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Two distance-regular graphs

Andries E. Brouwer · Dmitrii V. Pasechnik

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Abstract We construct two families of distance-regular graphs, namely the subgraph of the dual polar graph of type $B_3(q)$ induced on the vertices far from a fixed point, and the subgraph of the dual polar graph of type $D_4(q)$ induced on the vertices far from a fixed edge. The latter is the extended bipartite double of the former.

Keywords Distance-regular graph · Dual polar graph · Extended bipartite double

1 The extended bipartite double

We shall use $\sim$ to indicate adjacency in a graph. For notation and definitions of concepts related to distance-regular graphs, see [3]. We repeat the definition of extended bipartite double.

The *bipartite double* of a graph Γ with vertex set X is the graph with vertex set $\{x^+, x^- \mid x \in X\}$ and adjacencies $x^\delta \sim y^\epsilon$ iff $\delta\epsilon = -1$ and $x \sim y$. The bipartite double of a graph Γ is bipartite, and it is connected iff Γ is connected and not bipartite. If Γ has spectrum Φ, then its bipartite double has spectrum $(-\Phi) \cup \Phi$. See also [3], Theorem 1.11.1.

The *extended bipartite double* of a graph Γ with vertex set X is the graph with vertex set $\{x^+, x^- \mid x \in X\}$, and the same adjacencies as the bipartite double, except that also $x^- \sim x^+$ for all $x \in X$. The extended bipartite double of a graph Γ is bipartite, and it is connected iff Γ is connected. If Γ has spectrum Φ, then its extended bipartite double has spectrum $(-\Phi - 1) \cup (\Phi + 1)$. See also [3], Theorem 1.11.2.

A.E. Brouwer (✉)
Dept. of Math., Techn. Univ. Eindhoven, P.O. Box 513, 5600MB Eindhoven, Netherlands
e-mail: aeb@cwi.nl

D.V. Pasechnik
School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore 637371, Singapore
e-mail: dima@ntu.edu.sg

2 Far from an edge in the dual polar graph of type $D_4(q)$

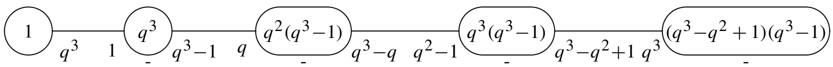
Let V be a vector space of dimension 8 over a field F, provided with a nondegenerate quadratic form of maximal Witt index. The maximal totally isotropic subspaces of V (of dimension 4) fall into two families $\mathcal{F}_1$ and $\mathcal{F}_2$, where the dimension of the intersection of two elements of the same family is even (4 or 2 or 0) and the dimension of the intersection of two elements of different families is odd (3 or 1).

The geometry of the totally isotropic subspaces of V, where $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ are incident when $\dim A \cap B = 3$ and otherwise incidence is symmetrized inclusion, is known as the geometry $D_4(F)$. The bipartite incidence graph on the maximal totally isotropic subspaces is known as the dual polar graph of type $D_4(F)$.

Below we take $F = \mathbf{F}_q$, the finite field with q elements, so that graph and geometry are finite. We shall use projective terminology, so that 1-spaces, 2-spaces and 3-spaces are called points, lines and planes. Two subspaces are called disjoint when they have no point in common, i.e., when the intersection has dimension 0.

Proposition 2.1 *Let Γ be the dual polar graph of type $D_4(\mathbf{F}_q)$. Fix elements $A_0 \in \mathcal{F}_1$ and $B_0 \in \mathcal{F}_2$ with $A_0 \sim B_0$. Let Δ be the subgraph of Γ induced on the set of vertices disjoint from A_0 or B_0. Then Δ is distance-regular with intersection array $\{q^3, q^3-1, q^3-q, q^3-q^2+1;\ 1, q, q^2-1, q^3\}$.*

The distance distribution diagram is



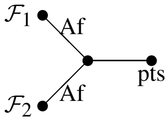
Proof There are q^6 elements $A \in \mathcal{F}_1$ disjoint from A_0 and the same number of $B \in \mathcal{F}_2$ disjoint from B_0, so that Δ has $2q^6$ vertices.

Given $A \in \mathcal{F}_1$, there are q^3+q^2+q+1 elements $B \in \mathcal{F}_2$ incident to it. Of these, q^2+q+1 contain the point $A \cap B_0$ and hence are not vertices of Δ. So, Δ has valency q^3.

Two vertices $A, A' \in \mathcal{F}_1$ have distance 2 in Δ if and only if they meet in a line, and the line $L = A \cap A'$ is disjoint from B_0. If this is the case, then L is in $q+1$ elements $B \in \mathcal{F}_2$, one of which meets B_0, so that A and A' have $c_2 = q$ common neighbours in Δ.

