Journal of Algebraic Combinatorics 3 (1994), 187–206

Hilbert Series of Group Representations and Gröbner Bases for Generic Modules

SHMUEL ONN*
EMAIL: ONN@DIMACS.RUTGERS.EDU
DIMACS, Rutgers University, Piscataway, NJ 08855-1179

Received March 1, 1993; Revised July 30, 1993

Abstract. Each matrix representation π: $G \longrightarrow \mathrm{GL}_n(\mathcal{K})$ of a finite Group G over a field $\mathcal{K}$ induces an action of G on the module $\mathcal{A}^n$ over the polynomial algebra $\mathcal{A} = \mathcal{K}[x_1, \ldots, x_n]$. The graded $\mathcal{A}$-submodule $M(\pi)$ of $\mathcal{A}^n$ generated by the orbit of $(x_1, \ldots, x_n)$ is studied. A decomposition of $M(\pi)$ into generic modules is given. Relations between the numerical invariants of π and those of $M(\pi)$, the latter being efficiently computable by Gröbner bases methods, are examined. It is shown that if π is multiplicity-free, then the dimensions of the irreducible constituents of π can be read off from the Hilbert series of $M(\pi)$. It is proved that determinantal relations form Gröbner bases for the syzygies on generic matrices with respect to any lexicographic order. Gröbner bases for generic modules are also constructed, and their Hilbert series are derived. Consequently, the Hilbert series of $M(\pi)$ is obtained for an arbitrary representation.

Keywords: Gröbner basis, linear representation, generic module, computational algebra, finite group, Hilbert series

1. Introduction

Each matrix representation $\pi : G \longrightarrow \mathrm{GL}_n(\mathcal{K})$ of a finite group G over a field $\mathcal{K}$ induces an action of G on the free $\mathbb{N}$-graded module $\mathcal{A}^n$ over the algebra $\mathcal{A} = \mathcal{K}[x_1, \ldots, x_n]$ of polynomials. In this article we consider the graded $\mathcal{A}$-submodule of $\mathcal{A}^n$ generated by the orbit

$$\pi(G)(x) = \left\{ \pi(g) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : g \in G \right\} \subset \mathcal{A}^n.$$

When $\mathcal{K}$ is algebraically closed and its characteristic does not divide the group order, we give a decomposition (Theorem 5.1) of this module $M(\pi)$, which reflects the decomposition of π into irreducible representations. The basic components of this decomposition are certain generic modules, an up-to-date exposition about which can be found in the monograph [3].

*Research was supported by Institut Mittag-Leffler, by the Mathematical Sciences Institute at Cornell University, and by the Center for Discrete Mathematics and Theoretical Computer Science at Rutgers University.

We are interested in the possibility of recovering numerical invariants of the representation π—the multiplicities and dimensions of its irreducible constituents—from numerical invariants of the module $M(\pi)$—its Hilbert series and Betti numbers. Since the latter can be computed quite efficiently by Gröbner bases methods, this may provide a way of computing the representation invariants as well. We use our decomposition theorem to derive expressions for the rank $\mathrm{rk}(\pi)$, the first and second Betti numbers $\beta_0(\pi)$, $\beta_1(\pi)$, and the Hilbert series $H(\pi, t)$ of $M(\pi)$. We show that, if the representation is multiplicity-free, then the Hilbert series suffices to recover the dimensions of the irreducible constituents of π.

Then we turn to study generic modules, those generated by the columns of a generic matrix, from a Gröbner bases perspective. We prove that determinantal relations form a Gröbner basis for the syzygies on a generic matrix with respect to any lexicographic order, and construct Gröbner bases for the generic modules themselves as well. As a by-product of our consideration of initial modules, we obtain the Hilbert series of a generic module. While the Hilbert series could be also computed from the free resolution (See [5] and [3, Section 2]), our derivation is direct and self-contained. As a result, we obtain the expression for the Hilbert series $H(\pi, t)$ of an arbitrary representation π.

The study of the module generated by the orbit $\pi(G)(x)$ is interesting also from a geometric point of view. If $\mathcal{F}$ is an extension field of $\mathcal{K}$ then a point in $\mathcal{F}^n$ is *generic* if its coordinates are algebraically independent over $\mathcal{K}$. The module structure of $\pi(G)(x)$ gives information on the structure of the orbit (under the induced action of G on $\mathcal{F}^n$) of any generic point. In this respect it is desirable, though will not be carried here, to study (for example, the rank of) the $\mathcal{A}$-submodules generated by arbitrary subsets of the orbit $\pi(G)(x)$. These submodules arise in the study of matroids and other objects defined on the G-orbits and in the study of the stratifications of $\mathcal{F}^n$ induced by them [11].

Though we assume here that the field $\mathcal{K}$ is algebraically closed, we note that in some cases of interest (see [11]), such as representations of the symmetric group, these results hold over the reals and rationals as well.

The paper is organized as follows. In the next section we fix some terminology. In Section 3, the module $M(\pi)$ of a linear representation is intrinsically defined by means of the symmetric algebra. In Section 4, we provide a decomposition of $M(\pi)$ in terms of the modules $M(\pi_i)$ of the isotypic components of π. In Section 5 we introduce generic modules and provide a decomposition of the module of an isotypic representation $\pi = m\chi$, and obtain Theorem 5.1, which provides the decomposition of $M(\pi)$ for an arbitrary π. As a result, in Corollary 5.1 we obtain $\mathrm{rk}(\pi)$ and $\beta_0(\pi)$, and express $H(\pi, t)$ in terms of the Hilbert series of generic modules. In Section 6 we show that the dimensions of all irreducible constituents of a multiplicity-free representation can be read off from its Hilbert series. In Section 7 we prove that determinantal relations form a Gröbner basis for the syzygies on the columns of a generic matrix with respect to any lexicographic order. As a result, in Corollary 7.1 we obtain $\beta_1(\pi)$. In Section 8 we construct

Gröbner bases for generic modules and compute their Hilbert series, resulting in an explicit expression for $H(\pi, t)$ given in Corollary 8.1. We conclude by raising some questions.

2. Terminology

We start by fixing some terminology on graded modules and finite group representations. Some general references for the former are [1, 9, 13], and for the latter [6, 16].

Let $\mathcal{K}$ be a field. A $\mathcal{K}$-algebra $\mathcal{A}$ *is graded* if as a $\mathcal{K}$-space it has a direct sum decomposition $\mathcal{A} = \oplus_{d\geq 0}\mathcal{A}_d$ such that $\mathcal{A}_0 = \mathcal{K}$ and $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ for all $i, j \in \mathbb{N}$. Typically $\mathcal{A}$ will be the algebra $\mathcal{K}[x_1, \ldots, x_n]$ of polynomials which is graded by letting $\mathcal{A}_d$ be the $\mathcal{K}$-space of d-forms in $\mathcal{A}$ (homogeneous polynomials of degree d). Given a graded $\mathcal{K}$-algebra, a *graded $\mathcal{A}$-module* is an $\mathcal{A}$-module M admitting a $\mathcal{K}$-space decomposition $M = \oplus_{d\geq 0}M_d$ such that $\mathcal{A}_iM_j \subseteq M_{i+j}$ $(i, j \in \mathbb{N})$. The elements in M_d are called *homogeneous of degree d*. A module which will be often used is the free $\mathcal{A}$-module $\mathcal{A}^m$ of m-tuples of elements from $\mathcal{A}$, equipped with the standard basis $\{u_1, \ldots, u_m\}$. A *graded submodule* N of M is one that is generated by homogeneous elements in M, in which case it inherits the grading $N_d = N \cap M_d$. If N is a graded submodule of M then its quotient is also graded by letting $(M/N)_d$ be the quotient of $\mathcal{K}$-spaces M_d/N_d. If M^1, M^2 are graded $\mathcal{A}$-modules, then their direct sum is graded by letting $(M^1 \oplus M^2)_d$ be the $\mathcal{K}$-space direct sum $M^1_d \oplus M^2_d$. If $f^1, \ldots, f^m$ are elements of an $\mathcal{A}$-module, M, then an element $g = \Sigma_{i=1}^m g_iu_i \in \mathcal{A}^m$ is a *syzygy* on $f^1, \ldots, f^m$ if $\Sigma_{i=1}^m g_if^i = 0$ in M. The set of such syzygies forms an $\mathcal{A}$-submodule of $\mathcal{A}^m$.

