Journal of Algebraic Combinatorics 3 (1994), 291–305
© 1994 Kluwer Academic Publishers, Boston. Manufactured in The Netherlands.

q-Hypergeometric Series and Macdonald Functions

NAIHUAN JING* JING@MERCURY.MATH.UKANS.EDU
Department of Mathematics, University of Kansas, Lawrence, Kansas 66045-2142

Received May 17, 1993; Revised August 16, 1993

Abstract. We derive a duality formula for two-row Macdonald functions by studying their relation with basic hypergeometric functions. We introduce two parameter vertex operators to construct a family of symmetric functions generalizing Hall-Littlewood functions. Their relation with Macdonald functions is governed by a very well-poised q-hypergeometric functions of type ${}_4\phi_3$, for which we obtain linear transformation formulas in terms of the Jacobi theta function and the q-Gamma function. The transformation formulas are then used to give the duality formula and a new formula for two-row Macdonald functions in terms of the vertex operators. The Jack polynomials are also treated accordingly.

Keywords: basic hypergeometric function, vertex operator, Macdonald symmetric function, Jack symmetric function

0. Introduction

Let Λ_R be the ring of symmetric functions with coefficients in a ring R in the variable $x_1, x_2, \ldots$, and $\mathcal{P}$ be the union of partitions of n: $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_1 \geq \lambda_2 \geq \cdots$. We will follow the usual notations in [7]. The ring $\Lambda_\mathbb{Z}$ has various $\mathbb{Z}$-bases indexed by partitions: the set of monomial symmetric functions $m_\lambda = \sum x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$; that of elementary symmetric functions $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$ with $e_n = m_{(1^n)}$; and that of Schur symmetric functions s_λ. There is also a $\mathbb{Q}$-basis of the power sum symmetric functions $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$, where $p_n = m_{(n)}$.

Let F be the field of rational functions in two independent indeterminates q, t. For any two partitions λ, $\mu \in \mathcal{P}$ define the scalar product on Λ_F by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} \prod_{i\geq 1} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \prod_{i\geq 1} i^{m_i} m_i! \tag{0.1}$$

where m_i is the occurrence of integer i in the partition λ, δ is the Kronecker symbol, and "$>$" is the dominance ordering in the set of partitions $\mathcal{P}$.

Macdonald has introduced a distinguished family of orthogonal symmetric functions $Q_\lambda(q, t)$ with respect to the scalar product and satisfying the following triangular relation [8]:

$$P_\lambda = m_\lambda + \sum_{\lambda > \mu} c_{\lambda\mu} m_\mu \tag{0.2}$$

in which $c_{\lambda\mu} \in F$ and λ, $\mu \in \mathcal{P}$.

*The research is partially supported by NSA grant MDA904-92-H03063.

We will mainly deal with the orthogonal symmetric functions $Q_\lambda(q, t)$ which are dual basis of $P_\lambda{}'s$ and thus proportional to $P_\lambda{}'s$. The polynomials Q_λ (or P_λ) are called Macdonald symmetric functions. They are generalizations of several familiar family of symmetric functions: the Schur functions ($q = t$); Hall-Littlewood functions ($q = 0$); Jack functions $Q_\lambda(\alpha)$ ($q = t^\alpha$, $t \to 1$) as well as elementary symmetric functions $e_{\lambda'}$ ($q = 1$) and the monomial symmetric functions m_λ ($t = 1$).

The Hall-Littlewood functions can be realized by certain vertex operator $H(z)$ associated with the infinite dimensional Heisenberg algebra $\mathfrak{h}$ over $\mathbb{Q}(t)$ generated by $h_n (n \in \mathbb{Z}^\times)$ and the central element c subject to the relation:

$$[h_m, h_n] = \delta_{m,-n}\frac{m}{1 - t^{|m|}}c. \tag{0.3}$$

In this context h_n corresponds to the power sum p_n, and the $Q_\lambda = H_{-\lambda_1} \cdots H_{-\lambda_k}.1$ in the basic representation space $V = Sym_{\mathbb{C}}(\mathfrak{h}^-) \simeq \Lambda_{\mathbb{C}}$ [5].

