}}) \odot (\gamma_{n_0}^{\otimes r'} \otimes \tilde{\rho}_{\mu^{(0)}}) \odot (\gamma_{n_1}^{\otimes(r'+1)} \otimes \tilde{\rho}_{\mu^{(1)}}) \odot \dots \\ &\quad \left. \odot (\gamma_{n_{r'-1}}^{\otimes(r-1)} \otimes \tilde{\rho}_{\mu^{(r'-1)}})) \right) \\ &= \text{Ind}_{G(r, n') \times G(r, n')}^{G(r, n)} (\rho_\mu \odot (\delta \otimes \rho_\mu)). \end{aligned}$$

We need to give an explicit description of this representation of $G(r, n)$. We consider the set Θ of two-rowed arrays $\begin{bmatrix} t_1 & \dots & t_{n'} \\ t_{n'+1} & \dots & t_n \end{bmatrix}$ such that $\{t_1, \dots, t_n\} = \{1, \dots, n\}$ and the t_i 's increase on each of the two rows. Each element in Θ can be identified with the permutation whose window notation is $[t_1, \dots, t_n]$.

Proposition 6.2 *Let $g \in G(r, n)$. There exists a unique $t' \in \Theta : g = t'(x_1, x_2)$ for an element $(x_1, x_2) \in G(r, n') \times G(r, n')$.*

Proof Existence. Let $g = [\sigma_1^{z_1}, \dots, \sigma_n^{z_n}]$, and let t be an element of Θ whose first and second lines are filled with the (reordered) integers $\sigma_1, \dots, \sigma_{n'}$, and $\sigma_{n'+1}, \dots, \sigma_n$, respectively. Since we need to obtain $g = t(x_1, x_2)$ with $(x_1, x_2) \in G(r, n') \times G(r, n')$, we have to check that $t^{-1}g \in G(r, n') \times G(r, n')$, i.e.,

$$1 \leq t^{-1}|g|(i) = t^{-1}\sigma_i \leq n' \quad \text{if } i \in [n'],$$

$$n' < t^{-1}|g|(i)t^{-1}\sigma_i \leq n \quad \text{if } i \in [n' + 1, n],$$

and this is an immediate consequence of the way t was constructed.

Uniqueness. This is due to cardinality reasons:

$$|\Theta| |G(r, n') \times G(r, n')| = \binom{n}{n'} (n'!r^{n'})^2 = |G(r, n)|. \quad \square$$

Thanks to Proposition 6.2, a set of coset representatives for

$$G(r, n)/(G(r, n') \times G(r, n'))$$

is given by Θ . Let T be the vector space spanned by the elements of Θ . The vector space associated to the representation we are dealing with can be identified with $T \otimes V \otimes V$, and the action of $\rho_{\mu, \mu}$ on it is given by

$$\rho_{\mu, \mu} : G(r, n) \longrightarrow GL(T \otimes V \otimes V)$$

$$x \longmapsto \rho_{\mu, \mu}(x) : T \otimes V \otimes V \longrightarrow T \otimes V \otimes V$$

$$t \otimes v_1 \otimes v_2 \longmapsto \delta(x_2)t' \otimes \rho_{\mu}(x_1)(v_1) \otimes \rho_{\mu}(x_2)(v_2),$$

where t', x_1 , and x_2 are uniquely determined by the relation $xt = t'(x_1, x_2)$ with $t \in \Theta$, $(x_1, x_2) \in G(r, n') \times G(r, n')$.

We are now ready for the second step.

Proposition 6.3 *The automorphism $S \in GL(T \otimes V \otimes V)$ defined by*

$$S(t \otimes v_1 \otimes v_2) = \hat{t} \otimes v_2 \otimes v_1,$$

where \hat{t} is the element of Θ obtained from t by exchanging its rows, is an associator for $T \otimes V \otimes V$.

Proof All we have to show is that S is an isomorphism of representations between $\rho_{\mu, \mu}$ and $\delta \otimes \rho_{\mu, \mu}$, i.e.,

$$S \circ \rho_{\mu, \mu}(g) = \delta(g)\rho_{\mu, \mu}(g) \circ S.$$

The set of permutations together with the diagonal matrix $[1^0, 2^0, \dots, n^1]$ generate $G(r, n)$, so this verification can be accomplished when g is one of these elements only. Just exploit Proposition 6.2 to write gt as a product $t'(g_1, g_2)$, and notice that $\hat{t} = ts$ if s is the tabloid $\begin{bmatrix} n'+1 & \dots & n \\ 1 & \dots & n' \end{bmatrix}$. Also, when $g = [1^0, 2^0, \dots, n^1]$, the equality $g\hat{t} = gts = t(g_1, g_2)s = ts(g_2, g_1) = \hat{t}(g_2, g_1)$ is used. □

Finally, the last step: let us compute $\Delta_{\mu,\mu}^1(g)$ for every g belonging to a split conjugacy class. Since $\Delta_{\mu,\mu}^1$ is a class function, we can choose g to be the normal element of each $G(r, p, n)$ -conjugacy class. In fact, even less is needed: it will be enough to choose the normal elements of the classes $cl_{2\alpha}^0$ only, because of the useful relation (see [13, Proposition 6.2])

$$\Delta_{\mu,\mu}^1(g h g^{-1}) = \delta(g) \Delta_{\mu,\mu}^1(h) \quad \forall g \in G(r, n), h \in G(r, p, n). \tag{8}$$

So we compute $\Delta_{\mu,\mu}^1(h)$, where h is the normal element of the class $cl_{2\alpha}^0$. By definition, $\Delta_{\mu,\mu}^1(h) = \text{tr}(S \circ h)$. Now, given $t' \in \mathcal{O}$ and $(h_1, h_2) \in G(r, n') \times G(r, n')$ satisfying $h t = t'(h_1, h_2)$, if v_i, v_j are vectors of a basis of V , then

$$\begin{aligned} S[h(t \otimes v_i \otimes v_j)] &= S[\delta(h_2)t' \otimes h_1 v_i \otimes h_2 v_j] \\ &= \delta(h_2) \hat{t}' \otimes h_2 v_j \otimes h_1 v_i \\ &= \hat{t}' \otimes h_2 v_j \otimes h_1 v_i, \end{aligned}$$

where the last equality depends on the special way we chose h . Namely, since $(h_1, h_2) = (t')^{-1} h t$ with $t, t' \in S_n$, the colors of (h_1, h_2) are the same as in h and they are simply permuted, so in (h_1, h_2) the colors are all even. So the trace we are computing is given by

$$\text{tr}(S \circ h) = \sum_{i,j=1,\dots,n', t=\hat{t}'} (\rho_\mu(h_2))_{i,j} (\rho_\mu(h_1))_{j,i} = \sum_{t=\hat{t}'} \chi^\mu(h_1 h_2).$$

Recall the way t' is constructed in the proof of Proposition 6.2: $t' = \hat{t}$ if and only if $|h|(t_i)$ belongs to $\{t_{n'+1}, \dots, t_{2n}\}$ for every $i \in [n']$:

$$\{t_{n'+1}, \dots, t_{2n}\} = \{|h|(t_i)\}_{1 \leq i \leq n'},$$

and, vice versa,

$$\{t_1, \dots, t_{n'}\} = \{|h|(t_i)\}_{n' < i \leq n}.$$

So the t 's satisfying $t = \hat{t}$ are those t such that, for every cycle of h , the consecutive numbers are in opposite rows. We have two possibilities for each cycle, so there are $2^{\ell(\alpha)}$ choices for t in total.

