Journal of Algebraic Combinatorics 3 (1994), 5–16

The Descent Monomials and a Basis for the Diagonally Symmetric Polynomials

E.E. ALLEN
Department of Mathematics and Computer Science, Wake Forest University, Winston-Salem, NC 27109

Received January 6, 1993; Revised June 17, 1993

Abstract. Let $\mathbf{R}(X) = \mathbf{Q}[x_1, x_2, \ldots, x_n]$ be the ring of polynomials in the variables $X = \{x_1, x_2, \ldots, x_n\}$ and $\mathbf{R}^*(X)$ denote the quotient of $\mathbf{R}(X)$ by the ideal generated by the elementary symmetric functions. Given a $\sigma \in S_n$, we let $g_\sigma(X) = \prod_{\sigma_i > \sigma_{i+1}} (x_{\sigma_1} x_{\sigma_2} \ldots x_{\sigma_i})$. In the late 1970s I. Gessel conjectured that these monomials, called the *descent monomials*, are a basis for $\mathbf{R}^*(X)$. Actually, this result was known to Steinberg [10]. A. Garsia showed how it could be derived from the theory of *Stanley-Reisner Rings* [3]. Now let $\mathbf{R}(X, Y)$ denote the ring of polynomials in the variables $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. The *diagonal action* of $\sigma \in S_n$ on polynomial $P(X, Y)$ is defined as $\sigma P(X, Y) = P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}, y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_n})$. Let $\mathbf{R}^\rho(X, Y)$ be the subring of $R(X, Y)$ which is invariant under the diagonal action. Let $\mathbf{R}^{\rho*}(X, Y)$ denote the quotient of $\mathbf{R}^\rho(X,Y)$ by the ideal generated by the elementary symmetric functions in X and the elementary symmetric functions in Y. Recently, A. Garsia in [4] and V. Reiner in [8] showed that a collection of polynomials closely related to the descent monomials are a basis for $\mathbf{R}^{\rho*}(X, Y)$. In this paper, the author gives elementary proofs of both theorems by constructing algorithms that show how to expand elements of $\mathbf{R}^*(X)$ and $\mathbf{R}^{\rho*}(X, Y)$ in terms of their respective bases.

Keywords: descent monomial, diagonally symmetric polynomials, polynomial quotient ring

1. Introduction

The basic purpose is to show that the methods introduced in [1] and [2] can also be used to give elementary proofs of Theorems 1.1 and 1.3 stated below. To be specific we need some notation. Let $\mathbf{R}(X) = \mathbf{Q}[x_1, x_2, \ldots, x_n]$ be the ring of polynomials in the variables $X = \{x_1, x_2, \ldots, x_n\}$. Given a $\sigma \in S_n$, we agree to represent σ as $\sigma_1\sigma_2\cdots\sigma_n$ where $\sigma_i = \sigma(i)$. The action of σ on a polynomial $P(X)$ is defined as

$$\sigma P(x_1, x_2, \ldots, x_n) = P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}).$$

A polynomial is said to be *symmetric* if $\sigma P(X) = P(X)$ for all $\sigma \in S_n$. Let us denote by $\mathbf{R}^{S_n}(X)$ the ring of symmetric polynomials in the alphabet X.

Recall that the *elementary symmetric function* $e_i(X)$ is defined to be

$$e_i(X) = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le n} x_{j_1} x_{j_2} \cdots x_{j_i}. \tag{1}$$

It is well known and not difficult to show that

(1) The $e_i(X)$ are algebraically independent.
(2) Every element of $\mathbf{R}^{S_n}(X)$ can be expressed as a polynomial in the $e_i(X)$.

Define $\mathbf{R}^*(X)$ to be the quotient of $\mathbf{R}(X)$ by the ideal generated by the elementary symmetric functions. In other words,

$$\mathbf{R}^*(X) = \frac{\mathbf{Q}[x_1, x_2, \dots, x_n]}{(e_1(X), e_2(X), \dots, e_n(X))}.$$

The *descent monomial* $g_\sigma(X)$ in the alphabet $X = \{x_1, x_2, \dots, x_n\}$ is defined to be

$$g_\sigma(X) = \prod_i (x_{\sigma_1} x_{\sigma_2} \dots x_{\sigma_i})^{\chi(\sigma_i > \sigma_{i+1})} \tag{2}$$

where we use the convention that if A is a statement then

$$\chi(A) = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases}$$

For example, if $\sigma = 2\ 1\ 4\ 3$ then

$$g_\sigma = (x_2)(x_2 x_1 x_4) = x_1 x_2^2 x_4.$$

Let us define $\mathcal{G} = \{g_\sigma : \sigma \in S_n\}$. We have the following theorem whose proof is given in Section 2.