The *rank* of a graded $\mathcal{A}$-module M is the largest size $\mathrm{rk}(M)$ of a subset of M admitting no nontrivial syzygies. If $\mathcal{A} = \mathcal{K}[x_1, \ldots, x_n]$ and M is a submodule of $\mathcal{A}^m$, then letting $\mathcal{F} = \mathcal{K}(x_1, \ldots, x_n)$ be the field of fractions of $\mathcal{A}$, the rank of M equals the dimension of the $\mathcal{F}$-subspace of $\mathcal{F}^m$ generated by $M \subset \mathcal{A}^m \subset \mathcal{F}^m$. The *Hilbert series* of M is the dimension generating function

$$H(M, t) = \sum_{d\geq 0} \dim_{\mathcal{K}}(M_d)t^d$$

in the ring $\mathbb{Z}[[t]]$ of univariate power series. For example, $H(\mathcal{K}[x_1, \ldots, x_n], t) = \frac{1}{(1-t)^n}$. If N is a graded submodule of M then $H(N, t) = H(M, t) - H(M/N, t)$. The *first Betti number* $\beta_0(M)$ of M is the minimum number of generators of M, and its *second Betti number* $\beta_1(M)$ is the minimum number of generators of the module of syzygies on any minimal generating set for M. If M^1 and M^2 are two graded $\mathcal{A}$-modules then $\mathrm{rk}(M^1 \oplus M^2) = \mathrm{rk}(M^1) + \mathrm{rk}(M^2)$ and, similarly,

$$H(M^1 \oplus M^2, t) = H(M^1, t) + H(M^2, t),$$
$$\beta_i(M^1 \oplus M^2) = \beta_i(M^1) + \beta_i(M^2) \quad (i = 0, 1).$$

If $\mathcal{B} = \mathcal{K}[x_1, \ldots, x_m] \subset \mathcal{A} = \mathcal{K}[x_1, \ldots, x_n]$ then $\mathcal{A}$ is a $\mathcal{B}$-module, so that given another $\mathcal{B}$-module M, we have he *scalar extension* $\mathcal{A} \otimes_{\mathcal{B}} M$ of M which is an $\mathcal{A}$-module. If M is a submodule of some $\mathcal{B}^d$ then $\mathrm{rk}(\mathcal{A} \otimes_{\mathcal{B}} M) = \mathrm{rk}(M)$ and

$$H(\mathcal{A} \otimes_{\mathcal{B}} M, t) = \frac{1}{(1-t)^n}(1-t)^m H(M, t), \quad \beta_i(\mathcal{A} \otimes_{\mathcal{B}} M) = \beta_i(M) \quad (i = 0, 1).$$

A *linear representation* of a finite group G over $\mathcal{K}$ is a homomorphism $\pi : G \longrightarrow \mathrm{GL}(V)$ where V is a $\mathcal{K}$-vector space. We will always assume that V is finite dimensional. The *dimension* of π is $\dim(V)$. Any choice of basis of V identifies it with $\mathcal{K}^n$ via the standard basis $U = \{u_1, \ldots, u_n\}$ of unit column vectors in $\mathcal{K}^n$, so any map in $L(V) = \mathrm{Hom}_{\mathcal{K}}(V, V)$ becomes a matrix in $\mathcal{K}^{n \times n}$ with respect to the chosen basis and the linear representation π becomes a *matrix representation* $\pi : G \longrightarrow \mathrm{GL}_n(\mathcal{K})$. A subspace U of V is *π-invariant* if $\pi(G)(U) \subseteq U$. The representation is *irreducible* over $\mathcal{K}$ if it has no proper invariant subspace. Two repersentation $\pi_i : G \longrightarrow \mathrm{GL}(V_i)$ $(i = 1, 2)$ are *isomorphic* if there is a $\mathcal{K}$-isomorphism $T : V_1 \longrightarrow V_2$ such that $\pi_2(g)(T(v)) = T(\pi_1(g)(v))$ for all $g \in G$ and $v \in V_1$.

Further facts on graded modules and group representations will be recalled when necessary.

3. A graded module of a linear representation

We now give an intrinsic definition of the module discussed in the introduction, making it apparent that its numerical invariants remain invariant under isomorphism of matrix representations.

For a finite dimensional $\mathcal{K}$-vector space U, let $\mathcal{S}U = \oplus_{d \geq 0} \mathcal{S}_d U$ be the graded *symmetric algebra* of the space U (see for example [9]). Thus, any basis $F = \{f_1, \ldots, f_n\}$ of U generates $\mathcal{S}U$ freely as a $\mathcal{K}$-algebra, and the dth summand $\mathcal{S}_d U$ is the $\mathcal{K}$-span of formal monomials $\Pi_{i=1}^n f_i^{a_i}$ of degree d. In particular $\mathcal{S}_1 U = U$ and $\mathcal{S}_0 U = \mathcal{K}$. Now, given another vector space V, consider the graded $\mathcal{S}U$-module $\mathcal{S}U \otimes V$ (tensor products are over $\mathcal{K}$ unless otherwise indicated), where the scalar multiplication is given by $s(t \otimes v) = (st) \otimes v$ for $s, t \in \mathcal{S}U$ and $v \in V$, and the grading is $(\mathcal{S}U \otimes V)_d = \mathcal{S}_d U \otimes V$. In particular, $(\mathcal{S}U \otimes V)_1 = U \otimes V$. It is a free module of rank n: any basis $E = \{e_1, \ldots, e_n\}$ of V gives an $\mathcal{S}U$-basis $\{1 \otimes e_1, \ldots, 1 \otimes e_n\}$ of $\mathcal{S}U \otimes V$. We will be mostly interested in the algebra and module above when $U = V^*$ is the dual space of V, and reserve the special notation $\mathcal{M} = \mathcal{S}V^* \otimes V$ for the corresponding $\mathcal{S}V^*$-module in this case. For a basis $E = \{e_1, \ldots, e_n\}$ of V we denote by $E^* = \{e_1^*, \ldots, e_n^*\}$ the unique dual basis of V^* satisfying $e_i^*(e_j) = \delta_{i,j}$ for all i, j. Using the natural isomorphism of $\mathcal{K}$-spaces between $V^* \otimes V$ and $L(V)$ which, for any basis $E = \{e_1, \ldots, e_n\}$ of V and its dual $E^* = \{e_1^*, \ldots, e_n^*\}$, is given by

$$\Psi : L(V) \longrightarrow V^* \otimes V : T \mapsto \sum_{i=1}^{n} e_i^* \otimes T(e_i),$$

we have that

$$\Psi(\pi(G)) = \{\Psi(\pi(g)) : g \in G\} \subset V^* \otimes V = (\mathcal{S}V^* \otimes V)_1 = \mathcal{M}_1,$$

and we can make the following definition.

Definition 3.1. The *graded module* $M(\pi)$ of a linear representation $\pi : G \longrightarrow \mathrm{GL}(V)$, where G is a finite group and V is a finite dimensional $\mathcal{K}$-space, will be the graded $\mathcal{S}V^*$-submodule of $\mathcal{M}$ generated by $\Psi(\pi(G))$. Its rank, first and second Betti numbers, and Hilbert series will be denoted by $\mathrm{rk}(\pi)$, $\beta_0(\pi)$, $\beta_1(\pi)$, and $H(\pi, t)$ respectively.

We now show that $M(\pi)$ is an intrinsic version of the module generated by $\pi(G)(x)$ discussed in the Introduction. Let $E = \{e_1, \dots, e_n\}$ be an arbitrary basis of V, so that V is identified with $\mathcal{K}^n$ and the linear representation becomes a matrix representation $\pi : G \longrightarrow \mathrm{GL}_n(\mathcal{K})$. Letting $E^* = \{e_1^*, \dots, e_n^*\}$ be the basis dual to E, the algebra $\mathcal{S}V^*$ is identified with $\mathcal{A} = \mathcal{K}[x_1, \dots, x_n]$ via the unit preserving isomorphism of graded $\mathcal{K}$-algebras

$$\mathcal{A} \longrightarrow \mathcal{S}V^* : x_i \mapsto e_i^*,$$

so that the module $\mathcal{M} = \mathcal{S}V^* \otimes V$ becomes an $\mathcal{A}$-module, and as such is identified with $\mathcal{A}^n$ via the isomorphism of graded $\mathcal{A}$-modules

$$\mathcal{M} \longrightarrow \mathcal{A}^n : 1 \otimes e_i \mapsto u_i.$$

For any $T \in \mathcal{K}^{n\times n} \cong L(V)$, this isomorphism takes $\Psi(T) \in \mathcal{M}$ to the element of $\mathcal{A}^n$

$$\sum_{i=1}^{n} x_i T u_i = T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Tx,$$

so the set $\Psi(\pi(G))$ is mapped bijectively onto the orbit

$$\pi(G)(x) = \{\pi(g)x : g \in G\} \subset \mathcal{A}^n.$$

Therefore, the module $M(\pi)$ generated by $\Psi(\pi(G))$ can be identified with the $\mathcal{A}$-submodule of $\mathcal{A}^n$ generated by $\pi(G)(x)$.

4. Decomposition induced by isotypic components

The observation below follows immediately from the definition of a graded module.