Furthermore, suppose that h contains a cycle of length $2k$ and color $2j$. Then, according to which of the two possible choices is made for t , a cycle of length k and color j will be contained either in h_1 or in h_2 . Thus, $h_1 h_2$ belongs to the $G(r', n')$ -class $cl_{\alpha^{(0)}, \alpha^{(2)}, \dots, \alpha^{(r-2)}}$. So our final result is

$$\Delta_{\mu,\mu}^1(h) = 2^{\ell(\alpha)} \chi^\mu(\alpha^{(0)}, \alpha^{(2)}, \dots, \alpha^{(r-2)}).$$

Let us now turn to the elements belonging to the other split conjugacy class $cl_{2\alpha}^1$. If h belongs to $cl_{2\alpha}^0$, thanks to Lemma 5.1,

$$g h g^{-1} \in cl_{2\alpha}^1 \quad \Rightarrow \quad z(g) = 1 \quad \text{mod } 2 \quad \Rightarrow \quad \delta(g) = -1,$$

and therefore (see (8))

$$\Delta^1_{\mu,\mu}(ghg^{-1}) = \delta(g)\Delta^1_{\mu,\mu}(h) = -2^{\ell(\alpha)}\chi^\mu(\alpha^{(0)}, \alpha^{(2)}, \dots, \alpha^{(r-2)}). \quad \square$$

7 The antisymmetric submodule

This section is the real heart of this paper. We study the irreducible decomposition of the antisymmetric submodule M_{Asym} (and hence also of the symmetric submodule M_{Sym}) of the Gelfand model $M(r, 1, p, n)$ of the group $G(r, p, n)$ constructed in Theorem 3.3. More precisely, we will show that M_{Asym} affords exactly one representation of each pair of split irreducible representations of $G(r, p, n)$; namely, the one labeled with 1.

If $\text{GCD}(p, n) = 1$, the antisymmetric submodule vanishes (and there are no split representations), so in this section we can always assume that $\text{GCD}(p, n) = 2$.

Theorem 7.1 *Let M_{Asym} be the antisymmetric submodule of the Gelfand model $M(r, 1, p, n)$ of $G(r, p, n)$. Then*

$$(M_{\text{Asym}}, \varrho) \cong \bigoplus_{[\mu, \mu] \in \text{Fer}(r, 1, p, n)} \rho^1_{[\mu, \mu]}.$$

Proof The strategy in this proof is the one outlined for the case of Weyl groups of type D . So we consider the two representations of $G(r, p, n)$,

$$(M_{\text{Asym}}, \phi^0) \quad \text{and} \quad (M_{\text{Asym}}, \phi^1),$$

given by

$$\phi^0(g)(C_v) \stackrel{\text{def}}{=} \zeta_r^{(g,v)} C_{|g|v|g|^{-1}}, \quad \phi^1(g)(C_v) \stackrel{\text{def}}{=} \zeta_r^{(g,v)} \zeta_r^{a(g,v)} C_{|g|v|g|^{-1}}$$

(notice that $\phi^1(g) = \varrho(g)|_{M_{\text{Asym}}}$). The main idea of this proof is to exploit Proposition 6.1 to show that

$$\chi_{\phi^0}(g) - \chi_{\phi^1}(g) = \sum_{[\mu] \in \text{Fer}(r', 1, p', n')} \chi^0_{[\mu, \mu]}(g) - \sum_{[\mu] \in \text{Fer}(r', 1, p', n')} \chi^1_{[\mu, \mu]}(g), \quad (9)$$

where we observe that if $[\mu]$ ranges in $\text{Fer}(r', 1, p', n')$, then $[\mu, \mu]$ ranges in $\text{Fer}(r, 1, p, n)$. First of all, we will compute the right-hand side of (9). We already know that it vanishes on every g belonging to an unsplit conjugacy class. So, let $g \in \text{cl}^\eta_{2\alpha}$.

Let χ_M denote the character of a model for the group $G(r', n')$. Then

$$\begin{aligned} \sum_{[\mu] \in \text{Fer}(r', 1, p', n')} (\chi^0_{[\mu, \mu]} - \chi^1_{[\mu, \mu]})(g) &= \frac{1}{p'} \sum_{\mu \in \text{Fer}(r', n')} (\chi^0_{[\mu, \mu]} - \chi^1_{[\mu, \mu]})(g) \\ &= \frac{1}{p'} (-1)^\eta \sum_{\mu \in \text{Fer}(r', n')} 2^{\ell(\alpha)} \chi_\mu(\alpha) \\ &= \frac{1}{p'} (-1)^\eta 2^{\ell(\alpha)} \chi_M(\alpha), \end{aligned} \quad (10)$$

where the first equality holds because the contribution of every $\mu \in \text{Fer}(r', n')$ provides p' copies of the same irreducible representation of $G(r, p, n)$, the second one follows from Proposition 6.1, and $\chi_M(\alpha)$ denotes the value of the character χ_M of a Gelfand model of $G(r', n')$ on any element of the conjugacy class cl_α .

Now, following [5], for $g \in G(r, n)$, we denote by $\Pi^{2,1}(g)$ the set of partitions of the set of disjoint cycles of g into:

- singletons;
- pairs of cycles having the same length.

If $\pi \in \Pi^{2,1}(g)$, we let $\ell(\pi)$ be the number of parts of π , and $\text{pair}_j(\pi)$ be the number of parts of π which are pairs of cycles of length j . Moreover, if $s \in \pi$ is a part of π , we let $z(s)$ be the sum of the colors of the (either 1 or 2) cycles in s .

If g and g' belong to the same conjugacy class cl_α , there is clearly a bijection between $\Pi^{2,1}(g)$ and $\Pi^{2,1}(g')$ preserving the statistics $\ell(\pi)$ and $\text{pair}_j(\pi)$, and the colors $z(s)$ of the parts of π ; therefore we sometimes write $\Pi^{2,1}(\alpha)$ to mean $\Pi^{2,1}(g)$ for some $g \in \text{cl}_\alpha$.

The set $\Pi^{2,1}(\alpha)$ can be used to describe the character of a Gelfand model of $G(r', n')$ (see [5, Proposition 3.6]):

$$\chi_M(\alpha) = \sum_{\pi} (r')^{\ell(\pi)} \prod_j j^{\text{pair}_j(\pi)} \tag{11}$$

for all $\alpha \in \text{Fer}(r', n')$, where the sum is taken over all elements of $\Pi^{2,1}(\alpha)$ having no singletons of even length and such that $z(s) = 0 \in \mathbb{Z}_{r'}$ for all $s \in \pi$.

