Foulkes characters, Eulerian idempotents, and an amazing matrix

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Abstract John Holte (Am. Math. Mon. 104:138–149, 1997) introduced a family of "amazing matrices" which give the transition probabilities of "carries" when adding a list of numbers. It was subsequently shown that these same matrices arise in the combinatorics of the Veronese embedding of commutative algebra (Brenti and Welker, Adv. Appl. Math. 42:545–556, 2009; Diaconis and Fulman, Am. Math. Mon. 116:788–803, 2009; Adv. Appl. Math. 43:176–196, 2009) and in the analysis of riffle shuffling (Diaconis and Fulman, Am. Math. Mon. 116:788–803, 2009; Adv. Appl. Math. 43:176–196, 2009). We find that the left eigenvectors of these matrices form the *Foulkes character table* of the symmetric group and the right eigenvectors are the *Eulerian idempotents* introduced by Loday (Cyclic Homology, 1992) in work on Hochschild homology. The connections give new closed formulae for Foulkes characters and allow explicit computation of natural correlation functions in the original carries problem.

1 Introduction

When n numbers are added in the usual way, "carries" accrue along the way. For example, working base $b = 10$, the display shows the carries along the top when

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$n = 3$ ten-digit numbers are added:

$$
\begin{array}{r}
\textit{2 22111 12120} \\
78667\ 51918 \\
65921\ 47787 \\
\underline{88424\ 99859} \\
2\ 33013\ 99564
\end{array}
$$

Here the carries (reading right to left in italic print) are $\kappa_0 = 0$, $\kappa_1 = 2$, $\kappa_2 = 1$, $\kappa_3 = 2, \dots$. When n numbers are added, the carries can be $0, 1, 2, \dots, n-1$. If the digits are chosen uniformly at random in $\{0, 1, \dots, b-1\}$, it is easy to see that the carries form a Markov chain: the chance that the next carry is j given the past carries only depends on the last carry. Thus the distribution of carries is determined by the transition matrix

$$M(i, j) = \text{chance}\{\text{next carry is } j \mid \text{last carry is } i\}.$$

The carries process was studied by Holte [17] who showed

$$M(i, j) = \frac{1}{b^n} \sum_{l=0}^{j - \lfloor i/b \rfloor} (-1)^l \binom{n+1}{l} \binom{n-1-i+(j+1-l)b}{n}, \quad 0 \le i, j \le n-1. \tag{1.1}$$

For example, when $n = 3$, the matrix is

$$\frac{1}{6b^2} \begin{bmatrix} b^2 + 3b + 2 & 4b^2 - 4 & b^2 - 3b + 2 \\ b^2 - 1 & 4b^2 + 2 & b^2 - 1 \\ b^2 - 3b + 2 & 4b^2 - 4 & b^2 + 3b + 2 \end{bmatrix}.$$

Holte found the eigenvalues, eigenvectors, and many amazing properties of these matrices.

Work of [4, 7, 8] shows that the same matrix arises in the analysis of the Gilbert–Shannon–Reeds method of shuffling cards and in the Hilbert series of the Veronese embedding of projective varieties.

The main results of the present paper identify a different area where the matrix appears. The left eigenvectors of the matrix are the *Foulkes characters* of the symmetric group. The right eigenvectors are the *Eulerian idempotents* that occur in the study of free Lie algebras and Hochschild homology. We obtain new closed-form expressions for these characters.

Section 2 gives background on Foulkes characters and presents some new results for left eigenvectors. Section 3 does the same for the right eigenvectors and applies some of the new formulae to the original carries process, giving the variance and covariance of the number of carries. Section 4 gives another connection between representation theory of the symmetric group (the RSK correspondence) and carries.

In follow-up work to the current paper, Novelli and Thibon [23] prove many of our results using the theory of noncommutative symmetric functions.

2 Foulkes characters

This section introduces the Foulkes characters of the symmetric group and some of their properties (Sect. 2.1). It shows that the Foulkes characters are the left eigenvectors of the transition matrix M of (1.1) (Sect. 2.2). This connection is used to prove a branching rule (from S_n to S_{n-1}) and a closed-form formula for Foulkes characters (Sect. 2.3).

2.1 Background on Foulkes characters

Foulkes characters were discovered by Foulkes [11] as part of the study of the descent patterns in the permutation group. They are developed in [19] and [18] gives a readable textbook treatment. Gessel and Reutenauer [15] use them to enumerate permutations by descents and conjugacy classes; see [9] for a probabilistic interpretation of these results. Stanley [28] uses Foulkes characters to develop enumerative results for alternating permutations by cycle type.