THEOREM 1.1. *The collection $\mathcal{G}$ is a basis for $\mathbf{R}^*(X)$.*

Let $\mathcal{M}(X)$ be a graded polynomial ring over the variables $X = \{x_1, x_2, \dots, x_n\}$ where the grading is obtained by the degrees of the variables and set $H_m(\mathcal{M}(X))$ to be the submodule of $\mathcal{M}(X)$ consisting of elements that are homogeneous of degree m in X. The Hilbert series of $\mathcal{M}(X)$ is defined to be

$$H(\mathcal{M})(X)) = \sum_{m \geq 0} t^m \dim H_m(\mathcal{M}(X))$$

An important ingredient in our proof is the following proposition, whose proof can be found in [3].

THEOREM 1.2. *Let $\mathbf{V}$ be a graded ring over a field F and $\mathcal{I} = \{i_1, i_2, \dots, i_n\}$ a set of homogeneous elements with $\deg(i_j) > 0$. Let $\mathbf{V}^*$ be the quotient of $\mathbf{V}$ by the ideal generated by $\mathcal{I}$. Let B be a finite collection of homogeneous elements such that the Hilbert series $H(V)$ equals*

$$\frac{\sum_{b \in B} t^{\deg(b)}}{\prod_{j=1}^{n} (1 - t^{\deg(i_j)})}. \tag{3}$$

Then the following are equivalent

(a)

$$\mathcal{BI} = \{bi_1^{p_1} i_2^{p_2} \cdots i_n^{p_n} : b \in \mathcal{B},\ p_i \geq 0\}$$

spans $\mathbf{V}$ *as a vector space.*

(b) $\mathcal{I}$ *are algebraically independent and* $\mathbf{V}$ *is a free module over* $F[i_1, \ldots, i_n]$ *with basis* $\mathcal{B}$.

(c) $\mathcal{B}$ *is a basis for* $\mathbf{V}^*$ *as a vector space.*

Now it is well known and not difficult to show that the Hilbert series of $\mathbf{R}(X)$ and the collections of the descent monomials and of the elementary symmetric functions satisfy (3). Thus once we have established that the collection

$$\mathcal{EG} = \{e_1^{p_1} e_2^{p_2} \cdots e_n^{p_n}(X) g_\sigma(X) : \sigma \in S_n\} \tag{4}$$

spans $\mathbf{R}(X)$ we will have a proof of Theorem 1.1.

Let μ be a vector of length n with nonnegative integer components. The *monomial symmetric function* $m_\mu(X)$ is defined to be

$$m_\mu(X) = \sum_{\sigma \in S_n/G_\mu} x_1^{\mu_{\sigma_1}} x_2^{\mu_{\sigma_2}} \ldots x_n^{\mu_{\sigma_n}}.$$

where G_μ is the stabilizer of μ. The fundamental theorem of symmetric functions implies that the product of the elementary symmetric functions $e_1^{p_1} e_2^{p_2} \cdots e_n^{p_n}(X)$ can be replaced in (4) by the monomial symmetric functions $m_\mu(X)$. Thus it will be sufficient to prove that the collection

$$\mathcal{MG} = \{m_\mu(X) g_\sigma(X) : \sigma \in S_n\} \tag{5}$$

spans $\mathbf{R}(X)$. This is precisely what we will do in Section 2.

Let $Y = \{y_1, y_2, \ldots, y_n\}$ and let $\mathbf{R}(X,Y)$ be the ring of polynomials in the variables X and Y. Furthermore, let us suppose that $\mathcal{M}(X, Y)$ is a bigraded module and let $H_{m,p}(\mathcal{M}(X, Y))$ denote the collection of polynomials of $\mathcal{M}(X, Y)$ that are homogeneous of degree m in X and p in Y. We define the Hilbert series of $\mathcal{M}(X, Y)$ to be

$$H(\mathcal{M}(X, Y)) = \sum_{m,p} t^m\, q^p\, \dim H_{m,p}(\mathcal{M}(X, Y))$$

We say that a polynomial $P(X,Y)$ is *doubly symmetric* if

$$P(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}, y_{\beta_1}, y_{\beta_2}, \ldots, y_{\beta_n}) = P(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$$

for all $\alpha, \beta \in S_n$. Let us denote the ring of doubly symmetric polynomials by $\mathbf{R}^{S_n \times S_n}(X, Y)$. Note that every polynomial in $\mathbf{R}^{S_n \times S_n}(X, Y)$ can be expressed as a polynomial in the collection $\{e_1(X), e_2(X), \ldots, e_n(X), e_1(Y), e_2(Y), \ldots, e_n(Y)\}$. We define the *diagonal action* of S_n on a polynomial $P(X,Y)$ by

$$\sigma P(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}, y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_n}).$$