In the present work we will introduce two-parameter vertex operators $X(q, t; z)$ generalizing $H(z)$ ($=X(0, t; z)$) associated to the Heisenberg algebra generated by h_n and the central element c:

$$[h_m, h_n] = m\delta_{m,-n}\frac{1 - q^{|m|}}{1 - t^{|m|}}c. \tag{0.4}$$

Then the symmetric function $X_{-\lambda_1} \cdots X_{-\lambda_k}.1$ is not the Macdonald function in general though $X_{-n}.1 = Q_n$. However they are related through q-hypergeometric series (at least for two row case) ${}_4\phi_3$.

One of the advantages of the vertex operator approach to the Hall-Littlewood functions is that we can derive various relations among the $Q_\lambda(t)$ by exploiting the associated contraction function $f(x) = \frac{1-x}{1-tx}$. For example the transformation relation between $f(x)$ and $f(x^{-1})$:

$$f(x) = -\frac{x - t}{1 - tx}f(x^{-1}) \tag{0.5}$$

is equivalent to the following commutation relation among the Hall-Littlewood functions:

$$Q_{r,s}(t) = tQ_{s,r}(t) - Q_{s-1,r+1}(t) + tQ_{r+1,s-1}(t), \tag{0.6}$$

where we only state the two-row case of the general relation for the $Q_\lambda(t)$, for details see [5].

In the case of the Macdonald polynomials the role of the rational function $\frac{1-x}{1-tx}$ is replaced by a very-well poised basic hypergeometric function ${}_4\phi_3(x)$ as mentioned above. Thus to generalize our method we will need to know the transformation relation for the function ${}_4\phi_3(x)$ and ${}_4\phi_3(x^{-1})$. In this case the relation will involve with other similar functions of type ${}_4\phi_3$ with coefficients expressed in terms of the Jacobi theta function and the q-Gamma function (see 2.4′). It is this transformation formula which reveals hidden duality relations satisfied by Macdonald functions.

Thus in Section 2 we make a detailed study of the appeared basic hypergeometric series and derive two transformation formulas, which are analogues of the well-known linear transformation formulas for the hypergeometric function ${}_2F_1$. We then use one of the transformation formulas to extend the definition of two row Macdonald functions and obtain a duality formula for the $Q_{r,s}$. We also give a raising operator formula for the two-row Macdonald function in terms of the symmetric functions $X_{-\lambda}.1$ obtained naturally from our two-parameter vertex operators.

The techniques we rely heavily on are those of the basic hypergeometric functions, which do not appear explicitly in the case of Hall-Littlewood functions. We have demonstrated another application of the theory of basic hypergeometric functions into those of symmetric functions and vertex operators. As a byproduct we also obtain some summation formulas for the nonterminating very-well poised series of type ${}_4\phi_3$.

I would like to thank Professor George Gasper for comments on my earlier manipulations of the Sears' transformation formula.

After this work was completed, Professor Igor Frenkel informed me that he also had the notion of the two parameter vertex operator (see the definition immediately after 1.1) in his private notes.

1. Deformed vertex operators

Let $\mathfrak{h}$ be the infinite dimensional Heisenberg algebra generated by h_n, $n \in \mathbb{Z}^\times$ and the central element c with the following relation:

$$[h_m, h_n] = m\delta_{m,-n}\frac{1-q^{|m|}}{1-t^{|m|}}c. \tag{1.1}$$

The algebra $\mathfrak{h}$ has a basic representation realized on the space $V = Sym(\mathfrak{h}^-)$, the symmetric algebra generated by the elements $h_{-n}, n \in \mathbb{N}$. For a positive integer n the element h_{-n} acts as the multiplication by h_{-n}, and h_n acts as the differentiation operator $((1-t^n)/n(1-q^n))\partial/\partial h_{-n}$. We will still use the same symbol h_n to denote the operators on the space V, which then satisfy the relation (1.1) with $c = 1$.