Observation 4.1. Let $\mathcal{A}$ be a graded $\mathcal{K}$-algebra and M a graded $\mathcal{A}$-module. If M is generated as an $\mathcal{A}$-module by elements from M_d for some d, then $S \subset M$ minimally generates M as an $\mathcal{A}$-module if and only if it is a $\mathcal{K}$-basis of M_d. In particular, $\beta_0(M) = \dim_{\mathcal{K}}(M_d)$.

The *enveloping space* of a linear representation $\pi : G \longrightarrow \mathrm{GL}(V)$ is the $\mathcal{K}$-subspace

$$\mathcal{E}(\pi) = span_{\mathcal{K}}(\pi(G)) \subseteq L(V).$$

Letting $\Psi : L(V) \longrightarrow V^* \otimes V = \mathcal{M}_1$ be the natural isomorphism as in Section 3, we obtain the following statement.

PROPOSITION 4.1. *The image $\Psi(S)$ of a subset $S \subset L(V)$ minimally generates $M(\pi)$ as an SV^*-module if and only if S is a $\mathcal{K}$-basis for the enveloping space $\mathcal{E}(\pi)$ of π.*

Proof. By definition, $M(\pi)$ is generated by $\Psi(\pi(G)) \subset M(\pi)_1$. Therefore, as observed above, a set T minimally generates $M(\pi)$ if and only if it is a $\mathcal{K}$-basis for $span_{\mathcal{K}}(\Psi\pi(G)) = M(\pi)_1$. Since Ψ is an isomorphism of $\mathcal{K}$-spaces, this is the case if and only if $T = \Psi(S)$ for a $\mathcal{K}$-basis S of $span_{\mathcal{K}}(\pi(G)) = \mathcal{E}(\pi)$. □

If $V = \oplus_{i=1}^k V_i$ is a decomposition of a $\mathcal{K}$-space V, then $V^* = \oplus_{i=1}^k V_i^*$ is a *dual decomposition* if for all i, each functional in V_i^* vanishes on $\oplus_{j \neq i} V_j$. Let $V = V_1 \oplus V_2$ and let $V^* = V_1^* \oplus V_2^*$ be a dual decomposition. The inclusion $V_1^* \longrightarrow V^*$ extends to an inclusion of algebras $\mathcal{S}V_1^* \longrightarrow \mathcal{S}V^*$, turning $\mathcal{S}V^*$ into a graded $\mathcal{S}V_1^*$-module. Given a subset $S \subset V^* \otimes V$, we write $\langle S \rangle$ for the $\mathcal{S}V^*$-submodule of $\mathcal{S}V^* \otimes V$ generated by it, and if further $S \subset V_1^* \otimes V$, we write $\langle S \rangle_{\mathcal{S}V_1^*}$ for the $\mathcal{S}V_1^*$-submodule of $\mathcal{S}V_1^* \otimes V$ generated by it. We omit the proofs of the next two simple statements.

PROPOSITION 4.2. *If $B_i \subset V^* \otimes V_i$ $(i = 1, 2)$, then $\langle B_1 \cup B_2 \rangle = \langle B_1 \rangle \oplus \langle B_2 \rangle$ as graded $\mathcal{S}V^*$-modules.*

PROPOSITION 4.3. *For any subset $S \subset V_1^* \otimes V$ we have that $\langle S \rangle$ is isomorphic as a graded $\mathcal{S}V^*$-module to the scalar extension $\mathcal{S}V^* \otimes_{\mathcal{S}V_1^*} \langle S \rangle_{\mathcal{S}V_1^*}$.*

We now need to review some more material from representation theory. Let $\pi = \oplus_{i=1}^k \pi_i$ be a *direct sum decomposition* of a linear representation $\pi : G \longrightarrow \mathrm{GL}(V)$, i.e., there is a corresponding decomposition $V = \oplus_{i=1}^k V_i$ of V into π-invariant subspaces, and π_i is the restriction of π to V_i. Let $V^* = \oplus_{i=1}^k V_i^*$ be a dual decomposition of V^*. Then for all i we have an embedding of $L(V_i)$ in $L(V)$ given via Ψ by

$$L(V_i) = \Psi^{-1}(V_i^* \otimes V_i) \subset \Psi^{-1}(V^* \otimes V) = L(V).$$

In particular, we have $\mathcal{E}(\pi_i) \subset \mathcal{E}(\pi)$. If each π_i in the decomposition of π is in turn a direct sum $\pi_i = \oplus_{j=1}^{m_i} \chi_i$ of copies of the same irreducible representation χ_i of G over $\mathcal{K}$ and the χ_i are pairwise nonisomorphic, then the π_i are the *isotypic components* of π. For proofs of the following fact consult, for example, [16, §4.5, Corollaries 1, 5; 6, Theorem of Frobenius and Schur (27.8)].

PROPOSITION 4.4. *If $\pi = \oplus_{i=1}^{k} \pi_i$ is a decomposition into isotypic components of a linear representation π over an algebraically closed field, then $\mathcal{E}(\pi) = \oplus_{i=1}^{k} \mathcal{E}(\pi_i)$.*

We can now prove the following lemma, which reduces the decomposition problem into that for isotypic representations.

LEMMA 4.1. *Let $\pi = \oplus_{i=1}^{k} \pi_i$ be a decomposition of a linear representation π over an algebraically closed field into isotypic components with corresponding decomposition $V = \oplus_{i=1}^{k} V_i$, and let $V^* = \oplus_{i=1}^{k} V_i^*$ be a dual decomposition of V^*. Then the representation module $M(\pi)$ has a decomposition into a direct sum of graded $\mathcal{S}V^*$-modules*

$$M(\pi) \cong \oplus_{i=1}^{k} \mathcal{S}V^* \otimes_{\mathcal{S}V_i}^{*} M(\pi_i).$$

Proof. For $i = 1, \dots, k$ let B_i be a $\mathcal{K}$-basis for $\mathcal{E}(\pi_i)$. By Proposition 4.4, $\mathcal{E}(\pi) = \oplus_{i=1}^{k} \mathcal{E}(\pi_i)$, so that $B = \cup_{i=1}^{k} B_i$ is a a $\mathcal{K}$-basis for $\mathcal{E}(\pi)$. By Proposition 4.1, $\Psi(B_i)$ generates $M(\pi_i)$ as an $\mathcal{S}V_i^*$-module, so by Proposition 4.3,

$$\langle \Psi(B_i) \rangle \cong \mathcal{S}V^* \otimes_{\mathcal{S}V_i^*} \langle \Psi(B_i) \rangle_{\mathcal{S}V_i^*} = \mathcal{S}V^* \otimes_{\mathcal{S}V_i^*} M(\pi_i).$$

Now, $\Psi(B_i) \subset V^* \otimes V_i$ so by Proposition 4.2, and using Proposition 4.1 once more,

$$M(\pi) = \langle \Psi(B) \rangle = \langle \cup_{i=1}^{k} \Psi(B_i) \rangle = \oplus_{i=1}^{k} \langle \Psi(B_i) \rangle \cong \oplus_{i=1}^{k} \mathcal{S}V^* \otimes_{\mathcal{S}V_i^*} M(\pi_i)$$

as claimed. □

5. Decomposition into generic modules

Having decomposed the representation module into the modules of the isotypic components of the representation, we now turn to obtain a decomposition of an isotypic representation.

For an isomorphism of $\mathcal{K}$-spaces $\phi : V \longrightarrow U$ let $\phi^* : V^* \longrightarrow U^*$ denote the dual isomorphism. If $\{e_1, \dots, e_n\}$ is any basis of V, $\{e_1^*, \dots, e_n^*\}$ the dual basis of V^*, $f_i = \phi(e_i)$, and $\{f_1^*, \dots, f_n^*\}$ the basis of U^* dual to the basis $\{f_1, \dots, f_n\}$ of U, then ϕ^* is given by $\phi^*(e_i^*) = f_i^*$. We denote by Φ the isomorphism of the tensor products induced from ϕ, which is given by

$$\Phi = \phi^* \otimes \phi : V^* \otimes V \longrightarrow U^* \otimes U : e_i^* \otimes e_j \mapsto f_i^* \otimes f_j.$$

Now let $\pi : G \longrightarrow \mathrm{GL}(V)$ be a linear representation with decomposition $\pi = \sum_{i=1}^{m} \pi_i \cong m\pi_1$, i.e., the π_i are isomorphic copies of a single representation π_1, and let $V = \oplus_{i=1}^{m} V_i$ and $V^* = \oplus_{i=1}^{m} V_i^*$ be corresponding decompositions of V and its dual. Let ϕ_1 be the identity on V_1. For $i = 2, \ldots, m$ let $\phi_i : V_1 \longrightarrow V_i$ be an isomorphism of $\mathcal{K}$-spaces yielding an isomorphism of the representations π_1 and π_i, i.e. $\phi_i(\pi_1(g)(v)) = \pi_i(g)(\phi_i(v))$ for all $g \in G$ and $v \in V_1$, and let Φ_i be the extension of ϕ_i to $V_i^* \otimes V_i$ as above. Let $\Psi : L(V) \longrightarrow V^* \otimes V$ and $\Psi_i : L(V_i) \longrightarrow V_i^* \otimes V_i$ $(i = 1, \ldots, m)$ be the natural isomorphisms of $\mathcal{K}$-spaces. Then for all i and all $g \in G$ we have $\Phi_i(\Psi_1(\pi_1(g))) = \Psi_i(\pi_i(g))$. Now, $V_i^* \otimes V_i \subset V^* \otimes V$ so the image of Ψ_i is contained in $V^* \otimes V$ for all i. Thus, we get

$$\Psi(\pi(g)) = \sum_{i=1}^{m} \Psi_i(\pi_i(g)) = \sum_{i=1}^{m} \Phi_i(\Psi_1(\pi_1(g))),$$

and we obtain the following statement.