Let us now evaluate $\chi_{\phi^0}(g) - \chi_{\phi^1}(g)$. To this aim, we recall some further notation used in [5]. Consider, for all $g \in G(r, p, n)$ and $\epsilon \in \mathbb{Z}_2$, the set

$$A^\epsilon(g) := \{w \in G(r, n) : w \text{ is antisymmetric, and } |g|w|g|^{-1} = (-1)^\epsilon w\}.$$

Any $w \in A^\epsilon(g)$ determines a partition $\pi(w) \in \Pi^{2,1}(g)$: a cycle c is a singleton of $\pi(w)$ if the restriction of $|w|$ to $\text{Supp}(c)$ is a permutation of $\text{Supp}(c)$, and $\{c, c'\}$ is a pair of $\pi(w)$ if the restriction of $|w|$ to $\text{Supp}(c)$ is a bijection between $\text{Supp}(c)$ and $\text{Supp}(c')$. Finally, if $\pi \in \Pi^{2,1}(g)$, we let $A_\pi^\epsilon \stackrel{\text{def}}{=} \{w \in A^\epsilon(g) : \pi(w) = \pi\}$. Then the set $A^\epsilon(g)$ can be decomposed into the disjoint union

$$A^\epsilon(g) = \bigcup_{\pi \in \Pi^{2,1}(g)} A_\pi^\epsilon. \tag{12}$$

With the above notation, we have

$$\chi_{\phi^0}(g) - \chi_{\phi^1}(g) = \frac{1}{p} \sum_{\pi \in \Pi^{2,1}(g)} \sum_{A_\pi^0 \cup A_\pi^1} \zeta_r^{(g,w)} (1 - \zeta_r^{a(g,w)}).$$

Since (see [5, Proof of Lemma 5.7])

$$a(g, w) = \begin{cases} 0 & \text{if } w \in A_\pi^0(g), \\ r' & \text{if } w \in A_\pi^1(g), \end{cases}$$

Table 1 The sets A_s^1

$ s $	A_s^1
$\{(i_1, \dots, i_d)\}$ with $d \equiv 2 \pmod 4$	$i_h \mapsto \zeta_r^k (-1)^h i_{h+\frac{d}{2}}$
$\{(i_1, \dots, i_d)\}$ with $d \not\equiv 2 \pmod 4$	\emptyset
$\{(i_1, \dots, i_d), (j_1, \dots, j_d)\}$, with d odd	\emptyset
$\{(i_1, \dots, i_d), (j_1, \dots, j_d)\}$, with d even	$i_h \mapsto \zeta_r^k (-1)^h j_{h+l}$ and $j_h \mapsto -\zeta_r^k (-1)^{h-l} i_{h-l}$

we find

$$\chi_{\phi^0}(g) - \chi_{\phi^1}(g) = \frac{1}{p^r} \sum_{\pi \in \Pi^{2,1}(g)} \sum_{w \in A_\pi^1} \zeta_r^{\langle g, w \rangle}. \tag{13}$$

It is shown in [5, Sect. 5] that if $\pi = \{s_1, \dots, s_h\}$, then the set A_π^1 has a natural decomposition $A_\pi^1 = A_{s_1}^1 \times \dots \times A_{s_h}^1$, i.e., every w in A_π^1 can be written as a product $w = w_1 \cdot \dots \cdot w_h$, with $w_i \in A_{s_i}^1$. The sets $A_{s_i}^1$ depend on the structure of $|s_i|$ only and are described in Table 1. In this table the indices of $i_1, \dots, i_d, j_1, \dots, j_d$ should be considered in \mathbb{Z}_d , and in any box of the table the parameters $k \in \mathbb{Z}_r$ and $l \in \mathbb{Z}_d$ are arbitrary but fixed. For example, if $s = \{(1, 2), (3, 4)\}$ and $r = 4$, then A_s^1 consists of eight elements having either the form $(1^{k+2}, 3^k)(2^k, 4^{k+2})$ or the form $(1^{k+2}, 4^k)(2^k, 3^{k+2})$ as k varies in $\{0, 1, 2, 3\}$. This allows us to focus on the single sets A_s^1 via the identity

$$\begin{aligned} \sum_{w \in A_\pi^1} \zeta_r^{\langle g, w \rangle} &= \sum_{w_1 \in A_{s_1}^1, \dots, w_h \in A_{s_h}^1} \zeta_r^{\langle g, w_1, \dots, w_h \rangle} \\ &= \sum_{w_1 \in A_{s_1}^1, \dots, w_h \in A_{s_h}^1} \zeta_r^{\sum_i \langle g_i, w_i \rangle} \\ &= \prod_{i=1}^h \sum_{w_i \in A_{s_i}^1} \zeta_r^{\langle g_i, w_i \rangle}, \end{aligned} \tag{14}$$

where $g_i \in G(r, \text{Supp}(s_i))$ is the restriction of g to $\text{Supp}(s_i)$ (if $s \in \Pi^{2,1}(g)$, we let $\text{Supp}(s)$, the support of s , be the union of the supports of the cycles in s).

Lemma 7.2 *If $g \in G(r, n)$ has at least one cycle c of odd length, then*

$$\sum_{w \in A_\pi^1} \zeta_r^{\langle g, w \rangle} = 0$$

for all $\pi \in \Pi^{2,1}(g)$.

Proof This is trivial since in these cases $A_{\pi}^1 = \emptyset$ (see Table 1). □

By Lemma 7.2 we can restrict our attention to those elements g having all cycles of even length.

Lemma 7.3 *If $g \in G(r, p, n)$ has at least one cycle c of odd color, then*

$$\sum_{\pi \in \Pi_{2,1}(g)} \sum_{w \in A_{\pi}^1} \zeta_r^{\langle g, w \rangle} = 0.$$

Proof Since the left-hand side is a class function (by (13)), we can assume that g is normal. We prove in this case the stronger statement that $\sum_{w \in A_{\pi}^1} \zeta_r^{\langle g, w \rangle} = 0$ for all $\pi \in \Pi_{2,1}(g)$. By Lemma 7.2, we can assume that the cycle c has even length. We split this result into two cases. Assume that the cycle c of odd color, say j , is a singleton $s_i = \{c\}$ of π . Then Table 1 displays the structure of $A_{s_i}^1$. In particular, if $\ell(c) \not\equiv 2 \pmod 4$, $A_{s_i}^1 = \emptyset$, and we are done; if $\ell(c) \equiv 2 \pmod 4$, we find

$$\sum_{w \in A_{s_i}^1} \zeta_r^{\langle g_i, w_i \rangle} = \sum_{k=0}^{r-1} \zeta_r^{jk} = 0,$$

since j is odd and cannot be a multiple of r .