We define a vertex operator on V by

$$\begin{aligned} X(z) &= \exp\left(\sum_{n=1}^{\infty}\frac{1-t^n}{n(1-q^n)}h_{-n}z^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{1-t^n}{n(1-q^n)}h_n z^{-n}\right)\\ &= \sum_{n\in\mathbf{Z}} X_n z^{-n} \end{aligned}$$

and its associated operator by

$$\begin{aligned} X^*(z) &= \exp\left(-\sum_{n=1}^{\infty}\frac{1-t^n}{n(1-q^n)}h_{-n}z^n\right)\exp\left(\sum_{n=1}^{\infty}\frac{1-t^n}{n(1-q^n)}h_n z^{-n}\right)\\ &= \sum_{n\in\mathbf{Z}} X^*_n z^n \end{aligned}$$

We also define the normal operator $: \quad :$ as the effect of moving $h_n{}'s$ to the right of $h_{-m}{}'s$. Then

$$
\begin{aligned}
&: X(z)X(w) : \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{1-t^n}{n(1-q^n)} h_{-n}(z^n + w^n)\right) \exp\left(-\sum_{n=1}^{\infty} \frac{1-t^n}{n(1-q^n)} h_n(z^{-n} + w^{-n})\right).
\end{aligned}
$$

Our vertex operators are only two parameter generalization of a special case of vertex operators. The latter belongs to the vertex operator algebras studied comprehensively in [1] to realize the Monster simple group.

To state our results we need the following notations from q-series [see, e.g., GR]:

$$
\begin{aligned}
(a;q)_\infty &= \prod_{n=0}^{\infty} (1 - aq^n), \\
(a;q)_n = (1-a)\cdots(1-aq^{n-1}) &= \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \\
(a_1, \ldots, a_m; q)_\infty &= (a;q)_\infty \cdots (a_m;q)_\infty.
\end{aligned}
$$

We also need the basic hypergeometric series:

$$
{}_{r+1}\phi_r\begin{pmatrix} a_1 & \cdots & a_{r+1} \\ b_1 & \cdots & b_r \end{pmatrix}; q, z \Bigg) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_n}{(b_1, \ldots, b_r; q)_n} \frac{z^n}{(q;q)_n}.
$$

Proposition 1.3.

$$
\begin{aligned}
X(z)X(w) &=: X(z)X(w) : \frac{(z/w;q)_\infty}{(tz/w;q)_\infty}, \\
X(z)X^*(w) &=: X(z)X^*(w) : \frac{(tz/w;q)_\infty}{(z/w;q)_\infty},
\end{aligned}
$$

where the function $\frac{(z/w;q)_\infty}{(tz/w;q)_\infty}$ *is meant* ${}_1\phi_0(t^{-1}; -; q, tz/w)$ *by the q-binomial theorem. The other series is understood similarly.*

Proof: We only show the first one as follows:

$$
\begin{aligned}
X(z)X(w) &=: X(z)X(w) : \exp\left(-\sum_{n,m=1}^{\infty} \frac{(1-t^n)(1-t^m)}{nm(1-q^n)(1-q^m)} [h_m, h_n] z^{-n} w^m\right) \\
&=: X(z)X(w) : \exp\left(-\sum_{n=1}^{\infty} \frac{1-t^n}{n(1-q^n)} \left(\frac{w}{z}\right)^n\right) \\
&=: X(z)X(w) : \prod_{m=0}^{\infty} \exp\left(-\sum_{n=1}^{\infty} \frac{1-t^n}{n} \left(q^m \frac{w}{z}\right)^n\right) = \prod_{m=0}^{\infty} \frac{1-q^m w/z}{1-q^m tw/z}.
\end{aligned}
$$