PROPOSITION 5.1. *Let $\pi = \sum_{i=1}^{m} \pi_i \cong m\pi_1$ and let Ψ_i and Φ_i be the associated $\mathcal{K}$-maps as defined above. If the set $S \subseteq L(V_1)$ spans $\mathcal{E}(\pi_1)$, then the set $\sum_{i=1}^{m} \Phi_i(\Psi_1(S))$ generates $M(\pi)$ as an SV^*-module.*

Proof. If S spans $\mathcal{E}(\pi_1)$ then by the above relation,

$$\begin{aligned} span_{\mathcal{K}}\left(\Psi^{-1}\sum_{i=1}^{m}\Phi_i\Psi_1(S)\right) &= \Psi^{-1}\sum_{i=1}^{m}\Phi_i\Psi_1(span_{\mathcal{K}}(S)) = \Psi^{-1}\sum_{i=1}^{m}\Phi_i\Psi_1(\mathcal{E}(\pi_1)) \\ &= \Psi^{-1}\sum_{i=1}^{m}\Phi_i\Psi_1(span_{\mathcal{K}}(\pi_1(G))) \\ &= span_{\mathcal{K}}\left(\Psi^{-1}\sum_{i=1}^{m}\Phi_i\Psi_1\pi_1(G)\right) \\ &= span_{\mathcal{K}}(\pi(G)) = \mathcal{E}(\pi), \end{aligned}$$

so by Proposition 4.1, $M(\pi)$ is generated by

$$\Psi\left(\Psi^{-1}\sum_{i=1}^{m}\Phi_i\Psi_1(S)\right) = \sum_{i=1}^{m}\Phi_i(\Psi_1(S)). \qquad \square$$

We need one more fact from representation theory (see again [16, §4.5, Corollaries 1, 5; 6, Theorem of Frobenius and Schur (27.8)]).

PROPOSITION 5.2. *If $\chi : G \longrightarrow GL(V)$ is an irreducible representation of a finite group over an algebraically closed field then $\mathcal{E}(\chi) = L(V) = \Psi^{-1}(V^* \otimes V)$.*

Now let $\pi : G \longrightarrow \mathrm{GL}(V)$ be an *isotypic representation* over an algebraically closed field $\mathcal{K}$, i.e., $\pi = \sum_{i=1}^{m} \chi_i \cong m\chi_1$ where χ_1 is an irreducible representation of G over $\mathcal{K}$. Let $V = \oplus_{i=1}^{m} V_i$ and $V^* = \oplus_{i=1}^{m} V_i^*$ be corresponding decompositions of V and its dual, and ϕ_i, Φ_i and Ψ_i as defined above. Combining Proposition 5.2 and Proposition 5.1, we get that, if B is any basis of $V_1^* \otimes V_1$, then $\sum_{i=1}^{m} \Phi_i(B)$ generates $M(\pi)$ as an $\mathcal{S}V^*$-module. Now, let $n = \dim(\chi_i) = \dim(V_i)$, and choose a basis

$$E = \{e_{i,j} : 1 \le i \le m,\ 1 \le j \le n\}$$

for V such that $\{e_{i,j} : 1 \le j \le n\}$ is a basis of V_i and so that the isomorphism ϕ which gives the isomorphism of χ_1 and χ_i as above, is given by

$$\phi_i : V_1 \longrightarrow V_i : e_{1,j} \mapsto e_{i,j}$$

for all i. Letting $E^* = \{e_{i,j}^* : 1 \le i \le m,\ 1 \le j \le n\}$ be the basis dual to E, we have $\Phi_k(e_{1,i}^* \otimes e_{1,j}) = e_{k,i}^* \otimes e_{k,j}$ for all k. We get the following Proposition.

PROPOSITION 5.3. *Let $\pi = \sum_{i=1}^{m} \chi_i \cong m\chi_1$ be an isotypic representation over an algebraically closed field, and let E and E^* be the bases of the representation space and its dual as above. Then the representation module $M(\pi)$ is minimally generated by the set*

$$S = \left\{ \sum_{k=1}^{m} e_{k,i}^* \otimes e_{k,j} : 1 \le i, j \le n \right\}.$$

Proof. Choose the basis $B = \{e_{1,i}^* \otimes e_{1,j} : 1 \le i, j \le n\}$ of $V_1^* \otimes V_1$. Then $M(\pi)$ is generated by

$$\sum_{k=1}^{m} \Phi_k(B) = \left\{ \sum_{k=1}^{m} \Phi_k(e_{1,i}^* \otimes e_{1,j}) : 1 \le i, j \le n \right\} = S. \qquad \square$$

For $j = 1, \ldots, n$ let

$$U_j = span_{\mathcal{K}}(\{e_{k,j} : 1 \le k \le m\}), \quad S_j = \left\{ \sum_{k=1}^{m} e_{k,i}^* \otimes e_{k,j} : 1 \le i \le n \right\}.$$

Then $V = \oplus_{j=1}^{k} U_j$ and $S_j \subset V^* \otimes U_j$, so by Proposition 4.2,

$$M(\pi) = \langle S \rangle = \langle \cup_{j=1}^{n} S_j \rangle = \oplus_{j=1}^{n} \langle S_j \rangle$$

as $\mathcal{S}V^*$-modules. Now let $\mathcal{B} = \mathcal{K}[x_{1,1,}, \ldots, x_{m,n}]$, so that all $\mathcal{S}V^*$-modules become $\mathcal{B}$-modules via the algebra isomorphism

$$\mathcal{B} \longrightarrow \mathcal{S}V^* : x_{k,i} \mapsto e_{k,i}^*.$$

Let $\mathcal{B}^m$ be the free $\mathcal{B}$-module with standard basis $\{u_1, \ldots, u_m\}$. Then for every j we have an isomorphism of graded $\mathcal{B}$-modules

$$\mathcal{S}V^* \otimes U_j \longrightarrow \mathcal{B}^m : 1 \otimes e_{k,j} \mapsto u_k.$$

This isomorphism takes the element $\sum_{k=1}^m e_{k,i}^* \otimes e_{k,j} \in S_j$ to the element $\sum_{k=1}^m x_{k,i} u_k$ of $\mathcal{B}^m$. Thus, S_j is mapped to the set of columns of an $m \times n$ generic matrix over $\mathcal{B}$, and the $\mathcal{B}$-module $\langle S_j \rangle$ is mapped isomorphically to a generic module $M(m, n)$ which we now define.

Definition 5.1. The generic $m \times n$ module $M(m, n)$ is the $\mathcal{B}$-submodule of $\mathcal{B}^m$ generated by the set $\{\sum_{k=1}^m x_{k,i} u_k : 1 \leq i \leq n\}$ of columns of the generic $m \times n$ matrix

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix}.$$

Remark. We point out a slight difference from terminology used elsewhere: while we reserve the term *generic module* for the *image* of the map $\mathcal{B}^n \rightarrow \mathcal{B}^m$ given by the matrix above, this term is often used (cf. [3]) for its cokernel $\mathcal{B}^m / M(m, n)$.

We have proved the following lemma.

LEMMA 5.1. *Let $\pi = m\chi$ be an isotypic representation over an algebraically closed field and let $n = \dim(\chi)$. Any choice of basis for the representation space yields an isomorphism of graded $\mathcal{B}$-modules*

$$M(\pi) \cong \oplus_{i=1}^n M(m, n) \cong nM(M, n).$$

Note the difference between the decompositions of π and $M(\pi)$: while π is a direct sum of m copies, the module $M(\pi)$ is a direct sum of n copies.