Recall that a permutation $\sigma \in S_n$ has a descent at i if $\sigma(i+1) < \sigma(i)$. The set of places where descents occur is $D(\sigma) \subseteq [n-1]$. For example, if $\sigma = 4512376$, $D(\sigma) = \{2, 6\}$. If $U \subseteq [n-1]$ is any set, Foulkes suggested constructing a ribbon shape (also called a rim hook) $R(U)$ beginning with a single box and sequentially adding the next box below the last box if $i \in U$, and to the left of the last box if $i \notin U$, $1 \le i \le n-1$. Thus, if $U = \{2, 6\}$, boxes are built up as follows:
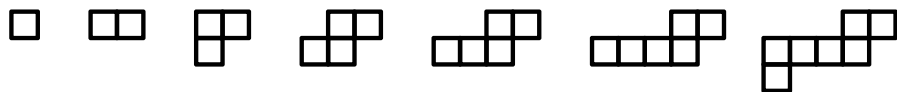
The final skew shape will have n boxes and be the lower rim of a partition α; in this example, α is 5,4,1, and the ribbon shape is 5,4,1\3:
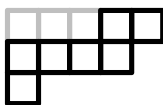
Labeling the boxes in the ribbon shape by all ways they can be sequentially removed from α and reading this from right to left and top to bottom gives all permutations with the original U as descent set. For example, removing boxes in the order shown as
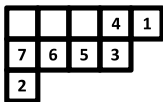
gives 1435672. The skew shape $R(U)$ corresponding to $U \subseteq [n-1]$ gives a skew character $\chi^{R(U)}$: if $R(U) = \alpha \backslash \beta$ and χ^λ is an irreducible character of the symmetric

Table 1 Foulkes character table for $n = 5$

	5	4	3	2	1
0	1	−1	1	−1	1
1	26	−10	2	2	−4
2	66	0	−6	0	6
3	26	10	2	−2	−4
4	1	1	1	1	1

group S_n, the coefficient of χ^λ in $\chi^{R(U)}$ is $\langle \chi^\beta \cdot \chi^\lambda | \chi^\alpha \rangle$ (see [22, Sect. 1.7]). From the development above, the dimension of $\chi^{R(U)}$ is the number of permutations with descent set U. Solomon [25, Sect. 6] describes a related construction of MacMahon in his work on Simon Newcomb's problem.

For fixed k, $0 \le k \le n - 1$, the Foulkes character $\chi^{n,k}$ is defined as the sum of $\chi^{R(U)}$ over all U with $n - k - 1$ descents. It follows that the dimension of $\chi^{n,k}$ is the *Eulerian number* $A(n, k)$, the number of permutations with k descents. Foulkes showed that $\chi^{n,k}(\sigma)$ only depends on σ through the number of cycles in σ. In particular,

$$\chi^{n,n-1} \text{ is the trivial character;} \tag{2.1}$$

$$\chi^{n,0} \text{ is the alternating character.} \tag{2.2}$$

Most importantly, letting $\chi_j^{n,k}$ denote the value of the Foulkes character on permutations with j cycles (so the dimension $\chi_n^{n,k} = A(n, k)$), one has from p. 306 of [18] that

$$\chi_j^{n,k} = \chi_j^{n-1,k-1} - \chi_j^{n-1,k} \quad \text{if } j < n, \ k > 0. \tag{2.3}$$

This, with the starting value $\chi_1^{1,0} = 1$, gives an efficient way to build a Foulkes character table. Let $k = 0, 1, \ldots, n - 1$ index the rows and $j = n, n - 1, \ldots, 1$ index the columns. Table 1 gives the example when $n = 5$.

Further properties of Foulkes characters appear in Kerber and Thürlings [19]:

$$\langle \chi^{n,k}, \chi^\lambda \rangle > 0 \quad \Longrightarrow \quad \lambda_1 \le k + 1, \ \lambda_1' \le n - k, \tag{2.4}$$

$$\langle \chi^{n,k}, \chi^{j+1,1^{n-j-1}} \rangle > 0 \quad \Leftrightarrow \quad j = k, \tag{2.5}$$

The $\chi^{n,k}$ are linearly independent, $\tag{2.6}$

If $\chi : S_n \to \mathbb{R}$ denotes a character, depending only on the number of cycles, then

$$\chi = \sum_i \frac{\langle \chi, \chi^{i+1,1^{n-i-1}} \rangle}{\chi^{i+1,1^{n-i-1}}(id)} \chi^{n,i}. \tag{2.7}$$

Thus the $\chi^{n,k}$ form a $\mathbb{Q}$ basis for the characters that only depends on the number of cycles. Hidden in the proof of (2.7): the hook character $\chi^{i+1,1^{n-i-1}}$ is the only hook occurring in $\chi^{n,i}$ and it occurs with multiplicity its degree $\binom{n-1}{i}$. A related fact

appears in Solomon [25, Theorem 4]. Kerber and Thürlings [19] further determine the permutation character for S_n acting on $\{1, 2, \ldots, N\}^n$:

$$\chi_N(\sigma) := N^{\# \text{ cycles in } (\sigma)} \qquad (2.8)$$

has the decomposition

$$\chi_N = \sum_k \binom{N + k}{n} \chi^{n,k}.$$

The $\chi^{n,k}$ are usually *not* irreducible, and [19] gives an interesting combinatorial rule for decomposing $\chi^{R(U)}$ (and thus $\chi^{n,k}$).