A polynomial $P(X, Y)$ is said to be *diagonally symmetric* if for all $\sigma \in S_n$

$$P(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = \sigma P(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n).$$

Clearly the ring of diagonally symmetric polynomials is spanned by the collection

$$\{\rho\, P(X, Y)\}$$

where $P(X, Y)$ is a polynomial in the variables X and Y and

$$\rho = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma.$$

Let us denote the ring of diagonally symmetric polynomials by $\mathbf{R}^{\rho}(X, Y)$. We wish to consider the quotient ring

$$\mathbf{R}^{\rho*}(X, Y) = \frac{\mathbf{R}^{\rho}(X, Y)}{(e_1(X), e_2(X), \ldots, e_n(X), e_1(Y), e_2(Y), \ldots, e_n(Y))}. \tag{6}$$

THEOREM 1.3. *The collection*

$$\mathcal{GR} = \{\rho g_\sigma(Y) r_{\sigma^{-1}}(X) : \sigma \in S_n\}$$

where

$$r_{\sigma^{-1}}(X) = \prod_i^n (x_1 \cdots x_i)^{\chi(\sigma_i^{-1} > \sigma_{i+1}^{-1})}$$

is a basis for $\mathbf{R}^{\rho*}(X, Y)$.

We will prove Theorem 1.3 by using the following two-parameter version of Theorem 1.2, namely

THEOREM 1.4. *Let* $\mathbf{V}$ *be a bigraded ring over a field* F *where the bigrading is obtained by the degree of the variables in* $X = \{x_1, x_2, \ldots, x_n\}$ *and* $Y = \{y_1, y_2, \ldots, y_n\}$ *respectively. For* $p = p(X, Y) \in \mathbf{V}$ *let* $\deg_X(p)$ *and* $\deg_Y(p)$ *be the degrees of* p *in* X *and* Y *respectively. Let* $\mathcal{I} = \{i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_n\}$ *be a set of homogeneous elements where* $\deg_X(i_k) > 0$, $\deg_Y(i_k) = 0$, $\deg_X(j_k) = 0$, *and* $\deg_Y(j_k) > 0$. *Let* $\mathbf{V}^*$ *be the quotient of* $\mathbf{V}$ *by the ideal generated by* $\mathcal{I}$. *Let* B *be a finite collection of homogeneous elements such that the Hilbert series* $H(V)$ *equals*

$$\frac{\sum_{b \in B} q^{\deg_Y(b)}\, t^{\deg_X(b)}}{\prod_{k=1}^n (1 - t^{\deg_X(i_k)})(1 - q^{\deg_Y(j_k)})}. \tag{7}$$

Then the following are equivalent

(a)

$$\mathcal{BI} = \{b i_1^{p_1} i_1^{p_2} \cdots i_n^{p_n} j_1^{m_1} j_2^{m_2} \cdots j_n^{m_n} : b \in \mathcal{B},\, p_i,\, m_i \geq 0\}$$

spans $\mathbf{V}$ *as a vector space.*

(b) $\mathcal{I}$ *is algebraically independent and* $\mathbf{V}$ *is a free module over* $F[i_1, \ldots, i_n, j_1, \ldots, j_n]$ *with basis* $\mathcal{B}$.

(c) $\mathcal{B}$ *is a basis for* $\mathbf{V}^*$ *as a vector space.*

That $\mathcal{GR}$, $\mathcal{I} = \{e_1(X), e_2(X), \ldots, e_n(X), e_1(Y), e_2(Y), \ldots, e_n(Y)\}$ and the Hilbert series of $H(\mathbf{R}^\rho(X, Y))$ satisfies (7) can be found in [4]. Furthermore, it is a corollary to Theorem 2.3 of [5]. Thus to prove Theorem 1.3 it is sufficient to show that the collection

$$\mathcal{EGR} = \{e_1^{p_1} e_2^{p_2} \cdots e_n^{p_n}(X)\; e_1^{q_1} e_2^{q_2} \cdots e_n^{q_n}(Y)\, \rho r_{\sigma^{-1}}(X)\; g_\sigma(Y) : \sigma \in S_n\} \tag{8}$$

spans $\mathbf{R}^\rho(X, Y)$. Once again we use the fundamental theorem of symmetric functions to substitute the monomial symmetric functions for the elementary symmetric functions in (8) and thus we need only show that the collection

$$\mathcal{MGR} = \{m_\mu(X) m_\nu(Y) \rho r_{\sigma^{-1}}(X) g_\sigma(Y) : \sigma \in S_n\} \tag{9}$$

spans $\mathbf{R}^\rho(X, Y)$. We will show this in Proposition 3.2.