Now let $\pi : G \longrightarrow \mathrm{GL}(V)$ be an arbitrary linear representation of a finite group G over an algebraically closed field $\mathcal{K}$, the characteristic of which does not divide the group order. By a theorem of Maschke (See e.g. [6]), π is *completely reducible*, i.e., has a decomposition $\pi = \sum_{i=1}^k m_i \chi_i$ where the χ_i are pairwise nonisomorphic irreducible representations of G over $\mathcal{K}$. Let $n = \dim(\pi)$ and $n_i = \dim(\chi_i)$. For $i = 1, \ldots, k$ let $\mathcal{A}(i) = \mathcal{K}[x_1, \ldots, x_{m_i n_i}]$ be the subalgebra of $\mathcal{A} = K[x_1, \ldots, x_n]$ generated by the first $m_i n_i$ indeterminates, so that the generic module $M(m_i, n_i)$ is an $\mathcal{A}(i)$-module. Recall from Section 3 that $M(\pi)$ can be turned into an $\mathcal{A}$-module by choosing an arbitrary basis for V. We can now state the decomposition theorem for our representation module, which follows from Lemmas 4.1 and 5.1.

THEOREM 5.1. *Let π be a linear representation of a finite group G over an algebraically closed field $\mathcal{K}$, the characteristic of which does not divide the group order. Let π have decomposition $\pi = \sum_{i=1}^k m_i\chi_i$ into irreducible representations of G over $\mathcal{K}$, let $n = dim(\pi)$ and $n_i = dim(\chi_i)$, and let $\mathcal{A}$ and $\mathcal{A}(i)$ be as above. Any choice of basis for the representation space yields an isomorphism of graded $\mathcal{A}$-modules*

$$M(\pi) \cong \oplus_{i=1}^k n_i(\mathcal{A} \otimes_{\mathcal{A}(i)} M(m_i, n_i)).$$

The rank of a generic $m \times n$ matrix over the field of fractions of $\mathcal{B} = \mathcal{K}[x_{1,1}, \ldots, x_{m,n}]$ is $\min\{m, n\}$, so (see Section 2) $\mathrm{rk}(M(m, n)) = \min\{m, n\}$. Also by Observation 4.1, $\beta_0(M(m, n)) = \dim_{\mathcal{K}}(M(m, n)_1) = n$ since $M(m, n)$ is generated by elements in $M(m, n)_1$. Let $H(m, n; t)$ be the Hilbert series of $M(m, n)$. By Theorem 5.1 and the behavior of the rank, Betti numbers and Hilbert series with respect to scalar extensions and direct sums (see Section 2), we have the following corollary.

COROLLARY 5.1. *The rank, first Betti number, and Hilbert series of a linear representation π as in the hypothesis of Theorem 5.1 satisfy, respectively*

$$\mathrm{rk}(\pi) = \sum_{i=1}^k n_i \cdot \min\{m_i, n_i\}, \quad \beta_0(\pi) = \sum_{i=1}^k n_i^2,$$

$$H(\pi, t) = \frac{1}{(1-t)^n} \sum_{i=1}^k n_i(1-t)^{m_i n_i} H(m_i, n_i; t).$$

In Sections 7 and 8 we will study generic modules in more detail, resulting in explicit expressions for $\beta_1(\pi)$ (Corollary 7.1) and $H(\pi, t)$ (Corollary 8.1) as well. First, however, we treat the case of multiplicity-free representations, for which a special nice property can be derived.

6. Multiplicity-free representations

A linear representation π is *multiplicity-free* if it has a decomposition $\pi = \sum_{i=1}^k \chi_i$ where the χ_i are pairwise nonisomorphic irreducible representations.

THEOREM 6.1. *Let π be a multiplicity-free linear representation of a finite group G over an algebraically closed field $\mathcal{K}$, the characteristic of which does not divide the group order. Let $\pi = \sum_{i=1}^k \chi_i$ be a decomposition of π into irreducible representations of G over $\mathcal{K}$, let $n = \dim(\pi)$ and let $n_i = \dim(\chi_i)$. Then the multiset $\{n_1, \ldots, n_k\}$ of dimensions of the χ_i can be recovered from the Hilbert series of π, which is given by*

$$H(\pi, t) = \frac{1}{(1-t)^n} \sum_{i=1}^k n_i(1-(1-t)^{n_i}) = \frac{1}{(1-t)^n}\left(n - \sum_{i=1}^k n_i(1-t)^{n_i}\right).$$

Proof. The generic $1 \times n$ module $M(1, n)$ is simply the unique maximal homogeneous ideal in $\mathcal{A} = \mathcal{K}[x_1, \dots, x_n]$ generated by $\{x_1, \dots, x_n\}$, and its Hilbert series is

$$H(1, n; t) = H(\oplus_{k \geq 1} \mathcal{A}_k, t) = \frac{1}{(1-t)^n} - 1.$$

Thus, by Corollary 5.1 we have

$$H(\pi, t) = \frac{1}{(1-t)^n} \sum_{i=1}^{k} n_i (1-t)^{n_i} H(1, n_i; t) = \frac{1}{(1-t)^n} \sum_{i=1}^{k} n_i (1 - (1-t)^{n_i}).$$

Now, let $l_1 < l_2 < \cdots < l_r$ be all distinct values taken by the n_i, and for $j = 1, \dots, r$ let m_j be the multiplicity of l_j in $\{n_1, \dots, n_k\}$. It suffices to show that the m_j and l_j can be read off from $H(\pi, t)$. Now, n is determined as the smallest positive integer for which $(1-t)^n H(\pi, t)$ is a polynomial in t. Then, letting $s = 1 - t$, we have

$$\sum_{j=1}^{r} m_j l_j s^{l_j} = n - s^n H(\pi, 1 - s),$$

and it is clear that the m_j and l_j are uniquely determined. $\square$

Example 6.1. Let $\pi_{(n-2,2)}$: $S_n \longrightarrow \mathrm{GL}(\mathbb{C}^{\binom{n}{2}})$ be the $(n-2, 2)$ Young representation of the symmetric group (see [10] and [11]). It is well known that it decomposes as

$$\pi_{(n-2,2)} = \chi_{(n)} \oplus \chi_{(n-1,1)} \oplus \chi_{(n-2,2)},$$

and the dimensions, which can be computed by the Hook formula [10], are 1, $n-1$, and $\frac{n(n-3)}{2}$, in the corresponding order. Thus, its Hilbert series is

$$H(\pi_{(n-2,2)}, t) = \frac{1}{(1-t)^{\binom{n}{2}}} \left(\binom{n}{2} - \left((1-t) + (n-1)(1-t)^{n-1} + \frac{n(n-3)}{2} (1-t)^{\frac{n(n-3)}{2}} \right) \right).$$

7. Gröbner bases for syzygies on generic matrices

The notion of a *Gröbner basis* is of great importance in contemporary computational algebra. In this section and the following one, we study generic modules from a Gröbner bases theoretical perspective. In the present section we prove that, for any m and n, determinantal relations form a Gröbner basis for the module of syzygies on the generic $m \times n$ matrix, with respect to any lexicographic monomial order on $\mathcal{A}^n$. In particular, this provides an elementary proof for the classical fact that these relations generate the syzygies, and that the second Betti number of $M(m, n)$ equals $\binom{n}{m+1}$, which gives the second Betti number of an arbitrary linear representation. In the next section we construct a Gröbner basis for the generic module $M(m, n)$ itself and derive an explicit expression for its Hilbert series, resulting in the Hilbert series of an arbitrary linear representation.