Marty Isaacs conjectured that $n!$ divides the determinant of the Foulkes character table. In fact, the following is true:

The determinant of the $n \times n$ matrix with k, j entry $\chi_j^{n,k}$ is $n!(n-1)! \cdots 2!$. (2.9)

Proof Construct an $n \times n$ matrix A from the $(n-1) \times (n-1)$ Foulkes character table by adding a left column consisting of the partial sums of the Eulerian numbers $A(n, 0), A(n, 0) + A(n, 1), \ldots, n!$ and filling out the rest of the top row with zeros. Thus, when $n = 5$,

$$A = \begin{pmatrix} 120 & 0 & 0 & 0 & 0 \\ 119 & 1 & -1 & 1 & -1 \\ 93 & 11 & -3 & -1 & 3 \\ 27 & 11 & 3 & -1 & -3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The 4×4 matrix in the lower right corner is the Foulkes character table for $n = 4$. The first column entries are the partial sums of the Eulerian numbers 1, 26, 66, 26, 1. In particular, the $(1, 1)$ entry is $n!$, so by induction the determinant of A is $n!(n-1)! \cdots 2!$.

The $n \times n$ Foulkes character table is constructed from A as follows: in A, subtract row 2 from row 1, then row 3 from row 2, and so on. The recurrence (2.3) shows this gives the $n \times n$ Foulkes character table. □

Marty Isaacs observes that while the $\{\chi^{n,k}\}$ are not disjoint, they sum up to the regular character of S_n; see [25, Th. 2] for a proof. Alas, this does not seem to be enough to have the nice theory of supercharacters [1] carry over, but the parallels are intriguing. Further properties of Foulkes characters are given in Sect. 2.3 after the connection with the carries transition matrix is developed.

Rim hook characters are a basic construction of representation theory of S_n; see [3, 20] and the references there. They are also available for other Coxeter groups [25]. Foulkes' innovation, showing that sums of these characters have interesting properties, has not been explored for general type.

2.2 The connection with carries

We noticed from Holte's paper [17] that the matrix of left (row) eigenvectors for the carries Markov chain on $\{0, 1, 2, 3, 4\}$ (e.g., adding 5 numbers) is

$$
\begin{pmatrix}
1 & 26 & 66 & 26 & 1 \\
1 & 10 & 0 & -10 & -1 \\
1 & 2 & -6 & 2 & 1 \\
1 & -2 & 0 & 2 & -1 \\
1 & -4 & 6 & -4 & 1
\end{pmatrix}.
$$

Comparing this with Table 1, the Foulkes character table, leads to the following result.

Theorem 2.1 *Let $v_{i,j}^n$ denote the jth entry of the ith left eigenvector of the carries matrix for addition of n numbers base b (here $0 \le i, j \le n - 1$, and the eigenvalues are $1/b^i$). Then*

$$
v_{i,j}^n = \chi_{n-i}^{n,n-j-1}.
$$

Proof The first case is that $i = 0$. From [17], $v_{0,j}^n = A(n, j)$. From the dimension formula, $\chi_n^{n,n-j-1} = A(n, n - 1 - j)$. By symmetry of the Eulerian numbers, $A(n, j) = A(n, n - 1 - j)$, so the theorem follows in the first case.

The second case is that $j = n - 1$. By (2.2), $\chi_j^{n,0} = (-1)^{n-j}$. Thus we need to show that $v_{n-j,n-1}^n = (-1)^{n-j}$. By Holte's formula for the left eigenvectors of the carries chain [17, p. 143], it follows that

$$
v_{n-j,n-1}^n = \sum_{r=0}^{n-1} (-1)^r \binom{n+1}{r} (n-r)^j.
$$

The result now follows by induction, since

$$
\sum_{r=0}^{n-1} (-1)^r \binom{n+1}{r} (n-r)^{j-1}(n-r)
$$

$$
= n \sum_{r=0}^{n-1} (-1)^r \binom{n+1}{r} (n-r)^{j-1} - \sum_{r=0}^{n-1} (-1)^r r \binom{n+1}{r} (n-r)^{j-1}
$$