2. The descent basis for $\mathbf{R}^*$

If $p = (p_1, p_2, \ldots, p_n)$ we will use the convention that $x^p = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$. We define the type $\tau(p)$ of x^p to be the rearrangement of entries of p in decreasing order. For example, if $p = (3, 1, 3, 0, 2, 0)$ then $\tau(p) = (3, 3, 2, 1, 0, 0)$.

We shall label the entries of p from smallest to largest, breaking ties from right to left. In other words, if $a_1 < a_2 < \cdots < a_k$ are the distinct entries in p, we first label all the entries of a_1 from right to left, then we label all the entries of a_2 from right to left, etc. From this labeling we construct $\gamma(p) = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ in the following manner. Replace the entry labeled by 1 with 0. Now recursively, if we have replaced the entry labeled t with s we replace the entry labeled $t + 1$ by s if it is left of the entry labeled t and by $s + 1$ otherwise. Let $\mu_i = p_i - \gamma_i$ so that $\mu(p)$ is the sequence

$$\mu(p) = p - \gamma(p) = (p_1 - \gamma_1, p_2 - \gamma_2, \ldots, p_n - \gamma_n). \tag{10}$$

For example, if

$$x^p = x_1^3 x_2^1 x_3^3 x_4^0 x_5^2 x_6^0$$

then we have

$$p = (3, 1, 3, 0, 2, 0) \text{ and } \textit{labeled } p = (3_6, 1_3, 3_5, 0_2, 2_4, 0_1)$$

and thus

$$\gamma(p) = (1, 0, 1, 0, 1, 0) \quad \text{and} \quad \mu(p) = (2, 1, 2, 0, 1, 0).$$

Note that we have defined a decomposition Φ of p into a pair of sequences $(\gamma(p), \mu(p))$. This decomposition is the usual P-partition encoding of p (see [7] and [9]). This gives the following theorem.

THEOREM 2.1. *Let $p = (p_1, p_2, \ldots, p_n)$. If $\gamma = \gamma(p)$, then x^γ is a descent monomial.*

We can define a total order on the set of monomials x^p in the following manner. We say that $x^q <_{ts} x^p$ if

$$\begin{aligned} &(1)\ \tau(q) <_L \tau(p);\ \text{or} \\ &(2)\ \text{if } \tau(q) = \tau(p) \text{ then } q <_L p \end{aligned} \tag{11}$$

where $<_L$ means that we are comparing the sequences in lexicographic order.

We will now show that the set $\mathcal{MG}$ is triangularly related to the set of monomials in $\mathbf{R}(X)$.

PROPOSITION 2.1. *Let $x^p = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$, $\gamma = \gamma(p)$ and $\mu = \mu(p)$. Then*

$$m_\mu(X)x^\gamma = x^p + \sum_{x^q <_{ts} x^p} c_q x^q \tag{12}$$

where m_μ is the monomial symmetric function corresponding to μ.

Proof. Recall that if G_μ is the stabilizer of μ then the monomial symmetric function $m_\mu(X)$ is defined as

$$m_\mu(X) = \sum_{\sigma \in S_n/G_\mu} x^{\sigma\mu}$$

where $\sigma\mu = (\mu_{\sigma_1}, \mu_{\sigma_2}, \ldots, \mu_{\sigma_n})$ and thus

$$m_\mu(X)x^\gamma = \sum_{\sigma \in S_n/G\mu} x^{\sigma\mu}x^\gamma = \sum_{\sigma \in S_n/G_\mu} x^{\sigma\mu+\gamma}.$$

Let $B_k = \{i : \gamma_i = k\}$ and define

$$S_B = S_{B_0} \times S_{B_1} \times \cdots \times S_{B_t}$$

where t is the largest entry in γ.

Now there are three cases

(a) $\sigma \in G_\mu$.
(b) $\sigma \in \beta G_\mu$ some $\beta \in S_B$ but $\sigma \notin G_\mu$.
(c) $\sigma \notin \beta G_\mu$ for all $\beta \in S_B$.

Note that in the first case,

$$\sigma\mu + \gamma = \mu + \gamma = (\mu_1 + \gamma_1, \ldots, \mu_n + \gamma_n) = (p_1\ p_2, \ldots, p_n) = p.$$

Thus $x^{\sigma\mu+\gamma} = x^p$.