We start by reviewing some material from the theory of monomial orders. Some references are [4; 8, Chapter 6]. Let $\mathcal{A}$ denote the $\mathcal{K}$-algebra of polynomials as usual and for any m, let the free $\mathcal{A}$-module $\mathcal{A}^m$ be equipped with its standard basis $\{u_1, \ldots, u_m\}$. Monomials in $\mathcal{A}$ will be denoted by s and t, while p and q will be polynomials, and we use $f = \sum_{i=1}^m f_i u_i$ and $g = \sum_{i=1}^m g_i u_i$ to denote elements of $\mathcal{A}^m$, where $f_i, g_i \in \mathcal{A}$. A *monomial* in $\mathcal{A}^m$ is an element of the form su_i. An element $f \in \mathcal{A}^m$ is *divisible* by $g \in \mathcal{A}^m$ if $f = pg$ for some $p \in \mathcal{A}$. Monomials form a $\mathcal{K}$-basis for $\mathcal{A}^m$, and we say that su_i *appears* in $f \in \mathcal{A}^m$ or that f *involves* su_i if it appears with a nonzero coefficient in the unique expression of f as a $\mathcal{K}$-linear combination of monomials. A total order $<$ on the set of monomials in $\mathcal{A}^m$ is a *monomial order* if for all monomials $s, t_1, t_2 \in \mathcal{A}$, the conditions $s \neq 1$ and $t_1 u_i < t_2 u_j$ imply $t_1 u_i < st_1 u_i < st_2 u_j$. This specializes to a definition of a monomial order on $\mathcal{A}$. Throughout this section and the next one we will assume that $\mathcal{A}^m$ is equipped with a monomial order. The *initial monomial* $\mathrm{in}(f)$ of f is then the largest monomial appearing in f. A subset $G \subset M$ is a *Gröbner basis* for a graded $\mathcal{A}$-submodule M of $\mathcal{A}^m$ if for every $f \in M$ there exists a $g \in G$ such that $\mathrm{in}(g)$ divides $\mathrm{in}(f)$. If G is a Gröbner basis for M then, in particular, it generates M as an $\mathcal{A}$-module. A *lexicographic monomial order* on $\mathcal{A}$ is one that is induced lexicographically from some total order on the indeterminates in $\mathcal{A}$: if $\mathcal{A} = \mathcal{K}[x_1, \ldots, x_n]$ and $x_1 > \cdots > x_n$, say, then a monomial $\prod_{i=1}^n x_i^{a_i}$ will be larger than another $\prod_{i=1}^n x_i^{b_i}$ if, letting i be the smallest index for which $a_i - b_i \neq 0$, we have $a_i - b_i > 0$. We shall call a monomial order on $\mathcal{A}^m$ *lexicographic* if it is obtained from some lexicographic order on $\mathcal{A}$ and some total order on $[m] = \{1, \ldots, m\}$ in the following way: $su_i > tu_j$ if either $s > t$ in $\mathcal{A}$, or $s = t$ and i is larger than j in the order on $[m]$.

In this section and the next we use X to denote both the generic $m \times n$ matrix as in Definition 5.1 and the set $\{x_{1,1}, \ldots, x_{m,n}\}$ of indeterminates appearing in it. If S is any subset of X then $\mathcal{K}[S]$ stands for the $\mathcal{K}$-algebra of polynomials generated by the indeterminates in S. In particular, we let $\mathcal{A} = \mathcal{K}[X] = \mathcal{K}[x_{1,1}, \ldots, x_{m,n}]$. The *S-content* of a monomial in $\mathcal{A}$ is the largest monomial in $\mathcal{K}[S]$ dividing it. By a *syzygy on* X we mean a syzygy on the columns of X, i.e., an element $f = \sum_{j=1}^n f_j u_j \in \mathcal{A}^n$ satisfying $\sum_{j=1}^n f_j x_{i,j} = 0$ for $i = 1, \ldots, m$. Finally, if $1 \leq i_1, \ldots, i_k \leq m$ and $1 \leq j_1, \ldots, j_k \leq n$ are indices, not necessarily distinct, we use the bracket $[i_1, \ldots, i_k | j_1, \ldots, j_k]$ to denote the corresponding minor of X, i.e., the determinant of the $k \times k$ matrix obtained by restricting X to the rows and columns indexed by the i_l and j_l respectively, reordered as designated in the bracket.

First we observe that there are no nontrivial syzygies on the columns of X if $n \leq m$.

Observation 7.1. If $n \leq m$ then the module $M(m, n)$ generated by the columns of X is free of rank n.

Proof. If $n \leq m$ then $\mathrm{rk}(M(m, n)) = \min\{m, n\} = n = \beta_0(M(m, n))$ (see end of Section 5), which proves the claim. $\square$

It is a classical fact that determinantal relations minimally generate the module of syzygies on X. We now prove that, moreover, they form a Gröbner basis for this module.

THEOREM 7.1. *Let $1 \leq m < n$, let X be a generic $m \times n$ matrix and let $\mathcal{A} = \mathcal{K}[X]$. The set of determinantal relations*

$$G = \left\{ \sum_{l=0}^{m} (-1)^l [1, \ldots, m | j_0, \ldots, \widehat{j_l}, \ldots, j_m] u_{j_l} : 1 \leq j_0 < j_1 < \cdots < j_m \leq n \right\}$$

both minimally generates the module of syzygies on X and is a Gröbner basis for it with respect to any lexicographic monomial order on $\mathcal{A}^n$.

Proof. First, note that for $i = 1, \ldots, m$ we have

$$\sum_{l=0}^{m} (-1)^l [1, \ldots, m | j_0, \ldots, \widehat{j_l}, \ldots, j_m] x_{i, j_l} = [i, 1, \ldots, m | j_0, \ldots, j_m] = 0,$$

which shows that all elements of G are indeed syzygies on X. Second, note that all elements of G are homogeneous of the same degree, so in order to prove that no proper subset of G generates the syzygies, it suffices to show that they are $\mathcal{K}$-linearly independent. This is easily seen to be the case, since each element $\sum_{l=0}^{m} (-1)^l [1, \ldots, m | j_0, \ldots, \widehat{j_l}, \ldots, j_m] u_{j_l}$ of G involves a monomial $\prod_{l=1}^{m} x_{l, j_l} u_{j_0}$ appearing in no other element of G.

It remains to prove that G is a Gröbner basis for the module of syzygies. Let a lexicographic order on $\mathcal{A}^n$ be given. Permuting rows and columns, modifying the total order on $[n]$ which gives the order on $\mathcal{A}^n$, and relabeling indeterminates if necessary, we may assume that $x_{1,1}$ is the $\mathcal{A}$-largest indeterminate. Given $n > m$ and a nontrivial syzygy $f = \sum_{l=1}^{n} f_l u_l$ on X, we will show that there is a determinantal relation on X whose initial monomial divides $\mathrm{in}(f)$. For any m, the proof will use induction on n. We claim that we need consider only syzygies f with all components f_l nonzero. For $n = m + 1$, this is always the case for a nontrivial f by Observation 7.1. For larger n, if some $f_j = 0$, then let $s u_k$ be the initial monomial of f, let $S = \{x_{1,j}, \ldots, x_{m,j}\}$, and let t be the S-content of s. Let $g = \sum_{l=1}^{n} g_l u_l$ be obtained from f by keeping exactly those terms of f involving monomials with S-content t. Let $\mathcal{B} = \mathcal{K}[X \backslash S]$. Then

$$h = \frac{1}{t} \sum_{l \in [n] \backslash \{j\}} g_l u_l \in \mathcal{B}^{n-1}$$

is a syzygy on the columns of the $m \times (n-1)$ generic matrix X' obtained from X by deleting the jth column. Letting $\mathcal{B}$ and $\mathcal{B}^{n-1}$ inherit the monomial orders from $\mathcal{A}$ and $\mathcal{A}^n$ respectively, we will have by induction that $\mathrm{in}(h) = \frac{1}{t}su_k$ is divisible by the initial monomial of some determinantal relation on X'. But this is also a determinantal relation on X, and $\mathrm{in}(f) = t \cdot \mathrm{in}(h)$ is divisible by its initial monomial as well. Thus, for any m and $n > m$ we will consider only syzygies f with all components f_l nonzero. For such syzygies, since $\sum_{l=1}^n x_{i,l} f_l = 0$ for $i = 1, \ldots, m$, we have that each indeterminate $x_{i,l}$ in X must divide some monomial of f.

We now proceed by induction on m. Consider first the case $m = 1$. Let $n > 1$ be arbitrary and consider any syzygy $f = \sum_{l=1}^n f_l u_l$ on X with all f_l nonzero, and let su_k be the initial monomial of f. Since the monomial order is lexicographic, no monomial in f has larger $x_{1,1}$-content than does s, so in particular $x_{1,1}$ divides s. Moreover, since $\sum_{l=1}^n f_l x_{1,l} = 0$, any monomial in f_1 must have smaller $x_{1,1}$-content than does s, so in particular, $k \neq 1$. Thus, the element

$$w = x_{1,k}u_1 - x_{1,1}u_k$$

is a determinantal relation in the set G, whose initial monomial is $x_{1,1}u_k$ which divides $su_k = \mathrm{in}(f)$.