$$
= n(-1)^{n-j+1} - (n+1) \sum_{r=1}^{n-1} (-1)^r \binom{n}{r-1} (n-r)^{j-1}
$$

$$
= n(-1)^{n-j+1} + (n+1) \sum_{r=0}^{n-2} (-1)^r \binom{n}{r} (n-1-r)^{j-1}
$$

$$
= n(-1)^{n-j+1} + (n+1)(-1)^{n-j}
$$

$$
= (-1)^{n-j}.
$$

For the remaining cases, $i > 0$ and $j < n - 1$. By the recursive formula (2.3), it is enough to show that

$$v_{i,j}^n = v_{i-1,j}^{n-1} - v_{i-1,j-1}^{n-1}$$

for $i > 0$, $j < n - 1$. From p. 144 of [17],

$$v_{i,j}^n = \text{Coefficient of } x^{j+1} \text{ in } (1-x)^{n+1}\left(x\frac{d}{dx}\right)^{n-i}(1-x)^{-1}.$$

Clearly,

$$\text{Coefficient of } x^{j+1} \text{ in } (1-x)^{n+1}\left(x\frac{d}{dx}\right)^{n-i}(1-x)^{-1}$$

$$= \text{ Coefficient of } x^{j+1} \text{ in } (1-x)^{n}\left(x\frac{d}{dx}\right)^{n-i}(1-x)^{-1}$$

$$- \text{ Coefficient of } x^{j} \text{ in } (1-x)^{n}\left(x\frac{d}{dx}\right)^{n-i}(1-x)^{-1},$$

which implies the result.   □

2.3 Some consequences

In [17], Holte gave a closed formula for the left eigenfunctions,

$$v_{i,j}^n = \sum_{r=0}^{j+1}(-1)^r\binom{n+1}{r}(j+1-r)^{n-i}. \qquad (2.10)$$

Thus we get an apparently new formula for the Foulkes characters.

Corollary 2.2

$$\chi_j^{n,k} = \sum_{r=0}^{n-k}(-1)^r\binom{n+1}{r}(n-k-r)^{j}.$$

In rereading Foulkes [11, Sect. 4], we found the formula (Theorem 4.1),

$$\chi_j^{n,k} = \sum_{r=0}^{n-j}(-1)^r\binom{n-j}{r}A(j,k+j+r-n).$$

This seems a little less direct than Corollary 2.2. Corollary 2.2 gives a direct proof of the following restriction formula of Foulkes characters from S_n to S_{n-1}.

Corollary 2.3 [11, Corollary 4.6]

$$\chi_{S_{n-1}}^{n,k} = (k+1)\chi^{n-1,k} + (n-k)\chi^{n-1,k-1}, \quad 0 \le k \le n-1.$$

Remarks We first learned Corollary 2.3 from Marty Isaacs, who both observed it and showed that it follows from Corollary 2.2. The well-known recursion formula for the Eulerian numbers $A(n,k) = (k+1)A(n-1,k) + (n-k)A(n-1,k-1)$ is the special case of evaluation at the identity. Thus Corollary 2.3 represents a "categorification" of this recurrence.

Proof The required formula translates to

$$\chi_j^{n,k} = (k+1)\chi_{j-1}^{n-1,k} + (n-k)\chi_{j-1}^{n-1,k-1}, \quad 0 \le k \le n-1, \; j \ge 1.$$

From Corollary 2.2,

$$(k+1)\chi_{j-1}^{n-1,k} + (n-k)\chi_{j-1}^{n-1,k-1}$$

$$= (k+1)\sum_{r=0}^{n-k-1}(-1)^r\binom{n}{r}(n-k-r-1)^{j-1} + (n-k)\chi_{j-1}^{n-1,k-1}$$

$$= (k+1)\sum_{r=1}^{n-k}(-1)^{r-1}\binom{n}{r-1}(n-k-r)^{j-1}$$

$$\quad + (n-k)^j + (n-k)\sum_{r=1}^{n-k}(-1)^r\binom{n}{r}(n-k-r)^{j-1}$$

$$= (n-k)^j + \sum_{r=1}^{n-k}(-1)^{r-1}\binom{n}{r-1}(n-k-r)^{j-1}$$

$$\quad + k\sum_{r=1}^{n-k}(-1)^{r-1}\binom{n}{r-1}(n-k-r)^{j-1}$$

$$\quad + n\sum_{r=1}^{n-k}(-1)^r\binom{n}{r}(n-k-r)^{j-1} - k\sum_{r=1}^{n-k}(-1)^r\binom{n}{r}(n-k-r)^{j-1}.$$