Given vertices $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ that are nonadjacent, i.e., that meet in a single point P, the neighbours A' of B at distance 2 to A in Δ correspond to the lines L on P in A disjoint from B_0 and nonorthogonal to the point $A_0 \cap B$. There are q^2+q+1 lines L on P in A, $q+1$ of which are orthogonal to the point $A_0 \cap B$, and one further of which meets B_0. (Note that the points $A_0 \cap B$ and $A \cap B_0$ are nonorthogonal since neither point is in the plane $A_0 \cap B_0$ and V does not contain totally isotropic 5-spaces.) It follows that $c_3 = q^2-1$, and also that Δ has diameter 4, and is distance-regular. □

The geometry induced by the incidence relation of $D_4(F)$ on the vertices of Δ, together with the points and lines contained in the planes disjoint from $A_0 \cup B_0$, has Buekenhout–Tits diagram (cf. [4])



that is, the residue of an object $A \in \mathcal{F}_1$ is an affine 3-space, where the objects incident to A in $\mathcal{F}_2$ play the rôle of points. Similar things hold more generally for $D_n(F)$ with arbitrary n, and even more generally for all diagrams of spherical type. See also [1], Theorem 6.1.

Let P be a nonsingular point, and let ϕ be the reflection in the hyperplane $H = P^\perp$. Then ϕ is an element of order two of the orthogonal group that fixes H pointwise, and consequently interchanges $\mathcal{F}_1$ and $\mathcal{F}_2$. For each $A \in \mathcal{F}_1$ we have $\phi(A) \sim A$. The quotient Γ/ϕ is the dual polar graph of type $B_3(q)$, and we see that more generally the dual polar graph of type $D_{m+1}(q)$ is the extended bipartite double of the dual polar graph of type $B_m(q)$. The quotient Δ/ϕ is a new distance-regular graph discussed in the next section. It is the subgraph consisting of the vertices at maximal distance from a given point in the dual polar graph of type $B_3(q)$. For even q we have $B_3(q) = C_3(q)$, and it follows that the symmetric bilinear forms graph on $\mathbf{F}_q^3$ is distance-regular, see [3] Proposition 9.5.10 and the diagram there on p. 286.

3 Far from a point in the dual polar graph of type $B_3(q)$

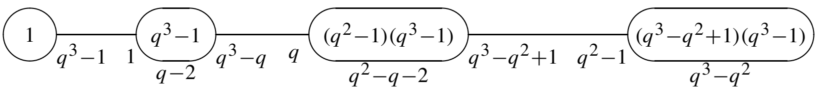
First a very explicit version of the graph of this section.

Proposition 3.1 (i) *Let W be a vector space of dimension 3 over the field $\mathbf{F}_q$, provided with an outer product $\times$. Let Z be the graph with vertex set $W \times W$ where $(u,u') \sim (v,v')$ if and only if $(u,u') \neq (v,v')$ and $u \times v + u' - v' = 0$. Then Z is distance-regular of diameter 3 on q^6 vertices. It has intersection array $\{q^3-1, q^3-q, q^3-q^2+1; 1, q, q^2-1\}$ and eigenvalues $q^3-1, q^2-1, -1, -q^2-1$ with multiplicities 1, $\frac{1}{2}q(q+1)(q^3-1)$, $(q^3-q^2+1)(q^3-1)$, $\frac{1}{2}q(q-1)(q^3-1)$, respectively.*

(ii) *The extended bipartite double $\hat{Z}$ of Z is distance-regular with intersection array $\{q^3, q^3-1, q^3-q, q^3-q^2+1;\ 1, q, q^2-1, q^3\}$ and eigenvalues $\pm q^3$, $\pm q^2$, 0 with multiplicities 1, $q^2(q^3-1)$, $2(q^3-q^2+1)(q^3-1)$, respectively.*

(iii) *The distance-1-or-2 graph $Z_1 \cup Z_2$ of Z, which is the halved graph of $\hat{Z}$, is strongly regular with parameters $(v,k,\lambda,\mu) = (q^6, q^2(q^3-1), q^2(q^2+q-3), q^2(q^2-1))$.*

The distance distribution diagram of Z is



Proof Note that the adjacency relation is symmetric, so that Z is an undirected graph. The computation of the parameters is completely straightforward. Clearly, Z has q^6 vertices. For $a, b \in W$ the maps $(u,u') \mapsto (u+a, u'+(a\times u)+b)$ are automorphisms of Z, so $\mathrm{Aut}(Z)$ is vertex-transitive.

The q^3-1 neighbours of $(0,0)$ are the vertices $(v,0)$ with $v \neq 0$. The common neighbours of $(0,0)$ and $(v,0)$ are the vertices $(cv,0)$ for $c \in \mathbf{F}_q$, $c \neq 0,1$. Hence $a_1 = q-2$.