Now assume that the cycle c belongs to a pair s_i of π . Let us call a and b the two colors of the cycles in s_i , with b odd. Again, looking at Table 1, we have

$$\begin{aligned} \sum_{w_i \in A_{s_i}^1} \zeta_r^{\langle g_i, w_i \rangle} &= \sum_{k=0}^{r-1} \sum_{l=0}^{d-1} \zeta_r^{ak+b(k+(l+1)r')} \\ &= \sum_{k=0}^{r-1} \left(\frac{d}{2} \zeta_r^{(a+b)k} + \frac{d}{2} \zeta_r^{ak+b(k+r')} \right) \\ &= \frac{d}{2} \sum_{k=0}^{r-1} \zeta_r^{(a+b)k} (1 + \zeta_r^{br'}) \\ &= \frac{d}{2} (1 + \zeta_r^{br'}) \sum_{k=0}^{r-1} \zeta_r^{(a+b)k}. \end{aligned}$$

Since b is odd, the factor $1 + \zeta_r^{br'}$ vanishes, and so does the whole sum. □

Lemma 7.4 *Let $g \in G(r, p, n)$ be normal and such that all cycles of g have even color and even length. Then, for all $\pi \in \Pi^{2,1}(g)$,*

$$\sum_{w \in A_{\pi}^1} \zeta_r^{\langle g, w \rangle} = \begin{cases} (-1)^{\text{sign}(g)} |A_{\pi}^1| & \text{if } z(s) = 0 \text{ for all } s \in \pi; \\ 0 & \text{otherwise.} \end{cases}$$

Proof We first assume that $\text{sign}(g) = 0$. If s_i is a singleton of π of color $2j$ and length $\not\equiv 2 \pmod 4$, then A_{s_i} is empty, and the result clearly follows. So we can assume that $\ell(s_i) \equiv 2 \pmod 4$, and we can derive the value of $\langle g_i, w_i \rangle$ from Table 1 and obtain

$$\sum_{w_i \in A_{s_i}^1} \zeta_r^{\langle g_i, w_i \rangle} = \sum_{k=0}^{r-1} \zeta_r^{2jk} = \begin{cases} 0 & \text{if } 2j \not\equiv 0 \pmod r; \\ r = |A_{s_i}^1| & \text{if } 2j \equiv 0 \pmod r. \end{cases}$$

Let now $s_i = \{c_1, c_2\}$ be a pair of cycles of length d , and colors respectively $2a$ and $2b$. We have

$$\begin{aligned} \sum_{w_i \in A_{s_i}^1} \zeta_r^{\langle g_i, w_i \rangle} &= \sum_{k=0}^{r-1} \sum_{l=0}^{d-1} \zeta_r^{2ak+2b(k+(l+1)r')} \\ &= d \sum_{k=0}^{r-1} \zeta_r^{2ak+2bk} = \begin{cases} 0 & \text{if } 2a + 2b \not\equiv 0 \pmod r; \\ dr = |A_{s_i}^1| & \text{if } 2a + 2b \equiv 0 \pmod r. \end{cases} \end{aligned}$$

The result follows from these computations together with (14). If $\text{sign}(g) = 1$, the proof is similar and is left to the reader. \square

We are now ready to prove Theorem 7.1. Let first g belong to an unsplit class. Then g has a cycle of odd length or a cycle of odd color, and then (13) and Lemmas 7.2 and 7.3 ensure that

$$\chi_{\phi^0}(g) - \chi_{\phi^1}(g) = 0 = \sum_{[\mu] \in \text{Fer}(r', 1, p', n')} \chi_{[\mu, \mu]}^0 - \sum_{[\mu] \in \text{Fer}(r', 1, p', n')} \chi_{[\mu, \mu]}^1(g). \tag{15}$$

So let g belong to the split conjugacy class of the form $c_{2\alpha}^\eta$. We are interested in the evaluation of the sum appearing in (13), and so we can assume that g is also a normal element. Thanks to Lemma 7.4, the only partitions $\pi \in \Pi^{2,1}(g)$ contributing to the sum (13) are those satisfying $z(s) = 0 \pmod r$ for all $s \in \pi$. Thus,

$$\begin{aligned} \chi_{\phi^0}(g) - \chi_{\phi^1}(g) &= \frac{1}{p'} (-1)^\eta \sum_{\pi \in \Pi^{2,1}(g)} |A_\pi^1| \\ &= \frac{1}{p'} (-1)^\eta \sum_{\pi \in \Pi^{2,1}(g)} r^{\ell(\pi)} \prod_j (2j)^{\text{pair}_{2j}(\pi)}, \end{aligned}$$

where, by Table 1, the sum is taken over all partitions of $\Pi^{2,1}(g)$ such that:

- singletons have length $\equiv 2 \pmod 4$;
- pairs have even length;
- $z(s) = 0 \pmod r$ for all $s \in \pi$.

Summarizing, if $g \in \text{cl}_{2\alpha}^\eta$, we have

$$\begin{aligned}
 \chi_{\phi^0}(g) - \chi_{\phi^1}(g) &= \frac{1}{p'}(-1)^\eta \sum_{\pi \in \Pi^{2,1}(2\alpha)} r^{\ell(\pi)} \prod_j (2j)^{\text{pair}_{2j}(\pi)} \\
 &= \frac{1}{p'}(-1)^\eta \sum_{\pi \in \Pi^{2,1}(2\alpha)} (2r')^{\ell(\pi)} \prod_j (2j)^{\text{pair}_{2j}(\pi)} \\
 &= \frac{1}{p'}(-1)^\eta 2^{\ell(\pi) + \sum_j \text{pair}_{2j}(\pi)} \sum_{\pi \in \Pi^{2,1}(2\alpha)} (r')^{\ell(\pi)} \prod_j j^{\text{pair}_{2j}(\pi)} \\
 &= \frac{1}{p'}(-1)^\eta 2^{\ell(\alpha)} \sum_{\pi \in \Pi^{2,1}(\alpha)} (r')^{\ell(\pi)} \prod_j j^{\text{pair}_j(\pi)},
 \end{aligned}$$

where α has to be considered as an element in $\text{Fer}(r', n')$, the last sum is taken over all partitions of $\Pi^{2,1}(\alpha)$ whose singletons have odd length (and pairs have any length), and $z(s) = 0 \in \mathbb{Z}_{r'}$ for all $s \in \pi$.

The above computation, together with (10), (11), and (15), leads to

$$\begin{aligned}
 &\sum_{[\mu] \in \text{Fer}(r', 1, p', n')} \chi_{[\mu, \mu]}^0(g) + \chi_{\phi^1}(g) \\
 &= \sum_{[\mu] \in \text{Fer}(r', 1, p', n')} \chi_{[\mu, \mu]}^1(g) + \chi_{\phi^0}(g) \quad \forall g \in G(r, p, n). \tag{16}
 \end{aligned}$$

Now, $\sum_{[\mu] \in \text{Fer}(r', 1, p', n')} \chi_{[\mu, \mu]}^0$ and $\sum_{[\mu] \in \text{Fer}(r', 1, p', n')} \chi_{[\mu, \mu]}^1$ are orthogonal characters. Therefore, by Theorem 2.2 and Proposition 3.2, we have that

$$\sum_{\mu \in \text{Fer}(r', 1, p', n')} \dim(\rho_{[\mu, \mu]}^0) = \sum_{\mu \in \text{Fer}(r', 1, p', n')} |\mathcal{ST}_{[\mu, \mu]}| = \dim(M_{\text{Asym}}) = \dim(\phi^0)$$

and, analogously, $\sum_{\mu \in \text{Fer}(r', 1, p', n')} \dim(\rho_{[\mu, \mu]}^1) = \dim(\phi^1)$, so we can conclude that

$$\sum_{\mu \in \text{Fer}(r', 1, p', n')} \chi_{[\mu, \mu]}^0(g) = \chi_{\phi^0}(g) \quad \text{and} \quad \sum_{\mu \in \text{Fer}(r', 1, p', n')} \chi_{[\mu, \mu]}^1(g) = \chi_{\phi^1}(g).$$