Combining the third and fifth terms using $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$, one obtains

$$= (n-k)^j + n\sum_{r=1}^{n-k}(-1)^r\binom{n}{r}(n-k-r)^{j-1} + \sum_{r=1}^{n-k}(-1)^{r-1}\binom{n}{r-1}(n-k-r)^{j-1}$$

$$\quad - k\sum_{r=1}^{n-k}(-1)^r\binom{n+1}{r}(n-k-r)^{j-1}.$$

Adding and subtracting

$$\sum_{r=1}^{n-k}(-1)^r(n-r)\binom{n+1}{r}(n-k-r)^{j-1}$$

yields

$$(n-k)^j + n\sum_{r=1}^{n-k}(-1)^r\binom{n}{r}(n-k-r)^{j-1} + \sum_{r=1}^{n-k}(-1)^{r-1}\binom{n}{r-1}(n-k-r)^{j-1}$$

$$+ \sum_{r=1}^{n-k}(-1)^r\binom{n+1}{r}(n-k-r)^j - \sum_{r=1}^{n-k}(-1)^r(n-r)\binom{n+1}{r}(n-k-r)^{j-1}$$

$$= \left[(n-k)^j + \sum_{r=1}^{n-k}(-1)^r\binom{n+1}{r}(n-k-r)^j\right]$$

$$+ n\sum_{r=1}^{n-k}(-1)^r\binom{n}{r}(n-k-r)^{j-1} + \sum_{r=1}^{n-k}(-1)^{r-1}\binom{n}{r-1}(n-k-r)^{j-1}$$

$$- \sum_{r=1}^{n-k}(-1)^r(n-r)\binom{n+1}{r}(n-k-r)^{j-1}$$

$$= \chi_j^{n,k} + \sum_{r=1}^{n-k}(-1)^r(n-k-r)^{j-1}\left[n\binom{n}{r} - \binom{n}{r-1} - (n-r)\binom{n+1}{r}\right]$$

$$= \chi_j^{n,k}.$$

The penultimate equality used Corollary 2.2 and the final equality used the identity

$$n\binom{n}{r} - \binom{n}{r-1} - (n-r)\binom{n+1}{r} = 0. \qquad \square$$

3 Riffle shuffle idempotents and right eigenvectors of the carries matrix

This section simplifies Holte's formula for the right eigenvectors of the carries matrix, and relates these eigenvectors to representation theory of the symmetric group (Sect. 3.1). The eigenfunctions are used to compute basic things about carries in Sect. 3.2.

3.1 Right eigenfunctions

To begin, let $u_j^n(i)$ denote the value of the jth right eigenvector of the carries chain ($0 \leq j \leq n-1$, eigenvalues $1/b^j$) evaluated at i ($0 \leq i \leq n-1$). We also let $s(n,k)$

be the Stirling number of the first kind, defined as $(-1)^{n-k}$ multiplied by the number of permutations on n symbols with k cycles. It can also be defined by the equation

$$x(x-1)\cdots(x-n+1) = \sum_{k \geq 0} s(n,k)x^k. \tag{3.1}$$

Theorem 4 of Holte [17] shows that

$$u_j^n(i) = \sum_{k=n-j}^{n} s(n,k)\binom{k}{n-j}(n-1-i)^{k-(n-j)}$$

where 0^0 is taken to be 1. Note that $u_j^n(i)$ is a polynomial in i of degree j. For example, $u_0^n(i) = 1$, $u_1^n(i) = n(n-1-i) - \binom{n}{2}$.

The next theorem gives a simpler formula for $u_j^n(i)$.

Theorem 3.1

$$u_j^n(i) = n! \sum_{k \geq 0} \frac{s(k,n-j)}{k!}\binom{n-i-1}{n-k}$$

$$= n! \cdot \text{Coefficient of } x^{n-j} \text{ in } \binom{x+n-i-1}{n}.$$

Proof By Theorem 4 of Holte [17], $u_j^n(i)$ is $n!$ multiplied by the i, j entry of the inverse of the matrix of left row eigenvectors. From this and (2.10), proving the first equality of the theorem is equivalent to proving that

$$n! \sum_{l \geq 0} \sum_{k \geq 0} \frac{s(k,n-l)}{k!}\binom{n-i-1}{n-k}\sum_{r=0}^{j+1}(-1)^r\binom{n+1}{r}(j+1-r)^{n-l} = n!\delta_{i,j}.$$