Next, let $m \geq 2$ and assume the claim is true for smaller values of m. Let $n > m$ be arbitrary and consider any syzygy $f = \sum_{l=1}^n f_l u_l$ on X with all f_l nonzero. Let $su_k = \mathrm{in}(f)$ and conclude as above that $x_{1,1}$ divides s. Now let $S = \{x_{1,1}, \ldots, x_{1,n}\} \cup \{x_{1,1}, \ldots, x_{m,1}\}$ and let t be the S-content of s. Since $\sum_{l=1}^n f_l x_{1,l} = 0$ we find again that any monomial in f_1 must have smaller $x_{1,1}$-content than does s, so in particular can not have S-content t. Thus, letting $g = \sum_{l=1}^n g_l u_l$ be obtained from f by keeping exactly those terms of f involving monomials with S-content t, we find that $g_1 = 0$. Let $\mathcal{B} = \mathcal{K}[X \backslash S]$. Then

$$h = \frac{1}{t}\sum_{l=2}^n g_l u_l \in \mathcal{B}^{n-1}$$

is a syzygy on the columns of the $(m-1) \times (n-1)$ generic matrix X' obtained from X by deleting the first row and first column. Letting $\mathcal{B}$ and $\mathcal{B}^{n-1}$ inherit the monomial orders from $\mathcal{A}$ and $\mathcal{A}^n$ respectively, we have by induction that $\mathrm{in}(h) = \frac{1}{t}su_k$ is divisible by the initial monomial of some determinantal relation on X'. Let this relation be

$$v = \sum_{l=1}^m (-1)^l [2, \ldots, m | j_1, \ldots, \widehat{j_l}, \ldots, j_m] u_{j_l},$$

where $2 \leq j_1 < \cdots < j_m \leq n$. Let $j_0 = 1$ and consider the determinantal relation on X

$$w = \sum_{l=0}^m (-1)^l [1, \ldots, m | j_0, \ldots, \widehat{j_l}, \ldots, j_m] u_{j_l}.$$

Using Laplace expansion on the first row for all the minors appearing in the expression for w, we find that the element of $\mathcal{A}^n$ obtained from w by keeping exactly those terms of w involving monomials which are divisible by $x_{1,1}$ is

$$\sum_{l=1}^{m}(-1)^l x_{1,1}[2, \ldots, m|j_1, \ldots, \hat{j}_l, \ldots, j_m]u_{j_l} = x_{1,1}v.$$

Since $x_{1,1}$ is the $\mathcal{A}$-largest indeterminate and the order on $\mathcal{A}^n$ is lexicographic, it is then the case that $\mathrm{in}(w) = x_{1,1}\mathrm{in}(v)$. Now, $x_{1,1}$ divides t and $\mathrm{in}(v)$ divides $\mathrm{in}(h)$, so we finally get that $\mathrm{in}(f) = t \cdot \mathrm{in}(h)$ is divisible by the initial monomial $\mathrm{in}(w)$ of the determinantal relation w in G. This completes the proof. □

Using the fact that $\beta_1(M(m, n)) = \binom{n}{m+1}$ (which can be deduced at once from Theorem 7.1), by Theorem 5.1 and the behavior of Betti numbers with respect to scalar extensions and direct sums we have the following corollary.

COROLLARY 7.1. *Let π be a linear representation of a finite group G over an algebraically closed field $\mathcal{K}$, the characteristic of which does not divide the group order. Let π have decomposition $\pi = \sum_{i=1}^{k} m_i\chi_i$ into irreducible representations of G over $\mathcal{K}$, and let $n = \dim(\pi)$ and $n_i = \dim(\chi_i)$. The second Betti number of π is*

$$\beta_1(\pi) = \sum_{i=1}^{k} n_i \binom{n_i}{m_i + 1}.$$

8. Gröbner bases and Hilbert series of generic modules

A *monomial module* is a submodule of $\mathcal{A}^m$ generated by monomials, e.g., $\mathcal{A}^m$ itself. The monomials in a monomial module always form a $\mathcal{K}$-basis for it. The *initial module* $\mathrm{in}(M)$ of a submodule M of $\mathcal{A}^m$ is the monomial module generated by the set $\{\mathrm{in}(f) : f \in M\}$ of initial monomials in M, so $\{\mathrm{in}(f) : f \in M\}$ also forms a $\mathcal{K}$-basis for $\mathrm{in}(M)$. Moreover, the (residue classes of) all other monomials form a $\mathcal{K}$-space basis for the quotient module $\mathcal{A}^m/M$ (Macaulay; see [8, 12]). Thus, if M is graded, then for all i

$$\begin{aligned}\dim_{\mathcal{K}}(M_i) &= \dim_{\mathcal{K}}(\mathcal{A}^m) - \dim_{\mathcal{K}}((\mathcal{A}^m/M)_i) = |\{\mathrm{in}(f) : f \in M_i\}| \\ &= \dim_{\mathcal{K}}(\mathrm{in}(M)_i).\end{aligned}$$

We record this fact below.

PROPOSITION 8.1. *Given any monomial order on $\mathcal{A}^m$, the Hilbert series of a graded submodule M of $\mathcal{A}^m$ satisfies $H(M, t) = H(\mathrm{in}(M), t)$.*

We shall call a monomial order on $\mathcal{A}^m$ *induced* if it is obtained from some monomial order on $\mathcal{A}$ as follows: $su_i > tu_j$ if either $i < j$, or $i = j$ and $s > t$ in

$\mathcal{A}$. For a submodule M of $\mathcal{A}^m$ and $k = 1, \dots, m$, let $M^k = M \cap \oplus_{i=k}^m \mathcal{A}u_i$ be the submodule of elements of M of the form $\sum_{i=k}^m f_i u_i$. Thus $M^m \subseteq \cdots \subseteq M^1 = M$. Consider the map of $\mathcal{A}$-modules

$$\phi_k : M^k \longrightarrow \mathcal{A} : \sum_{i=k}^m f_i u_i \mapsto f_k,$$

and let $I_k = \phi_k(M^k)$, an ideal of $\mathcal{A}$. We have the following pair of Propositions.

PROPOSITION 8.2. *Let M be a graded submodule of $\mathcal{A}^m$, and let M^k, ϕ_k and I_k be as above. Then $H(M, t) = \sum_{k=1}^m H(I_k, t)$, and $\operatorname{in}(M) = \oplus_{k=1}^m \operatorname{in}(I_k)u_k$ for any induced monomial order on $\mathcal{A}^m$.*

Proof. For the direct sum, it suffices (cf. Proposition 4.2) to prove that

$$\{\operatorname{in}(f) : f \in M\} = \cup_{k=1}^m \{\operatorname{in}(p)u_k : p \in I_k\}.$$

Consider $f \in M$. If u_k divides $\operatorname{in}(f)$ then, since the order is induced, $f \in M^k$ so $\phi_k(f) \in I_k$ and $\operatorname{in}(f) = \operatorname{in}(\phi_k(f))u_k$. Conversely, if $p \in I_k$ then $p = \phi_k(f)$ for some $f \in M^k$, and, excluding the trivial case $p = 0$, we have $\operatorname{in}(p)u_k = \operatorname{in}(f)$. For the Hilbert series, we have by Proposition 8.1 that

$$H(M, t) = H(\operatorname{in}(M), t) = \sum_{k=1}^m H(\operatorname{in}(I_k), t) = \sum_{k=1}^m H(I_k, t). \qquad \square$$

PROPOSITION 8.3. *Let M be a graded submodule of $\mathcal{A}^m$, let M^k, ϕ_k, and I_k be as above, and let $G_k \subset M^k$. If $P_k = \phi_k(G_k)$ is a Gröbner basis for I_k with respect to a monomial order on $\mathcal{A}$ for all k, then $\cup_{k=1}^m G_k$ is a Gröbner basis for M with respect to the induced monomial order on $\mathcal{A}^m$.*

Proof. Consider $f \in M$. If u_k divides $\operatorname{in}(f)$ then, since the order is induced, $f \in M^k$ so $f_k = \phi_k(f) \in I_k$ and $\operatorname{in}(f) = \operatorname{in}(f_k)u_k$. Since P_k is a Gröbner basis for I_k, there is some $p \in P_k$ such that $\operatorname{in}(p)$ divides $\operatorname{in}(f_k)$. Picking $g \in G_k$ such that $\phi_k(g) = p$, we find that $\operatorname{in}(g) = \operatorname{in}(p)u_k$ divides $\operatorname{in}(f) = \operatorname{in}(f_k)u_k$. $\square$

Now, let again X be the generic $m \times n$ matrix, let $\mathcal{A} = \mathcal{K}[X] = \mathcal{K}[x_{1,1}, \dots, x_{m,n}]$, and let $M = M(m, n)$ be the module generated by the columns of X. Denote the $k \times n$ generic matrix consisting of the first k rows of X by $X_{[k]}$, and the jth column $\sum_{i=1}^m x_{i,j}u_i$ of X by X^j. Let $G_1 = \{X^1, \dots, X^n\}$ and, for $2 \le k \le n$, let

$$G_k = \left\{ \sum_{l=1}^k (-1)^{l+1} [1, \dots, k-1 | j_1, \dots, \hat{j_l}, \dots, j_k] X^{j_l} : 1 \le j_1 < \cdots < j_k \le n \right\}.$$

For $k = 1, \ldots, m$ let $\mathcal{A}(k) = \mathcal{K}[X_{[k]}]$ be the subalgebra of $\mathcal{A}$ generated by the indeterminates appearing in $X_{[k]}$, and let $D_{k,n}$ be the determinantal ideal of $\mathcal{A}(k)$ generated by the set of maximal minors of $X_{[k]}$. This set is known to form a Gröbner basis for $D_{k,n}$ [14], and moreover, was recently shown to be a *universal* Gröbner basis, i.e., a Gröbner basis for $D_{k,n}$ with respect to any monomial order on $\mathcal{A}(k)$ [2, 15]. The Hilbert series $H(D_{k,n}, t)$ is known as well (cf. [7, Theorem 1]). We can now prove the following theorem.