Recalling that $\rho^1(g) = \varrho(g)|_{M_{\text{Asym}}}$, the above equality means that

$$(M_{\text{Asym}}, \varrho) \cong \bigoplus_{\mu \vdash m} \rho_{[\mu, \mu]}^1,$$

and Theorem 7.1 is proved. □

8 The antisymmetric classes

An antisymmetric element of $G(r, n)$ can be characterized by the structure of its cycles; namely, an element $v \in G(r, n)$ is antisymmetric if and only if every cy-

cle c of v has length 2 and is of the form $c = (a_1^{z_{a_1}}, a_2^{z_{a_2}})$ with $z_{a_2} = z_{a_1} + r'$. We say that the residue class of z_{a_1} and z_{a_2} modulo r' is the *type* of c . If the number of disjoint cycles of type i of an antisymmetric element v of $G(r, n)$ is t_i , then the integer vector $\tau(v) = (t_0, \dots, t_{r'-1})$ is called the *type* of v . It is clear that two antisymmetric elements in $G(r, n)$ are S_n -conjugate if and only if they have the same type. We denote by $AC(r, n)$ the set of types of antisymmetric elements in $G(r, n)$, i.e., the set of vectors $(t_0, \dots, t_{r'-1})$ with nonnegative integer entries such that $t_0 + \dots + t_{r'-1} = n'$. If $\text{GCD}(p, n) = 2$ we let γ be the cyclic permutation of $AC(r, n)$ defined by $\gamma(t_0, \dots, t_{r'-1}) = (t_{r/p}, t_{1+r/p}, \dots, t_{r'-1+r/p})$ where the indices must be intended as elements in \mathbb{Z}_r . We observe that γ has order p' , and so we have an action of the cyclic group $C_{p'}$ generated by γ on $AC(r, n)$. We denote the quotient set by $AC(r, p, n)^*$. The type of an antisymmetric element of $G(r, p, n)^*$ is then an element of $AC(r, p, n)^*$, and if $[\tau] \in AC(r, p, n)^*$, we let $c_{[\tau]}^1$ be the S_n -conjugacy class consisting of the antisymmetric absolute involutions in $G(r, p, n)^*$ of type $[\tau]$. The main result of this section, Theorem 8.4, provides a compatibility between the coefficients of τ and the sizes of the indices of the irreducible components of the module $M(c_{[\tau]}^1)$. For this, the following criterion will be helpful.

Proposition 8.1 *Let $v = (n_0, \dots, n_{r-1})$ be a composition of n into r parts, and ρ a representation of $G(r, n)$. Then the following are equivalent:*

- (1) *The irreducible subrepresentations of ρ are all of the form $\rho_{\lambda^{(0)}, \dots, \lambda^{(r-1)}}$ with $|\lambda^{(i)}| = n_i$ for all $i \in [0, r - 1]$;*
- (2) *There exists a representation ϕ of $G(r, v)$ such that $\rho = \text{Ind}_{G(r, v)}^{G(r, n)}(\phi)$ and $\phi(g) = \zeta_r^{\sum i z(g_i)} \phi(|g|)$ for all $g = (g_0, \dots, g_{r-1}) \in G(r, v)$.*

Proof In proving that (1) implies (2), we can clearly assume that ρ is irreducible, and in this case the result is straightforward from the description in Proposition 2.1. In proving that (2) implies (1), we can assume that ϕ is irreducible. Then it is clear that $\phi \downarrow_{S_v}$ is also irreducible, where $S_v = S_{n_0} \times \dots \times S_{n_{r-1}}$. In particular, there exist $\lambda^{(0)}, \dots, \lambda^{(r-1)}$ with $|\lambda^{(i)}| = n_i$ such that $\phi \downarrow_{S_v} \cong \rho_{\lambda^{(0)}} \odot \dots \odot \rho_{\lambda^{(r-1)}}$. Now we can conclude that

$$\phi \cong (\gamma_{n_0}^0 \otimes \tilde{\rho}_{\lambda^{(0)}}) \odot \dots \odot (\gamma_{n_{r-1}}^{r-1} \otimes \tilde{\rho}_{\lambda^{(0)}}),$$

and so the result follows again from Proposition 2.1. □

We now concentrate on the special case $p = 2$, so that $p' = 1$. The general case will then be a direct consequence. Since the index of $G(r, 2, n)$ in $G(r, n)$ is 2, the induction $\psi = \text{Ind}_{G(r, 2, n)}^{G(r, n)}(M(c_\tau^1), \varrho)$ of the $G(r, 2, n)$ -representation $M(c_\tau^1)$ to $G(r, n)$ is a representation on the direct sum $V \oplus V'$ of two copies of $V \stackrel{\text{def}}{=} M(c_\tau^1)$. So a basis of $V \oplus V'$ consists of all the elements C_v, C'_v as v varies in c_τ^1 . If $x = [I^{r-1}, 2^0, \dots, n^0]$ is taken as a representative of the nontrivial coset of $G(r, 2, n)$ and we impose that $x \cdot C_v = C'_v$, the representation ψ of $G(r, n)$ on $V \oplus V'$ will be

as follows:

$$g \cdot C_v = \begin{cases} \zeta_r^{\langle g, \tilde{v} \rangle} \zeta_r^{z_1(\tilde{v}) - z_{|g|^{-1}(1)}(\tilde{v})} C_{|g|v|g|^{-1}} & \text{if } g \in G(r, 2, n), \\ \zeta_r^{\langle g, \tilde{v} \rangle} \zeta_r^{z_1(\tilde{v})} C'_{|g|v|g|^{-1}} & \text{if } g \notin G(r, 2, n), \end{cases}$$

and

$$g \cdot C'_v = \begin{cases} \zeta_r^{\langle g, \tilde{v} \rangle} C'_{|g|v|g|^{-1}} & \text{if } g \in G(r, 2, n), \\ \zeta_r^{\langle g, \tilde{v} \rangle} \zeta_r^{-z_{|g|^{-1}(1)}(\tilde{v})} C_{|g|v|g|^{-1}} & \text{if } g \notin G(r, 2, n); \end{cases}$$

where \tilde{v} is any lift of v in $G(r, n)$. Now we want to show that this representation ψ of $G(r, n)$ is actually also induced from a particular representation of $G(r, (\tau, \tau)) = G(r, t_0) \times \dots \times G(r, t_{r'-1}) \times G(r, t_0) \times \dots \times G(r, t_{r'-1})$. With this in mind, we let \mathcal{C} be the set of elements $v \in c_t$ having a lift \tilde{v} in $G(r, n)$ satisfying the following condition: if $(a^i, b^{i+r'})$ is a cycle of \tilde{v} of type i , then $a \in [t_0 + \dots + t_{i-1} + 1, t_0 + \dots + t_i]$ and $b \in [n' + t_0 + \dots + t_{i-1} + 1, n' + t_0 + \dots + t_i]$. Then, if $z \stackrel{\text{def}}{=} \min\{j : t_j \neq 0\}$, we let

$$W \stackrel{\text{def}}{=} \bigoplus_{v \in \mathcal{C}} \mathbb{C}(C_v + \zeta_r^z C'_v) \subseteq V \oplus V'.$$

Lemma 8.2 *The subspace W is invariant under the restriction of ψ to $G(r, (\tau, \tau))$.*