Now

$$n! \sum_{l=n-k}^{n} \sum_{k \geq 0} \frac{s(k,n-l)}{k!}\binom{n-i-1}{n-k}\sum_{r=0}^{j+1}(-1)^r\binom{n+1}{r}(j+1-r)^{n-l}$$

$$= n! \sum_{k \geq 0} \binom{n-i-1}{n-k}\sum_{r=0}^{j+1}(-1)^r\binom{n+1}{r}\frac{1}{k!}\sum_{l=n-k}^{n} s(k,n-l)(j+1-r)^{n-l}$$

$$= n! \sum_{k \geq 0} \binom{n-i-1}{n-k}\sum_{r=0}^{j+1}(-1)^r\binom{n+1}{r}\frac{1}{k!}\sum_{t=0}^{k} s(k,t)(j+1-r)^t$$

$$= n! \sum_{k \geq 0} \binom{n-i-1}{n-k}\sum_{r \geq 0}(-1)^r\binom{n+1}{r}\binom{j+1-r}{k}$$

$$= n! \sum_{r \geq 0}(-1)^r\binom{n+1}{r}\sum_{k \geq 0}\binom{n-i-1}{n-k}\binom{j+1-r}{k}$$

$$= n! \sum_{r \geq 0} (-1)^r \binom{n+1}{r} \binom{n+j-i-r}{n}$$

$$= n! \delta_{i,j}.$$

The third equality used (3.1), the fifth used the basic identity $\sum_i \binom{a}{i} \binom{b}{n-i} = \binom{a+b}{n}$ [26, p. 12], and the final equality is from p. 147 of [17].

To prove the second equality of the theorem, write

$$\binom{x+n-i-1}{n} = \sum_k \binom{x}{k} \binom{n-i-1}{n-k}$$

and use (3.1).   □

Remark Let $E_{n,k}$ be elements of the symmetric group algebra defined by the equation

$$\sum_{k=1}^n x^k E_{n,k} = \sum_{w \in S_n} \binom{x+n-d(w)-1}{n} w$$

where $d(w)$ denotes the number of descents of w. By work of Garsia and Reutenauer [13], these are orthogonal idempotents of the symmetric group algebra whose sum is the identity. They also arise in the theory of riffle shuffling [2] and in Hochschild homology [16]. Their images under the sign map are known as *Eulerian idempotents*. In this version, they were discovered by Gerstenhaber–Schack [14] to give Hodge decompositions of Hochschild homology. They have been developed by Loday [21] for cyclic homology. Patras [24] gives an unusual treatment involving decompositions of the n-cube into simplices. For a textbook treatment, see Weibel [29, Sect. 9.4.3].

The eigenvectors of the carries and descent matrix lift to eigenvectors of the full riffle shuffle matrix. These in turn are identified in Denham [6] and Diaconis–Pang–Ram [10].

Clearly, the value of $E_{n,k}$ on a permutation depends only on its number of descents. Letting $E_{n,k}(d)$ denote the value of $E_{n,k}$ on a permutation with d descents, we have the following corollary of Theorem 3.1.

Corollary 3.2

$$u_j^n(i) = n! E_{n,n-j}(i).$$

It would be nice to have a more conceptual proof of Corollary 3.2. The paper [23] is a development along these lines.

3.2 Applications

This section gives some applications of the explicit form of the right eigenvectors of the carries chain for the addition of n numbers base b. We note that another application (to lower bounding the convergence rate of the carries chain) appears in [8].

The transition matrix of the carries chain is viewed as a linear operator on functions in the usual way: $Kf(x) = \sum_y K(x, y)f(y)$. We let κ_r denote the value of the carry from column $r - 1$ to column r. We also recall that the covariance $\text{Cov}(Z_1, Z_2)$ between two random variables Z_1, Z_2 is $E(Z_1Z_2) - E(Z_1)E(Z_2)$.

Proposition 3.3 *Suppose that the carries chain is started from its stationary distribution π. Then for $n \geq 2$,*

$$\text{Cov}(\kappa_0, \kappa_r) = \frac{1}{b^r} \frac{n+1}{12} > 0.$$

Proof Since $P(\kappa_0 = i) = \pi(i)$,

$$E(\kappa_0\kappa_r) = \sum_i i\pi(i)E(\kappa_r|\kappa_0 = i).$$

By Theorem 3.1, u_1^n is a right eigenvector of the carries chain with eigenvalue $1/b$. Dividing by $-1/n$, we have the eigenvector $f(i) = i - \frac{n-1}{2}$,

$$E(\kappa_r|\kappa_0 = i) = K^r\left(f + \frac{n-1}{2}\right)(i) = \frac{1}{b^r}\left(i - \frac{n-1}{2}\right) + \frac{n-1}{2}.$$

It follows that

$$E(\kappa_0\kappa_r) = \sum_i i\left[\frac{1}{b^r}\left(i - \frac{n-1}{2}\right) + \frac{n-1}{2}\right]\pi(i). \tag{3.2}$$