In case (b), we have $\sigma = \beta g$ where $\beta \in S_B$ and $g \in G_\mu$. Recall that every element of S_B fixes γ and g fixes μ and thus

$$\sigma\mu + \gamma = \beta g\mu + \beta\gamma = \beta\mu + \beta\gamma = \beta(\mu + \gamma) = \beta p$$

and therefore $\tau(\sigma\mu + \gamma) = \tau(p)$. Note, however, if $j, k \in B_i$ and $j \leq k$ then $\mu_j \geq \mu_k$. Thus $\sigma\mu <_L \mu$ and

$$\sigma\mu + \gamma <_L \mu + \gamma = p.$$

In case (c), where $\sigma \notin \beta G_\mu$ for all $\beta \in S_B$ we have that $\tau(\sigma\mu + \gamma) < \tau(p)$. In either case (b) or case (c) we have that $x^{\sigma\mu+\gamma} <_{ts} x^p$. □

Proposition 2.1 shows that the collection $\mathcal{MG}$ spans $\mathbf{R}(X)$. Thus we have proven Theorem 1.1.

The proof of Proposition 2.1 implies an algorithm for the expansion of an element of $\mathbf{R}^*$ as a linear combination of elements of $\mathcal{G}$. As an example, suppose that our alphabet is $X = \{x_1, x_2, x_3, x_4\}$ and that $x^p = x_2^2x_3x_4$. Then $p = (0, 2, 1, 1)$, $\gamma = \gamma(p) = (0, 1, 1, 1)$, and $\mu = \mu(p) = (0, 1, 0, 0)$. Now

$$m_\mu(X) = x_1 + x_2 + x_3 + x_4$$

thus

$$m_\mu(X)x^\gamma = x_1x_2x_3x_4 + x_2^2x_3x_4 + x_2x_3^2x_4 + x_2x_3x_4^2$$

or

$$x_2^2x_3x_4 = m_\mu(X)x^\gamma - x_1x_2x_3x_4 - x_2x_3^2x_4 - x_2x_3x_4^2.$$

Now both $m_\mu(X)x^\gamma$ and $x_1x_2x_3x_4$ are elements of the ideal generated by the elementary symmetric functions. The terms $x_2x_3^2x_4$ and $x_2x_3x_4^2$ are elements of $\mathcal{G}$, namely the descent monomials that correspond to 3241 and 4231 respectively. Thus in $\mathbf{R}^*$ we have

$$x_2^2x_3x_4 = -x_2x_3^2x_4 - x_2x_3x_4^2.$$

3. A basis for $R^{\rho*}(X, Y)$

Suppose that $y^q = g_\sigma(Y)$. We say that x^p is *minimal* with respect to y^q if $p = (p_1, p_2, \ldots, p_n)$ satisfies the following three conditions:

$$\begin{aligned} &\text{(a) } p_n = 0; \\ &\text{(b) if } q_i \geq q_{i+1} \text{ then } p_i = p_{i+1}; \\ &\text{(c) if } q_i < q_{i+1} \text{ then } p_i = p_{i+1} + 1; \end{aligned} \tag{13}$$

As an example, let us suppose that $\sigma = 41253$. Thus if $y^q = g_\sigma(Y)$ then $q = (1, 1, 0, 2, 1)$ and the minimal monomial with respect to y^q would be x^p where $p = (1, 1, 1, 0, 0)$.

PROPOSITION 3.1. *Suppose that $y^q = g_\sigma(Y)$ for some $\sigma \in S_n$. Then x^p is minimal with respect to y^q if and only if $x^p = r_{\sigma^{-1}}(X)$ where*

$$r_{\sigma^{-1}}(X) = \prod_i^n (x_1 \cdots x_i)^{\chi(\sigma_i^{-1} > \sigma_{i+1}^{-1})}. \tag{14}$$

Proof. Suppose $x^p = r_{\sigma^{-1}}(X)$. Note $p_n = 0$. Now,

$$\begin{aligned} q_i < q_{i+1} &\Rightarrow i \text{ is to the right of } i+1 \text{ in } \sigma \\ &\Rightarrow \sigma_i^{-1} > \sigma_{i+1}^{-1} \\ &\Rightarrow p_i = p_{i+1} + 1 \end{aligned}$$

Thus $x^p = r_{\sigma^{-1}}(X)$ is minimal with respect to y^q.