Proof It is clear that if $v \in \mathcal{C}$ and $g \in G(r, (\tau, \tau))$, then $|g|v|g|^{-1} \in \mathcal{C}$. We observe that, by definition, $|g|$ permutes the elements in \tilde{v} having the same color, and in particular $z_1(\tilde{v}) = z_{|g|^{-1}(1)}(\tilde{v})$. Moreover, by definition, we also have $z_1(\tilde{v}) = z$. In particular, if $g \in G(r, 2, n) \cap G(r, (\tau, \tau))$, we have

$$\begin{aligned} g \cdot (C_v + \zeta_r^z C'_v) &= \zeta_r^{\langle g, \tilde{v} \rangle} \zeta_r^{z_1(\tilde{v}) - z_{|g|^{-1}(1)}(\tilde{v})} C_{|g|v|g|^{-1}} + \zeta_r^z \zeta_r^{\langle g, \tilde{v} \rangle} C'_{|g|v|g|^{-1}} \\ &= \zeta_r^{\langle g, \tilde{v} \rangle} \left(C_{|g|v|g|^{-1}} + \zeta_r^z C'_{|g|v|g|^{-1}} \right), \end{aligned}$$

and if $g \in G(r, (\tau, \tau))$, but $g \notin G(r, 2, n)$, we have

$$\begin{aligned} g \cdot (C_v + \zeta_r^z C'_v) &= \zeta_r^{\langle g, \tilde{v} \rangle} \zeta_r^{z_1(\tilde{v})} C'_{|g|v|g|^{-1}} + \zeta_r^z \zeta_r^{\langle g, \tilde{v} \rangle} \zeta_r^{-z_{|g|^{-1}(1)}(\tilde{v})} C_{|g|v|g|^{-1}} \\ &= \zeta_r^{\langle g, \tilde{v} \rangle} \left(\zeta_r^z C'_{|g|v|g|^{-1}} + C_{|g|v|g|^{-1}} \right). \end{aligned}$$

The proof is now complete. □

Lemma 8.3 *We have $V \oplus V' = \text{Ind}_{G(r, (\tau, \tau))}^{G(r, n)}(W)$.*

Proof For this, we need to prove that

$$V \oplus V' = \bigoplus_{g \in K} g \cdot W, \tag{17}$$

where K is any set of coset representatives of $G(r, (\tau, \tau))$ in $G(r, n)$. But

$$[G(r, n) : G(r, (\tau, \tau))] = \frac{n!r^n}{(\tau!)^2r^n} = \frac{n!}{(\tau!)^2},$$

where $\tau! = t_0! \cdots t_{r'-1}!$. Moreover, $\dim(V \oplus V') = 2 \dim V = 2 \frac{\binom{n}{2} \binom{n-2}{2} \cdots \binom{2}{2} 2^{n'}}{\tau!2} = \frac{n!}{\tau!}$ and $\dim W = \tau!$, so that $[G(r, n) : G(r, (\tau, \tau))] = \frac{\dim(V \oplus V')}{\dim W}$, and hence to prove (17) it is enough to show that $V \oplus V' \subset G(r, n)W$. To show this, we take $\sigma = [n' + 1, n' + 2, \dots, n, 1, 2, \dots, n'] \in S_n$. Then it follows that conjugation by σ stabilizes \mathcal{C} , although $\sigma \notin G(r, (\tau, \tau))$. Then we have

$$\begin{aligned} \sigma \cdot (\zeta_r^z C_v + C'_v) &= \zeta_r^z C_{\sigma v \sigma^{-1}} + \zeta_r^{z-z_{n'+1}(\tilde{v})} C'_{\sigma v \sigma^{-1}} \\ &= \zeta_r^z C_{\sigma v \sigma^{-1}} + \zeta_r^{r'(\tilde{v})} C'_{\sigma v \sigma^{-1}} \\ &= \zeta_r^z C_{\sigma v \sigma^{-1}} - C'_{\sigma v \sigma^{-1}}. \end{aligned}$$

Since also $\zeta_r^z C_{\sigma v \sigma^{-1}} + C'_{\sigma v \sigma^{-1}} \in W$, we conclude that both $C_{\sigma v \sigma^{-1}}$ and $C'_{\sigma v \sigma^{-1}}$ belong to $G(r, n)W$ for all $v \in \mathcal{C}$, and the proof is complete. \square

We are now ready to prove the main result of this section.

Theorem 8.4 *Let $\text{GCD}(p, n) = 2$ and $[\tau] = [t_0, \dots, t_{r'}] \in AC(r, p, n)^*$. Then*

$$M(c_{[\tau]}^1) = \bigoplus_{\substack{[\lambda^{(0)}, \dots, \lambda^{(r'-1)}] \in \text{Fer}(r', 1, p', n') : \\ |\lambda^{(i)}| = t_i \forall i \in [0, r'-1]}} \rho_{[\lambda^{(0)}, \dots, \lambda^{(r'-1)}, \lambda^{(0)}, \dots, \lambda^{(r'-1)}]}^1.$$

Proof If $p = 2$, we need to look closer at the $G(r, (\tau, \tau))$ -representation W . From the proof of Lemma 8.2 we have that $gD_v = \zeta_r^{(g,v)} D_{|g|v|g|^{-1}} = \prod_i \zeta_r^{iz(g_i)} D_{|g|v|g|^{-1}}$, where $D_v \stackrel{\text{def}}{=} C_v + \zeta_r^z C'_v$ for $v \in \mathcal{C}$, are the basis elements of W . In particular, condition (2) of Proposition 8.1 is satisfied, and the result is a straightforward consequence of Theorem 7.1.

If $p > 2$, we simply have to observe that the $G(r, p, n)$ -module $M(c_{[\tau]}^1)$ is a quotient of the restriction to $G(r, p, n)$ of the $G(r, 2, n)$ -module $M(c_{[\tau]}^1)$. Since $\text{GCD}(p, n) = 2$, the irreducible representations of $G(r, 2, n)$ restricted to $G(r, p, n)$ remain irreducible (and are indexed in the “same” way). The result is then a consequence of the case $p = 2$ and Theorem 7.1. \square

9 The symmetric classes

In this section we complete our discussion with the description of the $G(r, p, n)$ -module $M(c)$, where c is any S_n -conjugacy class of symmetric absolute involutions in $G(r, p, n)^*$. Despite the case $p = 1$ considered in [6] and the case of antisymmetric

classes treated in Sect. 8, where a self-contained proof of the irreducible decomposition of the module $M(c)$ was given, we will need here to make use of all the main results that we have obtained so far, namely the construction of the complete Gelfand model in [5], the study of the submodules $M(c)$ for wreath products in [6], and the discussion of the antisymmetric submodule in Sect. 7.

We first observe that, by Theorems 3.3 and 7.1, the symmetric submodule has the following decomposition into irreducible representations:

$$M_{\text{Sym}} \cong \bigoplus_{[\lambda] \in \text{Fer}(r, 1, p, n)} \rho_{[\lambda]}^0.$$

If v is a symmetric absolute involution in $G(r, p, n)^*$, we denote by $\text{Sh}(v)$ the element of $\text{Fer}(r, 1, p, n)$ which is the shape of the multitableaux of the image of v , via the projective Robinson–Schensted correspondence. Namely, we let

$$\text{Sh}(v) \stackrel{\text{def}}{=} [\lambda] \in \text{Fer}(r, 1, p, n),$$

where

$$v \xrightarrow{RS} (P, P)$$

with $P \in \mathcal{ST}_{[\lambda]}$. For notational convenience, if c is an S_n -conjugacy class of symmetric absolute involutions in $G(r, p, n)^*$, we also let $\text{Sh}(c) = \cup_{v \in c} \text{Sh}(v) \subset \text{Fer}(r, 1, p, n)$.