From [17] or [8], the stationary distribution of the carries chain is $\pi(i) = A(n, i)/n!$ (here $A(n, i)$ is the number of permutations on n symbols with i descents). It is known [5] that for $n \geq 2$ the mean and variance of the Eulerian numbers are $\frac{n-1}{2}$ and $\frac{n+1}{12}$. This, together with (3.2), gives

$$E(\kappa_0\kappa_r) = \frac{1}{b^r}\left[\text{Var}(\kappa_0) + E(\kappa_0)^2\right] - \frac{1}{b^r}\left(\frac{n-1}{2}\right)^2 + \left(\frac{n-1}{2}\right)^2$$

$$= \frac{1}{b^r}\frac{n+1}{12} + \left(\frac{n-1}{2}\right)^2,$$

and the result follows since $E(\kappa_0)E(\kappa_r) = (\frac{n-1}{2})^2$, which is true since the chain started from its stationary distribution π. □

A similar calculation allows us to compute the covariance started from the state 0, as in the usual carries process.

Proposition 3.4 *Suppose that the carries chain is started from 0, and that $n \geq 2$. Then*

1. $E(\kappa_s) = (1 - \frac{1}{b^s})\frac{n-1}{2}.$

2. $\text{Var}(\kappa_s) = (1 - \frac{1}{b^{2s}})\frac{n+1}{12}$.
3. $\text{Cov}(\kappa_s, \kappa_{s+r}) = \frac{1}{b^r}\frac{n+1}{12}(1 - \frac{1}{b^{2s}}) > 0$.

Proof Parts 1 and 2 are proved in Theorem 4.1 of [7]. For part 3, arguing as in the proof of Proposition 3.3,

$$E(\kappa_s \kappa_{s+r}) = \sum_i i P(\kappa_s = i) E(\kappa_{s+r}|\kappa_s = i)$$

$$= \sum_i i P(\kappa_s = i)\left[\frac{1}{b^r}\left(i - \frac{n-1}{2}\right) + \frac{n-1}{2}\right]$$

$$= \frac{1}{b^r}\left[\text{Var}(\kappa_s) + E(\kappa_s)^2\right] + \left(1 - \frac{1}{b^r}\right)\frac{n-1}{2}E(\kappa_s).$$

By parts 1 and 2, this is equal to

$$\frac{1}{b^r}\left[\left(1 - \frac{1}{b^{2s}}\right)\frac{n+1}{12} + \left(1 - \frac{1}{b^s}\right)^2\frac{(n-1)^2}{4}\right] + \left(1 - \frac{1}{b^r}\right)\frac{(n-1)^2}{4}\left(1 - \frac{1}{b^s}\right)$$

$$= \frac{1}{b^r}\frac{n+1}{12}\left(1 - \frac{1}{b^{2s}}\right) + \frac{(n-1)^2}{4}\left(1 - \frac{1}{b^s}\right)\left[\frac{1}{b^r}\left(1 - \frac{1}{b^s}\right) + \left(1 - \frac{1}{b^r}\right)\right]$$

$$= \frac{1}{b^r}\frac{n+1}{12}\left(1 - \frac{1}{b^{2s}}\right) + \frac{(n-1)^2}{4}\left(1 - \frac{1}{b^s}\right)\left(1 - \frac{1}{b^{s+r}}\right)$$

$$= \frac{1}{b^r}\frac{n+1}{12}\left(1 - \frac{1}{b^{2s}}\right) + E(\kappa_s)E(\kappa_{s+r}). \qquad \square$$

The eigenvectors of the carries matrix can be used to give a simple proof that the sum of $\chi^{n,k}$ gives the regular character. Holte [17] shows that if V is the matrix whose rows are left eigenvectors and U is the matrix whose columns are right eigenvectors, then $VU = n! \times I$. Just looking at products involving the first column of U (which is identically one) we get that the $\chi^{n,k}$ sum up to the regular character.

It is instructive to see the problems encountered in trying to use the available eigenstructure to bound the rate of convergence of the carries chain to its stationary distribution $A(n, j)/n!$. Let $M_b(i, j)$ be the transition matrix (1.1) corresponding to adding n numbers base b. Let $M_b^k(i, j)$ be the kth power of this matrix. From elementary linear algebra,

$$M_b^k(i, j) = \frac{A(n, j)}{n!} + \sum_{a=1}^{n-1}\frac{1}{b^{ak}}r_a(i)l_a(j)$$

where l_a, r_a are the left and right eigenvectors of M_b normed so that $l_0(j) = \frac{A(n,j)}{n!}$ (and $r_0(j) = 1$) for $0 \le j \le n - 1$. Here

$$l_a(j) = \frac{1}{n!} \chi_{n-a}^{n,n-j-1} \quad \text{with } \chi_{n-a}^{n,n-j-1} \text{ the Foulkes characters of Sect. 2.}$$

$$r_a(j) = u_a^n(j) \quad \text{with } u_a^n(j) \text{ given in Theorem 3.1.}$$

The total variation distance to stationarity, starting at $i = 0$, after k steps is

$$\frac{1}{2} \sum_{j=0}^{n-1} \left| M_b^k(0, j) - \frac{A(n, j)}{n!} \right|.$$