On the other hand, if we suppose that x^p is minimal with respect to y^q then we see that $p_n = 0$ and

$$\begin{aligned} p_i = p_{i+1} + 1 &\Rightarrow q_i < q_{i+1} \\ &\Rightarrow \sigma_i^{-1} > \sigma_{i+1}^{-1} \end{aligned}$$

Thus if x^p is minimal with respect to y^q then

$$\begin{aligned} x^p &= \prod_{j=1}^n \prod_{i \geq j} (x_j)^{\chi(\sigma_i^{-1} > \sigma_{i+1}^{-1})} \\ &= \prod_{i=1}^n \prod_{j \leq i} (x_j)^{\chi(\sigma_i^{-1} > \sigma_{i+1}^{-1})} \\ &= \prod_{i=1}^n (x_1 \cdots x_i)^{\chi(\sigma_i^{-1} > \sigma_{i+1}^{-1})}. \end{aligned}$$

Thus $x^p = r_{\sigma^{-1}}(X)$. □

Note that the above proposition also may be also found in [5] or [6]. Recall that $\mathbf{R}^\rho(X, Y)$ is the ring of diagonally symmetric polynomials in the alphabets X and Y. Note that within each polynomial $\rho x^p y^q$ there is a unique monomial $x^{p'} y^{q'}$ such that

$$\binom{p'_1}{q'_1} \geq_L \binom{p'_2}{q'_2} \geq_L \cdots \geq_L \binom{p'_n}{q'_n}$$

where

$$\binom{p'_i}{q'_i} \geq_L \binom{p'_j}{q'_j}$$

means

(1) $p'_i > p'_j$; or
(2) if $p'_i = p'_j$ then $q'_i \geq q'_j$

(see [5]). Since $\rho x^p y^q = \rho x^{p'} y^{q'}$ we see that $\mathbf{R}^\rho(X, Y)$ is spanned by the collection

$$\mathcal{PXY} = \{\rho x^p y^q : (p, q) \in \mathcal{DS}\}$$

where

$$\mathcal{DS} = \left\{(p, q) : \binom{p_1}{q_1} \geq_L \binom{p_2}{q_2} \geq_L \cdots \geq_L \binom{p_n}{q_n}\right\}.$$

Now if $\tau(p) = (p'_1, p'_2, \ldots, p'_n)$ and $\tau(q) = (q'_1, q'_2, \ldots, q'_n)$ then we define the *xy-type* $\tau_{xy}(p, q)$ of (p, q) to be $(p'_1, p'_2, \ldots, p'_n, q'_1, q'_2, \ldots, q'_n)$. We now define a total order on the polynomials $\rho x^p y^q$ (where $(p, q) \in \mathcal{DS}$) and say that $\rho x^p y^q >_{xy} \rho x^u y^v$ if and only if

(a) $\tau_{xy}(p, q) >_L \tau_{xy}(u, v)$; or
(b) if $\tau_{xy}(p, q) = \tau_{xy}(u, v)$ then $(p, q) >_L (u, v)$.

Given this, we can now prove the following proposition:

PROPOSITION 3.2. *The collection*

$$\mathcal{MGR} = \{m_\nu(X)\, m_\mu(Y)\, \rho\; r_{\sigma^{-1}}(X)\, g_\sigma(Y) : \sigma \in S_n\} \tag{15}$$

is triangularly related to the set $\mathcal{PXY}$, and thus $\mathcal{MGR}$ spans $\mathbf{R}^\rho(X, Y)$.

Proof. Let $(p, q) \in \mathcal{DS}$. As was seen in Section 2, q decomposes into (μ, γ). Now let $\sigma \in S_n$ be such that $g_\sigma(Y) = y^\gamma$. Set $x^\delta = r_{\sigma^{-1}}(X)$. Note that $(\delta, \gamma) \in \mathcal{DS}$. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ where $\nu_i = p_i - \delta_i$.

Now define $D_i = \{j : \delta_j = i\}$ and $C_i = \{j : \gamma_j = i\}$. Suppose that d is the largest entry of δ and g is the largest entry of γ. Set

$$S_D = S_{D_0} \times S_{D_1} \times \cdots \times S_{D_d}$$

and

$$S_C = S_{C_0} \times S_{C_1} \times \cdots \times S_{C_g}$$

Now

$$m_\nu(X)m_\mu(Y)\rho\, r_{\sigma^{-1}}(X)g_\sigma(Y) = \frac{1}{|G_\nu|\,|G_\mu|}\sum_{\alpha\in S_n}\sum_{\beta\in S_n} x^{\alpha\nu+\delta}y^{\beta\mu+\gamma}$$

where $|G_\nu|$ and $|G_\mu|$ denote the cardinality of G_ν and G_μ, respectively. There are three cases that we wish to consider.