We are now ready to state the main result of this section.

Theorem 9.1 *Let c be a S_n -conjugacy class of symmetric absolute involutions in $G(r, p, n)^*$, and $\text{GCD}(p, n) = 1, 2$. Then the following decomposition holds:*

$$M(c) \cong \bigoplus_{[\lambda] \in \text{Sh}(c)} \rho_{[\lambda]}^0.$$

Before proving this theorem, we need some further preliminary observations. Fix an arbitrary S_n -conjugacy class c of symmetric absolute involutions in $G(r, p, n)^*$, and let c_1, \dots, c_s be the S_n -conjugacy classes of $G(r, n)$ which are lifts of c in $G(r, n)$ (one may observe that s can be either p or $p/2$, though this is not needed). We will need to consider the following restriction to $G(r, p, n)$ of the submodule of the Gelfand model for $G(r, n)$ associated to the classes c_1, \dots, c_s :

$$\tilde{M}(c) \stackrel{\text{def}}{=} (M(c_1) \oplus \dots \oplus M(c_s)) \downarrow_{G(r, p, n)}.$$

Now the crucial observation is the following.

Lemma 9.2 *The $G(r, p, n)$ -module $M(c)$ is a quotient (and hence is isomorphic to a subrepresentation) of $\tilde{M}(c)$.*

Proof Let $K(c)$ be the vector subspace of $\tilde{M}(c)$ spanned by the elements $C_v - C_{\zeta^{r/p}v}$ as v varies among all elements in c_1, \dots, c_s . Then it is clear that, as a vector

space, $M(c)$ is the quotient of $\tilde{M}(c)$ by the vector subspace $K(c)$. Moreover, if $g \in G(r, p, n)$, then

$$\begin{aligned} \varrho(g)(C_v - C_{\zeta^{r/p}v}) &= \zeta_r^{\langle g, v \rangle} (-1)^{\text{inv}_v(g)} C_{|g|v|g|^{-1}} \\ &\quad - \zeta_r^{\langle g, \zeta^{r/p}v \rangle} (-1)^{\text{inv}_{\zeta^{r/p}v}(g)} C_{|g|\zeta^{r/p}v|g|^{-1}} \\ &= \zeta_r^{\langle g, v \rangle} (-1)^{\text{inv}_v(g)} \left(C_{|g|v|g|^{-1}} - C_{\zeta^{r/p}|g|v|g|^{-1}} \right), \end{aligned}$$

since $g \in G(r, p, n)$ implies $\langle g, \zeta^{r/p}v \rangle = \langle g, v \rangle$. In particular, we deduce that $K(c)$ is also a submodule of $\tilde{M}(c)$ (as $G(r, p, n)$ -modules). The fact that $M(c) \cong \tilde{M}(c)/K(c)$ is now a direct consequence of the definition of the structures of $G(r, p, n)$ -modules. \square

We are now ready to complete the proof of the main result of this section.

Proof of Theorem 9.1 By Theorem 3.4 the $G(r, n)$ -module $M(c_1) \oplus \dots \oplus M(c_s)$ is the sum of all representations ρ_λ with $\lambda \in \text{Sh}(c_i)$ for some $i \in [s]$ or, equivalently, with $[\lambda] \in \text{Sh}(c)$. It follows that the restriction $\tilde{M}(c)$ of this representation to $G(r, p, n)$ has the following decomposition:

$$\tilde{M}(c) \cong \bigoplus_{\substack{[\lambda] \in \text{Sh}(c): \\ m_p(\lambda)=1}} (\rho_{[\lambda]}^0)^{\oplus p} \oplus \bigoplus_{\substack{[\lambda] \in \text{Sh}(c): \\ m_p(\lambda)=2}} (\rho_{[\lambda]}^0 \oplus \rho_{[\lambda]}^1)^{\oplus p/2}. \tag{18}$$

We recall that $M(c)$ is a submodule of a Gelfand model for $G(r, p, n)$ and also a submodule of $\tilde{M}(c)$ by Lemma 9.2, and hence, by (18), we have that $M(c)$ is isomorphic to a subrepresentation of

$$\bigoplus_{\substack{[\lambda] \in \text{Sh}(c): \\ m_p(\lambda)=1}} \rho_{[\lambda]}^0 \oplus \bigoplus_{\substack{[\lambda] \in \text{Sh}(c): \\ m_p(\lambda)=2}} (\rho_{[\lambda]}^0 \oplus \rho_{[\lambda]}^1).$$

Furthermore, we already know that the split representations $\rho_{[\lambda]}^1$ appear in the antisymmetric submodule by Theorem 7.1, and so they cannot appear in $M(c)$. To complete the proof, it is now sufficient to observe that, if c and c' are two distinct S_n -conjugacy classes of symmetric absolute involutions in $G(r, p, n)^*$, then the two sets $\text{Sh}(c)$ and $\text{Sh}(c')$ are disjoint. \square

We can also give an explicit combinatorial description of the set $\text{Sh}(c)$ for a given S_n -conjugacy class of symmetric absolute involutions in $G(r, p, n)^*$. For this, we let $SC(r, n) = \{(f_0, \dots, f_{r-1}, q_0, \dots, q_{r-1}) \in \mathbb{N}^{2r} : f_0 + \dots + f_{r-1} + 2(q_0 + \dots + q_{r-1}) = n\}$. The set $SC(r, n)$ has already been used in [6, Sect. 6] to parameterize the S_n -conjugacy classes of absolute involutions in $G(r, n)$. Let γ be the permutation of $SC(r, n)$ defined by

$$\begin{aligned} \gamma(f_0, \dots, f_{r-1}, q_0, \dots, q_{r-1}) \\ = (f_{r/p}, f_{1+r/p} \dots, f_{r-1+r/p}, q_{r/p}, q_{1+r/p} \dots, q_{r-1+r/p}), \end{aligned}$$

where the indices are treated as elements in \mathbb{Z}_r . We observe that γ has order p , and so we have an action of the cyclic group C_p generated by γ on $SC(r, n)$. We denote the quotient set by $SC(r, p, n)^*$. The set $SC(r, p, n)^*$ parameterizes the S_n -conjugacy classes of symmetric absolute involutions in $G(r, p, n)^*$ in the following way. Let $v \in I(r, p, n)^*$ be symmetric, and \tilde{v} be any lift of v in $I(r, n)$. Then the *type* of v is given by

$$[f_0(\tilde{v}), \dots, f_{r-1}(\tilde{v}), q_0(\tilde{v}), \dots, q_{r-1}(\tilde{v})] \in SC(r, p, n)^*,$$

where

$$f_i(\tilde{v}) = |\{j \in [n] : \tilde{v}_j = j^i\}|,$$

$$q_i(\tilde{v}) = |\{(h, k) : 1 \leq h < k \leq n, \tilde{v}_h = k^i \text{ and } \tilde{v}_k = h^i\}|.$$

It is clear that this is well defined, and we have that two symmetric elements in $I(r, p, n)^*$ are S_n -conjugate if and only if they have the same type (see also [6, Sect. 6] for the special case $p = 1$).