□

Proposition 1.4. *For a partition* $\lambda = (\lambda_1, \dots, \lambda_k)$,

$$X_{-\lambda_1} \cdots X_{-\lambda_k}.1 = \prod_{1\le i<j\le k} \frac{(R_{ij};q)_\infty}{(tR_{ij};q)_\infty} Q_{\lambda_1} \cdots Q_{\lambda_k}$$

where R_{ij} *is the raising operator acting on the monomials in the* Q_n *such that* $R(Q_m Q_n) = Q_{m+1}Q_{n-1}$ *and*

$$\sum_{n=0}^{\infty} Q_n z^n = \exp\left(\sum_{n=1}^{\infty} \frac{1-t^n}{n(1-q^n)} h_{-n} z^n\right) = Q(z).$$

Proof: By the properties of normal product we have that

$$X(z_1)\cdots X(z_k) =: X(z_1)\cdots X(z_k): \prod_{1\le i<j\le k} \frac{(z_j/z_i;q)_\infty}{(tz_j/z_i;q)_\infty}.$$

Thus

$$\begin{aligned} &X_{-\lambda_1} \cdots X_{-\lambda_k}.1 \\ &= \frac{1}{(2\pi\sqrt{-1})^k} \int X(z_1)\cdots X(z_k).1 z_1^{\lambda_1} \cdots z_k^{\lambda_k} \frac{dz_1}{z_1} \cdots \frac{dz_k}{z_k} \\ &= \frac{1}{(2\pi\sqrt{-1})^k} \int Q(z_1^{-1}) \cdots Q(z_k^{-1}) \prod_{i<j} \frac{(z_j/z_i;q)_\infty}{(tz_j/z_i;q)_\infty} \frac{dz_1}{z_1^{1-\lambda_1}} \cdots \frac{dz_k}{z_k^{1-\lambda_k}} \\ &= \prod_{1\le i<j\le k} \frac{(R_{ij};q)_\infty}{(tR_{ij};q)_\infty} Q_{\lambda_1} \cdots Q_{\lambda_k} \end{aligned}$$

where the contours of the integrals are around the origin such that $|z_1| > \cdots > |z_k| > 0$. □

If we identify the element h_{-n} to the power sum symmetric function p_n, then the scalar product is realized as the Hermitian structure on V given by

$$h_n^* = h_{-n}, \qquad n \in \mathbb{Z}^\times.$$

It is easy to check that for $h_{-\lambda} = h_{-\lambda_1} \cdots h_{-\lambda_k}$,

$$\langle h_{-\lambda}, h_{-\mu} \rangle = \delta_{\lambda,\mu} \prod_{i\ge 1} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \prod_{i\ge 1} i^{m_i} m_i!. \tag{1.5}$$

Under the identification the elements $X_{-\lambda}.1 = X_{-\lambda_1} \cdots X_{-\lambda_k}.1$ form a basis of symmetric functions in Λ_F. It is clear from the construction that our symmetric functions $X_{-\alpha}.1$ satisfy the triangularity condition in terms of monomial functions as Macdonald functions do.

In [5] we have shown that $X_{-\lambda}.1 = Q_\lambda(0, t)$ in the case of the Hall-Littlewood functions, and in the case of $(q, t) = (0, -1)$ we have another realization of the Schur Q-functions by certain twisted vertex operators in [4]. However it is not true in general that $X_{-\lambda}.1$ will be the Macdonald polynomials though this is the case when λ is a one row partition (which is why we still use Q_n to represent $X_{-n}.1$ in the above). In fact, we know that

$$Q_{r,s} = {}_4\phi_3\left(\begin{matrix} q^{r-s}, & t^{-1}, & q^{1+(r-s)/2}, & -q^{1+(r-s)/2} \\ & q^{r-s+1}t, & q^{(r-s)/2}, & -q^{(r-s)/2} \end{matrix}; q, tR_{12}\right)Q_rQ_s \quad (1.6)$$

as discovered by Jing-Józefiak [6].