(a) $\alpha \notin S_D$ implies that $\tau(\alpha\nu + \delta) <_L \tau(p)$ and thus $\tau(\alpha\nu + \delta, \beta\mu + \gamma) <_L (p, q)$.

(b) Suppose $\alpha \in S_D$ and $\beta \notin S_C$. $\alpha \in S_D$ implies that $\tau(\alpha\nu + \delta) = \tau(p)$ but $\beta \notin S_C$ implies that $\tau(\beta\mu + \gamma) <_L \tau(q)$. Thus once again we have that $\tau(\alpha\nu + \delta, \beta\mu + \gamma) <_L \tau(p, q)$.

(c) Suppose $\alpha \in S_D$ and $\beta \in S_C$. Note that the elements of S_D and S_G fix δ and γ respectively. Thus

$$\alpha\nu + \delta = \alpha(\nu + \delta) = \alpha(p)$$

and

$$\beta\mu + \gamma = \beta(\mu + \gamma) = \beta(q)$$

and thus $\tau_{xy}(\alpha\nu + \delta, \beta\mu + \gamma) = \tau_{xy}(p, q)$. Now

$$\begin{aligned}\rho x^{\alpha\nu+\delta}y^{\beta\mu+\gamma} &= \rho x^{\alpha(p)}y^{\beta(q)}\\ &= \rho x^{p}y^{\alpha^{-1}\beta(q)}\\ &= \rho x^{p}y^{t}\end{aligned}$$

where $(p, t) \in \mathcal{DS}$. Note that $t = \sigma\alpha^{-1}\beta(q)$ where $\sigma \in S_D$. Let us suppose that $t \neq q$. Let s be the smallest integer such that $t_s \neq q_s$. $i, j \in C_k$ and $i < j$ imply that $q_i \geq q_j$. $(p, q) \in \mathcal{DS}$, $i, j \in D_k$ and $i < j$ also imply that $q_i \geq q_j$. Thus $t_s < q_s$. Therefore, we have that $(p, t) \leq_L (p, q)$. Thus

$$m_\nu(X)\, m_\mu(Y)\, \rho\, r_{\sigma^{-1}}(X)\, g_\sigma(Y) = c_{p,q}x^py^q + \sum_{(p',q')<_{xy}(p,q)} c_{p',q'}x^{p'}y^{q'}$$

where $c_{p,q} > 0$. □

This proof provides an algorithm for expanding elements of $\mathbf{R}^\rho(X, Y)$ in terms of the basis $\mathcal{MGR}$ and thus elements of $\mathbf{R}^{*\rho}(X, Y)$ into the basis $\mathcal{GR}$. Suppose that

$$x^p y^q = x_1^2 x_2 y_2^2 y_3 y_4$$

Note

$$\binom{2}{0} \geq_L \binom{1}{2} \geq_L \binom{0}{1} \geq_L \binom{0}{1}.$$

In Section 1, we saw that

$$y_2^2 y_3 y_4 = m_1(Y) y_2 y_3 y_4 - m_4(Y) - y_2 y_3^2 y_4 - y_2 y_3 y_4^2.$$

Thus

$$x_1^2 x_2 y_2^2 y_3 y_4 = m_1(Y) x_1^2 x_2 y_2 y_3 y_4 - m_4(Y) x_1^2 x_2 - x_1^2 x_2 y_2 y_3^2 y_4 - x_1^2 x_2 y_2 y_3 y_4^2$$

and therefore

$$\rho x_1^2 x_2 y_2^2 y_3 y_4 = m_1(Y) \rho x_1^2 x_2 y_2 y_3 y_4 - m_4(Y) \rho x_1^2 x_2 - \rho x_1^2 x_2 y_2 y_3^2 y_4 - \rho x_1^2 x_2 y_2 y_3 y_4^2.$$

Note that $\rho x_1^2 x_2 = m_{2,1}(X)$ and that $\rho x_1^2 x_2 y_2 y_3^2 y_4$ and $\rho x_1^2 x_2 y_2 y_3 y_4^2$ are elements of our basis $\mathcal{GR}$ (hence $\mathcal{MGR}$). Thus the only term that is not already expanded into elements of our basis is $m_1(Y) \rho x_1^2 x_2 y_2 y_3 y_4$. Now $y_2 y_3 y_4$ is a descent monomial, namely it is the descent monomial of 2341, but $x_1^2 x_2$ is not minimal with respect to $y_2 y_3 y_4$. The minimal monomial that corresponds to the sequence (0, 1, 1, 1) is the monomial that corresponds to the sequence (1, 0, 0, 0), or x_1. Thus we must have

$$\nu = p - (1, 0, 0, 0) = (2, 1, 0, 0) - (1, 0, 0, 0) = (1, 1, 0, 0).$$

Thus

$$m_\nu(X) x_1 = x_1^2 x_2 \ + \ x_1^2 x_3 \ + \ x_1^2 x_4 \ + \ x_1 x_2 x_3 \ + \ x_1 x_2 x_4 \ + \ + x_1 x_3 x_4.$$