By [6, Proposition 3.1] we can now conclude that if $[v] = [f_0, \dots, f_{r-1}, q_0, \dots, q_{r-1}] \in SC(r, p, n)^*$ and $c = \{v \in I(r, p, n)^* : v \text{ is symmetric of type } [v]\}$, then

$$\text{Sh}(c) = \left\{ \begin{array}{l} [\lambda^{(0)}, \dots, \lambda^{(r-1)}] \in \text{Fer}(r, p, n)^* : \text{for all } i \in [0, r-1], \\ |\lambda_i| = f_i + 2q_i, \text{ and } \lambda^{(i)} \text{ has exactly } f_i \text{ columns of odd length} \end{array} \right\}.$$

For example, if we consider $v = [1^0, 3^1, 2^1, 4^1, 5^1, 7^2, 6^2, 8^3, 10^4, 9^4, 11^4, 12^4, 14^5, 13^5,] \in G(6, 6, 14)^*$, then the type of v is $[v] = [1, 2, 0, 1, 2, 0; 0, 1, 1, 0, 1, 1]$. Therefore, if c is the S_n -conjugacy class of v in $G(6, 6, 14)^*$, we have that $\text{Sh}(c)$ is given by all elements $[\lambda^{(0)}, \dots, \lambda^{(5)}] \in \text{Fer}(6, 1, 6, 14)$ such that $\lambda^{(0)}$ and $\lambda^{(3)}$ have 1 box (and 1 column of odd length), $\lambda^{(1)}$ and $\lambda^{(4)}$ have 4 boxes and 2 columns of odd length, and $\lambda^{(2)}$ and $\lambda^{(5)}$ have 2 boxes and no columns of odd length, i.e.,

$$\text{Sh}(c) = \left\{ \left[\begin{array}{cccccc} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array} \right], \left[\begin{array}{cccccc} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array} \right], \left[\begin{array}{cccccc} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array} \right] \right\}$$

Therefore we have the following decomposition of $M(c)$ into irreducible representations:

$$M(c) \cong \rho \left[\begin{array}{cccccc} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array} \right] \oplus \rho^0 \left[\begin{array}{cccccc} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array} \right] \oplus \rho^0 \left[\begin{array}{cccccc} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array} \right]$$

10 A final survey and a further generalization

The aim of this section is to provide a statement containing all the results that we have collected in this paper and holding for all the groups $G(r, p, n)$ with $\text{GCD}(p, n) = 1, 2$. Furthermore, we use this occasion to observe that the above results apply, in fact, to all the projective groups $G(r, p, q, n)$ with $\text{GCD}(p, n) = 1, 2$.

If v is an absolute involution in $G(r, q, p, n)$ and c is any (symmetric or anti-symmetric) S_n -conjugacy class of absolute involutions in $G(r, q, p, n)$, we define $\text{Sh}(v) \in \text{Fer}(r, q, p, n)$ and $\text{Sh}(c) \subset \text{Fer}(r, q, p, n)$ as in Sect. 9. Moreover, we let $\iota(c) = 0$ if the elements of c are symmetric and $\iota(c) = 1$ if the elements of c are antisymmetric.

Theorem 10.1 *Let $G = G(r, p, q, n)$ with $\text{GCD}(p, n) = 1, 2$, and let $(M(r, q, p, n), \varrho)$ be its Gelfand model defined in Theorem 3.3. Given an S_n -conjugacy class c of absolute involutions in G^* , let $M(c) = \mathbb{C} - \text{span}\{C_v : v \in c\}$ so that $M(r, q, p, n)$ naturally splits as a G -module into the direct sum*

$$M(r, q, p, n) = \bigoplus_c M(c).$$

Then the submodule $M(c)$ has the following decomposition into irreducibles:

$$M(c) \cong \bigoplus_{[\lambda] \in \text{Sh}(c)} \rho_{[\lambda]}^{\iota(c)}.$$

Proof We have already established this result for $q = 1$. In fact, if $\iota(c) = 0$, this is the content of Theorem 9.1, and, if $\iota(c) = 1$, the result follows directly from Theorem 8.4 with the further observation that if v is an antisymmetric element of type $[t_0, \dots, t_{r'-1}]$, then $\text{Sh}(v) = [\lambda^{(0)}, \dots, \lambda^{(r-1)}]$ with $|\lambda^{(i)}| = |\lambda^{(i+r')}| = t_i$.

If $q \neq 1$, the result is straightforward since an S_n -conjugacy class of absolute involutions in $G(r, q, p, n)$ is also an S_n -conjugacy class of absolute involutions in $G(r, 1, p, n)$ and the definitions of the Gelfand models for $G(r, p, q, n)$ and $G(r, p, 1, n)$ are compatible with the projection $G(r, p, 1, n) \rightarrow G(r, p, q, n)$. \square

Acknowledgement The first author would like to thank Christian Stump and Itaru Terada for useful conversations during the preparation of this work.

References

1. Adin, R., Postnikov, A., Roichman, Y.: A Gelfand model for wreath products. *Isr. J. Math.* **179**, 381–402 (2010)
2. Baddeley, R.W.: Models and involution models for wreath products and certain Weyl groups. *J. Lond. Math. Soc.* **44**, 55–74 (1991)
3. Bernstein, I.N., Gelfand, I.M., Gelfand, S.I.: Models of representations of compact Lie groups. *Funk. Anal. Prilozh.* **9**, 61–62 (1975) (in Russian)
4. Caselli, F.: Projective reflection groups. *Isr. J. Math.* **185**, 155–187 (2011)
5. Caselli, F.: Involution reflection groups and their models. *J. Algebra* **24**, 370–393 (2010)
6. Caselli, F., Fulci, R.: Refined Gelfand models for wreath products. *Eur. J. Comb.* **32**, 198–216 (2011)
7. Inglis, N.F.J., Richardson, R.W., Saxl, J.: An explicit model for the complex representations of S_n . *Arch. Math.* **54**, 258–259 (1990)
8. Klyachko, A.A.: Models for complex representations of groups $\text{GL}(n, q)$. *Mat. Sb.* **120**, 371–386 (1983) (in Russian)
9. Marberg, E.: Automorphisms and generalized involution models of finite complex reflection groups. *J. Algebra* **334**, 295–320 (2011)
10. Marberg, E.: Generalized involution models for wreath products, *Isr. J. Math.* (2011, in press). [arXiv:1007.5078](https://arxiv.org/abs/1007.5078)

11. Pfeiffer, G.: Character tables of Weyl Groups in GAP. Bayreuth. Math. Schr. **47**, 165–222 (1994)
12. Shephard, G.C., Todd, J.A.: Finite unitary reflection groups. Can. J. Math. **6**, 274–304 (1954)
13. Stembridge, J.R.: On the eigenvalues of representations of reflection groups and wreath products. Pac. J. Math. **140**(2), 353–396 (1989)
14. Stanley, R.P.: Enumerative Combinatorics, vol. 2. Cambridge Studies in Advanced Mathematics, vol. 62. Cambridge University Press, Cambridge (1999)
15. Stanton, D.W., White, D.E.: A Schensted algorithm for rim hook tableaux. J. Comb. Theory, Ser. A **40**, 211–247 (1985)