From the formulae above,

$$\left| M_b^k(0, j) - \frac{A(n, j)}{n!} \right| = \left| \sum_{a=1}^{n-1} \frac{1}{b^{ak}} r_a(0) l_a(j) \right|.$$

While bounding this is feasible, it is a bit of a mess. In [8, Sect. 3], a different representation is used to prove that the carries chain is close to stationarity after order $\frac{1}{2} \log_b(n)$ steps.

4 Carries and the RSK correspondence

In this section, we use the RSK correspondence to derive a generating function for descents after a b^r-riffle shuffle on a deck of n cards. Recall that an a-riffle shuffle is given by first cutting the n cards into packet sizes $k_1, \ldots, k_a$ with multinomial probability $\frac{n!}{a^n \prod_i k_i!}$. The cards are then dropped one at a time with probability proportional to packet size. So if the packet sizes are $X_1, \ldots, X_a$ at a given time, the next card drops from packet i with probability $X_i/(X_1 + \cdots + X_a)$. For further background on riffle shuffles, the reader can see [2].

By a main result of [7], the generating function for descents after a b^r-riffle shuffle on a deck of n cards is equal to the generating function for the rth carry κ_r when n numbers are added base b (and one can give another proof of Theorem 4.1 using Holte's formula for $P(\kappa_r = i)$).

Theorem 4.1 *Let $P(w)$ be the probability of w following a b^r-riffle shuffle on a deck of n cards started at the identity, and let $d(w)$ denote the number of descents of w. Then*

$$\sum_{w \in S_n} P(w) t^{d(w)+1} = \frac{(1-t)^{n+1}}{b^{rn}} \sum_{k \geq 1} t^k \binom{b^r k + n - 1}{n}. \tag{4.1}$$

Proof Let w be a permutation produced by a b^r riffle shuffle. The RSK correspondence associates to w a pair of standard Young tableaux $(P(w), Q(w))$ of the same shape. Moreover, there is a notion of descent set for standard Young tableaux, and by Lemma 7.23.1 of [27], the descent set of w is equal to the descent set of $Q(w)$. It is known from [12] that if w is produced by a b^r shuffle, then the chance that $Q(w)$ is equal to any particular standard Young tableau of shape λ is $s_\lambda(\frac{1}{b^r}, \ldots, \frac{1}{b^r})$, where

there are b^r variables. Letting $f_\lambda(a)$ denote the number of standard Young tableaux of shape λ with a descents, it follows that

$$P\big(d(w) = a\big) = \sum_{|\lambda|=n} f_\lambda(a) s_\lambda\left(\frac{1}{b^r}, \ldots, \frac{1}{b^r}\right).$$

By (7.96) of [27],

$$\sum_{a \geq 0} f_\lambda(a) t^{a+1} = (1-t)^{n+1} \sum_{k \geq 1} s_\lambda(1, \ldots, 1) t^k$$

where in the kth summand, $s_\lambda(1, \ldots, 1)$ denotes the Schur function with k variables specialized to 1. Thus

$$\sum_{a \geq 0} P\big(d(w) = a\big) \cdot t^{a+1}$$

$$= \sum_{a \geq 0} \sum_{|\lambda|=n} f_\lambda(a) s_\lambda\left(\frac{1}{b^r}, \ldots, \frac{1}{b^r}\right) \cdot t^{a+1}$$

$$= (1-t)^{n+1} \sum_{k \geq 1} t^k \sum_{|\lambda|=n} s_\lambda\left(\frac{1}{b^r}, \ldots, \frac{1}{b^r}\right) s_\lambda(1, \ldots, 1)$$

$$= (1-t)^{n+1} \sum_{k \geq 1} t^k [u^n] \sum_{n \geq 0} \sum_{|\lambda|=n} s_\lambda\left(\frac{1}{b^r}, \ldots, \frac{1}{b^r}\right) s_\lambda(1, \ldots, 1) \cdot u^n$$

where $[u^n] g(u)$ denotes the coefficient of u^n in a power series $g(u)$. Applying the Cauchy identity for Schur functions [27, p. 322], this becomes

$$(1-t)^{n+1} \sum_{k \geq 1} t^k [u^n] \big(1 - u/b^r\big)^{-b^r k} = \frac{(1-t)^{n+1}}{b^{rn}} \sum_{k \geq 1} t^k \binom{b^r k + n - 1}{n}$$

as desired.   □

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