Letting $m_{1^2}(X) = m_{(1,1,0,0)}(X)$ we have

$$\begin{aligned} m_{1^2}(X) x_1 y_2 y_3 y_4 = {} & x_1^2 x_2 y_2 y_3 y_4 \ + \ x_1^2 x_3 y_2 y_3 y_4 \ + \ x_1^2 x_4 y_2 y_3 y_4 \\ & + x_1 x_2 x_3 y_2 y_3 y_4 \ + \ x_1 x_2 x_4 y_2 y_3 y_4 \ + \ x_1 x_3 x_4 y_2 y_3 y_4. \end{aligned}$$

Recognizing that

$$\rho x_1^2 x_2 y_2 y_3 y_4 = \rho x_1^2 x_3 y_2 y_3 y_4 = \rho x_1^2 x_4 y_2 y_3 y_4$$

and that

$$\rho x_1 x_2 x_3 y_2 y_3 y_4 = \rho x_1 x_2 x_4 y_2 y_3 y_4 = \rho x_1 x_3 x_4 y_2 y_3 y_4$$

gives us that

$$m_{1^2}(X) \rho x_1 y_2 y_3 y_4 = 3 \rho x_1^2 x_2 y_2 y_3 y_4 + 3 \rho x_1 x_2 x_3 y_2 y_3 y_4$$

or better

$$\rho x_1^2 x_2 y_2 y_3 y_4 = \frac{1}{3}\rho m_{(1,1,0,0)} x_1 y_2 y_3 y_4 - \rho x_1 x_2 x_3 y_2 y_3 y_4$$

Now,

$$\rho x_1 x_2 x_3 y_2 y_3 y_4 = \rho x_1 x_2 x_3 y_1 y_2 y_4$$

and both $\rho m_{1^2}(X) x_1 y_2 y_3 y_4$ and $\rho x_1 x_2 x_3 y_1 y_2 y_4$ are elements of $\mathcal{MGR}$. Thus, we have that

$$\begin{aligned}\rho x_1^2 x_2 y_2^2 y_3 y_4 = {} & \frac{1}{3} m_1(Y) \rho m_{1^2}(X) x_1 y_2 y_3 y_4 - m_1(Y) \rho x_1 x_2 x_3 y_1 y_2 y_4 \\ & - m_4(Y) m_{2,1}(X) - \rho x_1^2 x_2 y_2 y_3^2 y_4 - \rho x_1^2 x_2 y_2 y_3 y_4^2.\end{aligned}$$

Now in $\mathbf{R}^{*\rho}(X, Y)$ we have

$$\rho x_1^2 x_2 y_2^2 y_3 y_4 = -\rho x_1^2 x_2 y_2 y_3^2 y_4 - \rho x_1^2 x_2 y_2 y_3 y_4^2.$$

References

1. E. Allen, "A conjecture of Procesi and a new basis for the graded left regular representation of S_n," Ph.D. Thesis, University of California, San Diego, CA, 1991.
2. E. Allen, "A conjecture of Procesi and the straightening algorithm of G.C. Rota," *Proc. Nat. Acad. Sci.* **89** (1992), 3980–3984.
3. A. Garsia, "Combinatorial methods in the theory of Cohen-Macaulay rings," *Adv. Math.* **38** (1980), 229–266.
4. A. Garsia, "Unpublished classroom notes," Winter 1991, University of California, San Diego, CA.
5. A. Garsia and I. Gessel, "Permutation statistics and partitions," *Adv. Math.* **31** (1979), 288–305.
6. B. Gordon, "Two theorems on multipartite partitions," *J. London Math. Soc.* **38** (1963), 459–464.
7. P.A. MacMahon, *Combinatory Analysis I-II*, Cambridge University Press, London/New York, 1916; Chelsea, New York, 1960.
8. V. Reiner, "Quotients of Coxeter complexes and P-partitions," Mem. Amer. Math. Soc. **95** (460), 1992.
9. R. Stanley, "Ordered structures and partitions," Mem. Amer. Math. Soc. **119**, 1972.
10. R. Steinberg, "On a theorem of Pittie," *Topology* **14** (1975), 173–177.