THEOREM 8.1. *For any induced monomial order on $\mathcal{A}^m$, a Gröbner basis for the generic module $M(m, n)$ is provided by the set*

$$\bigcup_{k=2}^{\min\{m,n\}} \left\{ \sum_{l=1}^{k} (-1)^{l+1} [1, \ldots, k-1 | j_1, \ldots, \hat{j}_l, \ldots, j_k] X^{j_l} : 1 \le j_1 < \cdots < j_k \le n \right\} \cup \{X^1, \ldots, X^n\}.$$

The Hilbert series of $M(m, n)$ is

$$H(m, n; t) = \frac{1}{(1-t)^{mn}} \sum_{k=1}^{\min\{m,n\}} \sum_{d=k}^{n} (-1)^{k+d} \binom{d-1}{k-1} \binom{n}{d} t^d.$$

Proof. First note that $M^1 = M$ is generated by G_1, so $I_1 = \phi_1(M^1)$ is the ideal of $\mathcal{A}$ generated by

$$P_1 = \phi_1(G_1) = \{x_{1,1}, \ldots, x_{1,n}\}.$$

Second note that for $2 \le k \le m$, an element $\sum_{j=1}^{n} f_j X^j$ of M is in M^k if and only if $f = \sum_{j=1}^{n} f_j u_j \in \mathcal{A}^n$ is a syzygy on the generic submatrix $X_{[k-1]}$ of X. If $n < k < m$ then by Observation 7.1, there are no nontrivial syzygies on $X_{[k-1]}$, so $M^k = \{0\}$. Finally, if $2 \le k \le \min\{m, n\}$ then, by Theorem 7.1, the submodule M^k is generated by G_k. We conclude that if $n < k < m$ then $I_k = \phi_k(M^k) = \{0\}$, whereas if $2 \le k \le \min\{m, n\}$ then $I_k = \phi_k(M^k)$ is the ideal of $\mathcal{A}$ generated by the set

$$\begin{aligned} P_k = \phi_k(G_k) &= \left\{ \sum_{l=1}^{k} (-1)^{l+1} [1, \ldots, k-1 | j_1, \ldots, \hat{j}_l, \ldots, j_k] x_{k,j_l} : 1 \le j_1 < \cdots < j_k \le n \right\} \\ &= \{(-1)^{k+1} [1, \ldots, k | j_1, \ldots, j_k] : 1 \le j_1 < \cdots < j_k \le n\}, \end{aligned}$$

which is, up to sign, the set of maximal minors of $X_{[k]}$. Thus, for $k = 1, \ldots, \min\{m, n\}$ the ideal I_k of $\mathcal{A}$ is generated by the set of maximal minors, so $I_k = \mathcal{A} \otimes_{\mathcal{A}(k)} D_{k,n}$ is a scalar extension of $D_{k,n}$.

For the statement on Gröbner bases, consider any monomial order on $\mathcal{A}$. It is easily seen that, since P_k is a Gröbner basis for $D_{k,n}$ with respect to this order, it is also a Gröbner basis for I_k. It then follows from Proposition 8.3 that $\bigcup_{k=1}^{min\{m,n\}} G_k$, which is the set in the statement of the theorem, is a Gröbner basis for M with respect to the induced order on $\mathcal{A}^m$.

For the statement on Hilbert series, by Proposition 8.2

$$\begin{aligned} H(m, n; t) = \sum_{k=1}^{m} H(I_k, t) &= \sum_{k=1}^{min\{m,n\}} H(\mathcal{A} \otimes_{\mathcal{A}(k)} D_{k,n}, t) \\ &= \sum_{k=1}^{min\{m,n\}} \frac{1}{(1-t)^{mn-kn}} H(D_{k,n}, t). \end{aligned}$$

The claim follows by substituting the known expressions for the $H(D_{k,n}, t)$. □

Note that for $n \leq m$ the above expression for the Hilbert series reduces to $\frac{nt}{(1-t)^{mn}}$ as it should (cf. Observation 7.1), and for $m < n$ it can be somewhat simplified to

$$H(m, n; t) = \frac{t}{(1-t)^{mn}} \left(n - \sum_{d=m}^{n-1} (-1)^{m+d} \binom{d-1}{m-1} \binom{n}{d+1} t^d \right).$$

Combining Corollary 5.1 with Theorem 8.1, we finally obtain the Hilbert series of a linear representation.

COROLLARY 8.1. *Let π be a linear representation of a finite group G over an algebraically closed field $\mathcal{K}$, the characteristic of which does not divide the group order. Let π have decomposition $\pi = \sum_{i=1}^{k} m_i \chi_i$ into irreducible representations of G over $\mathcal{K}$, and let $n = \dim(\pi)$ and $n_i = \dim(\chi_i)$. The Hilbert series of π is*

$$H(\pi, t) = \frac{1}{(1-t)^n} \sum_{i=1}^{k} n_i \sum_{r=1}^{min\{m_i,n_i\}} \sum_{d=r}^{n_i} (-1)^{r+d} \binom{d-1}{r-1} \binom{n_i}{d} t^d.$$

9. Discussion

A first interesting possible continuation is to find Gröbner bases for all kernels appearing in a free resolution of a generic module.

Second, we raise the following question. Is the set of determinantal relations on a generic matrix a universal Gröbner basis for the module of syzygies on its columns? If true, this may provide an alternative way of proving the remarkable universality of the ideal of maximal minors of a generic matrix [2, 15].

Returning to linear representations, we note that the Hilbert series and Betti numbers of a representation π depend only on the set of pairs $\{(m_1, n_1), \ldots, (m_k, n_k)\}$ of the multiplicities and dimensions of irreducible constituents of π. We raise the following question. For what representations π, or, equivalently, sets of pairs as above, do the Betti numbers $\beta_d(M(\pi))$ of all dimensions determine all the m_i and n_i (the multiplicities and dimensions of the irreducible constituents of π)?

Acknowledgments

I am indebted to L.J. Billera for discussions which originated this study and helped it evolve, and to R.P. Stanley and J.R. Stembridge for enlightening discussions and for their help in observing Proposition 4.1. I also thank I. Borosh, A. Machiavelo, B. Sturmfels, W. Vasconcelos, and A. Zelevinsky for helpful conversations, R. Adin for a helpful remark on the formulas for the Hilbert series, and a referee for valuable comments and references. Finally, I express my gratitude to Institut Mittag-Leffler for its hospitality while this work was initiated.

References

1. M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison Wesley, Reading, MA, 1969.
2. D. Bernstein and A. Zelevinsky, "Combinatorics of maximal minors," *J. Alg. Comb.* **2** (1993), 111–121.
3. W. Bruns and U. Vetter, *Determinantal Rings*, Lecture Notes in Math. 1327, Springer-Verlag, Berlin, 1988.
4. B. Buchberger, "Gröbner bases—An algorithmic method in polynomial ideal theory," in *Multidimensional Systems Theory*, N.K. Bose, ed. D. Reidel, 1985.
5. D.A. Buchsbaum and D. Eisenbud, "Generic free resolutions and a family of generically perfect ideals," *Adv. Math.* **18** (1975), 245–301.
6. C.W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience, New York, NY, 1962.
7. J.A. Eagon and D.G. Northcott, "A note on the Hilbert functions of certain ideals which are defined by matrices," *Mathematika* **9** (1962), 118–126.
8. D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Lecture Notes 4, Brandeis University, Waltham, MA., Spring, 1989.
9. H. Federer, *Geometric Measure Theory*, Springer-Verlag, Berlin, 1969.
10. G.D. James. *The Representation Theory of the Symmetric Group*, Lecture Notes in Mathematics 682, Springer-Verlag, Berlin, 1978.
11. S. Onn, "Geometry, complexity, and combinatorics of permutation polytopes," *J. Combin. Theory Series A* **64** (1993), 31–49.
12. R.P. Stanley, "Hilbert functions of graded algebras," *Adv. Math.* **28** (1978), 57–83.
13. R.P. Stanley, *Combinatorics and Commutative Algebra*, Birkhäuser, Boston, 1983.
14. B. Sturmfels, "Gröbner bases and Stanley decompositions of determinantal rings," *Math. Z.* **205** (1990), 137–144.
15. B. Sturmfels and A. Zelevinsky, "Maximal minors and their leading terms," *Adv. Math.* **98** (1993), 65–112.
16. E.B. Vinberg, *Linear Representations of Groups*, Birkhäuser, Boston, 1989.