The $(q^3-1)(q^2-1)$ vertices at distance 2 from $(0,0)$ are the vertices (u,u') with $u,u' \neq 0$ and $u' \perp u$.[1] The common neighbours of $(0,0)$ and (u,u') are the $(v,0)$ with $v \times u = u'$, and together with $(v,0)$ also $(v+cu,0)$ is a common neighbour, so $c_2 = q$. Vertices (u,u') and (v,v'), both at distance 2 from $(0,0)$, are adjacent when $0 \neq v \perp u'$ and $v \neq u$ and $v \times u \neq u'$ and $v' = u \times v + u'$, so that $a_2 = q^2-q-2$.

The remaining $(q^3-1)(q^3-q^2+1)$ vertices have distance 3 to $(0,0)$. They are the (w,w') with $w \not\perp w'$ or $w = 0 \neq w'$. The neighbours (u,u') of (w,w') that lie at distance 2 to $(0,0)$ satisfy $0 \neq u \perp w'$ and $(0 \neq) u' = w \times u + w'$, so that $c_3 = q^2-1$. This shows that Z is distance-regular with the claimed parameters. The spectrum follows.

The fact that the extended bipartite double is distance-regular, and has the stated intersection array, follows from [3], Theorem 1.11.2(vi).

The fact that Z_3 is strongly regular follows from [3], Proposition 4.2.17(ii) (which says that this happens when Z has eigenvalue -1). □

For $q=2$, the graphs here are (i) the folded 7-cube, (ii) the folded 8-cube, (iii) the halved folded 8-cube. All are distance-transitive. For $q>2$ these graphs are not distance-transitive.

When q is a power of two, the graphs $\hat{Z}$ have the same parameters as certain Kasami graphs, but for $q>2$ these are nonisomorphic.

Next, a more geometric description of this graph.

Let H be a vector space of dimension 7 over the field $\mathbf{F}_q$, provided with a nondegenerate quadratic form. Let Γ be the graph of which the vertices are the maximal totally isotropic subspaces of H (of dimension 3), where two vertices are adjacent when their intersection has dimension 2. This graph is known as the dual polar graph of type $B_3(q)$. It is distance-regular with intersection array $\{q(q^2+q+1), q^2(q+1), q^3; 1, q+1, q^2+q+1\}$. (See [3], §9.4.)

Proposition 3.2 *Let Γ be the dual polar graph of type $B_3(q)$. Fix a vertex π_0 of Γ, and let Δ be the subgraph of Γ induced on the collection of vertices disjoint from π_0. Then Δ is isomorphic to the graph Z of Proposition 3.1. Its extended bipartite double $\hat{\Delta}$ (or $\hat{Z}$) is isomorphic to the graph of Proposition 2.1.*

Proof Let V be a vector space of dimension 8 over $\mathbf{F}_q$ (with basis $\{e_1,\dots,e_8\}$), provided with the nondegenerate quadratic form $Q(x) = x_1x_5 + x_2x_6 + x_3x_7 + x_4x_8$. The point $P = (0,0,0,1,0,0,0,-1)$ is nonisotropic, and $P^\perp$ is the hyperplane H defined by $x_4 = x_8$. Restricted to H the quadratic form becomes $Q(x) = x_1x_5 + x_2x_6 + x_3x_7 + x_4^2$.

The D_4-geometry on V has disjoint maximal totally isotropic subspaces $E = \langle e_1,e_2,e_3,e_4\rangle$ and $F = \langle e_5,e_6,e_7,e_8\rangle$. Fix E and consider the collection of all maximal totally isotropic subspaces disjoint from E. This is precisely the collection of

[1] With orthogonality relation compatible with $\times$, so that $u \perp (u \times v)$ for all u, v.

images F_A of F under matrices $\binom{I\ A}{0\ I}$, where A is alternating with zero diagonal (cf. [3], Proposition 9.5.1(i)). Hence, we can label the q^6 vertices $F_A \cap H$ of Δ with the q^6 matrices A.

Two vertices are adjacent when they have a line in common, that is, when they are the intersections with H of maximal totally isotropic subspaces in V, disjoint from E, that meet in a line contained in H. Let

$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}.$$

Then $\det A = (af - be + cd)^2$, and if $\det A = 0$ but $A \neq 0$, then $\ker A$ has dimension 2, and is spanned by the four vectors $(0, f, -e, d)^\top$, $(-f, 0, c, -b)^\top$, $(e, -c, 0, a)^\top$, $(-d, b, -a, 0)^\top$. Writing the condition that matrices A and A' belong to adjacent vertices we find the description of Proposition 3.1 if we take $u = (c, e, f)$ and $u' = (-d, b, -a)$. □

4 History

In 1991 the second author constructed the graphs from Sect. 2 and the first author those from Sect. 3. Both were mentioned on the web page [2], but not published thus far. These graphs have been called the Pasechnik graphs and the Brouwer–Pasechnik graphs, respectively, by on-line servers.

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