Let $q = t^\alpha$, $t \to 1$, the relation in the Heisenberg algebra becomes

$$[h_m, h_n] = m\alpha\delta_{m,-n}, \quad (1.7)$$

and the vertex operator $X(q, t; z)$ degenerates to

$$X(\alpha; z) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{\alpha n}h_{-n}z^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{1}{\alpha n}h_n z^{-n}\right) \quad (1.8)$$

which contains some of the vertex operators in the representation theory of the affine Lie algebra $\hat{sl}(2)$ [1].

Meanwhile Proposition (1.4) specializes to

$$X(\alpha)_{-\lambda_1}\cdots X(\alpha)_{-\lambda_k}.1 = \prod_{1\le i<j\le k}(1-R_{ij})^{1/\alpha}Q(\alpha)_{\lambda_1}\cdots Q(\alpha)_{\lambda_k}, \quad (1.9)$$

since ${}_1\phi_0(q^{-1/\alpha}; -; q, q^{1/\alpha}z) \to (1-z)^{1/\alpha}$ as $q \to 1^-$.

2. The very-well poised hypergeometric series ${}_4\phi_3$

In this section we will study the very well-poised basic hypergeometric series ${}_4\phi_3$ in great details to prepare for our further investigations of Macdonald functions. Follow the conventional notations of the q-series in [3], we denote the very-well poised series ${}_4\phi_3$ by

$${}_4W_3(a; b; q, z) = {}_4\phi_3\left(\begin{matrix} a, & a^{1/2}q, & -a^{1/2}q, & b \\ & a^{1/2}, & -a^{1/2}, & aq/b \end{matrix}; q, z\right). \quad (2.1)$$

The very well-poised series ${}_4\phi_3$ can be expressed as a linear combination of the series ${}_2\phi_1$. In fact we have

$${}_4W_3(a; b; q, z) = \frac{1}{1-a}[{}_2\phi_1(a, b; aq/b; q, z) - a\,{}_2\phi_1(a, b; aq/b; q, q^2z)]. \quad (2.2)$$

To study its transformation property we recall the Watson's transformation formula for ${}_2\phi_1$ [3]:

$$\begin{aligned} &{}_2\phi_1(a, b; c; q, z) \\ &\quad = \frac{(b, c/a; q)_\infty(az, q/az; q)_\infty}{(c, b/a; q)_\infty(z, q/z; q)_\infty}{}_2\phi_1(a, aq/c; aq/b; q, cq/abz) \\ &\quad + \frac{(a, c/b; q)_\infty(bz, q/bz; q)_\infty}{(c, a/b; q)_\infty(z, q/z; q)_\infty}{}_2\phi_1(b, bq/c; bq/a; q, cq/abz) \end{aligned} \quad (2.3)$$

provided that $|\arg(-z)| < \pi$, c and a/b are not integral powers of q, and $a,\ b,\ z \neq 0$.

Theorem 2.4. *If a/b is not an integral power of q and $|arg(-z)| < \pi$ and $a,\ b \neq 0$, we have that*

$$\begin{aligned}
&{}_4W_3(a;b;q,\ z)\\
&\quad= -\frac{1}{a}\frac{(b,\ q/b;q)_\infty}{(aq/b,\ b/a;q)_\infty}\frac{(az,\ q/az;q)_\infty}{(z,\ q/z;q)_\infty}{}_4W_3(a;b;q,\ 1/b^2z)\\
&\qquad+\frac{b^2-a}{b^2(1-a)}\frac{(a,\ aq/b^2;q)_\infty}{(aq/b,\ a/b;q)_\infty}\frac{(bz,\ q/bz;q)_\infty}{(z,\ q/z;q)_\infty}{}_4W_3(b^2/a;b;q,\ 1/b^2z).
\end{aligned}$$

Proof: It follows from Watson's transformation formula (2.3) that

$$\begin{aligned}
{}_4W_3(a;b;q,\ z) &= \frac{1}{1-a}[{}_2\phi_1(a,\ b;aq/b;q,\ z) - a\,{}_2\phi_1(a,\ b;aq/b;q,\ q^2z)]\\
&= \frac{1}{1-a}\left[\frac{(b,\ q/b;q)_\infty}{(aq/b,\ b/a;q)_\infty}\frac{(az,\ q/az;q)_\infty}{(z,\ q/z;q)_\infty}{}_2\phi_1(a,\ b;aq/b;q,\ q^2/b^2z)\right.\\
&\qquad\left.+\frac{(a,\ aq/b^2;q)_\infty}{(aq/b,\ a/b;q)_\infty}\frac{(bz,\ q/bz;q)_\infty}{(z,\ q/z;q)_\infty}{}_2\phi_1(b,\ b^2/a;bq/a;q,\ q^2/b^2z)\right]\\
&\quad-\frac{a}{1-a}\left[\frac{(b,\ q/b;q)_\infty}{(aq/b,\ b/a;q)_\infty}\frac{(aq^2z,\ 1/aqz;q)_\infty}{(q^2z,\ 1/qz;q)_\infty}{}_2\phi_1(a,\ b;aq/b;q,\ 1/b^2z)\right.\\
&\qquad+\frac{(a,\ aq/b^2;q)_\infty}{(aq/b,\ a/b;q)_\infty}\frac{(bq^2z,\ 1/bqz;q)_\infty}{(q^2z,\ 1/qz;q)_\infty}\\
&\qquad\qquad\left.\times{}_2\phi_1(b,\ b^2/a;bq/a;q,\ 1/b^2z)\right]\\
&= \frac{1}{1-a}\frac{(b,\ q/b;q)_\infty}{(aq/b,\ b/a;q)_\infty}\frac{(aq^2z,\ q/az;q)_\infty}{(q^2z,\ q/z;q)_\infty}\\
&\quad\times\left[\frac{(1-az)(1-aqz)}{(1-z)(1-qz)}{}_2\phi_1(a,\ b;aq/b;q,\ q^2/b^2z)\right.\\
&\qquad\left.-a\frac{(1-1/aqz)(1-1/az)}{(1-1/qz)(1-1/z)}{}_2\phi_1(a,\ b;aq/b;q,\ 1/b^2z)\right]\\
&\quad+\frac{1}{1-a}\frac{(a,\ aq/b^2;q)_\infty}{(aq/b,\ a/b;q)_\infty}\frac{(bq^2z,\ q/bz;q)_\infty}{(q^2z,\ q/z;q)_\infty}\\
&\qquad\times\left[\frac{(1-bz)(1-bqz)}{(1-z)(1-qz)}{}_2\phi_1(b,\ b^2/a;bq/a;q,\ q^2/b^2z)\right.\\
&\qquad\left.-a\frac{(1-1/bqz)(1-1/bz)}{(1-1/qz)(1-1/z)}{}_2\phi_1(b,\ b^2/q;bq/a;q,\ q^2/b^2z)\right]\\
&= \frac{1}{1-a}\frac{(b,\ q/b;q)_\infty}{(aq/b,\ b/a;q)_\infty}\frac{(az,\ q/az;q)_\infty}{(z,\ q/z;q)_\infty}\left[{}_2\phi_1(a,\ b;aq/b;q,\ q^2/b^2z)\right.\\
&\qquad\qquad\left.-\ a^{-1}{}_2\phi_1(q,\ b;aq/b;q,\ 1/b^2z)\right]\\
&\quad+\frac{1}{1-a}\frac{(a,\ aq/b^2;q)_\infty}{(aq/b,\ a/b;q)_\infty}\frac{(bz,\ q/bz;q)_\infty}{(z,\ q/z;q)_\infty}\left[{}_2\phi_1(b^2/a,\ b;bq/a;q,\ q^2/b^2z)\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad - \frac{a}{b^2}{}_2\phi_1(b^2/a,\, b; bq/a; q,\, 1/b^2z)\Big] \\
&= -\frac{1}{a}\frac{(b,\, q/b; q)_\infty}{(aq/b,\, b/a; q)_\infty}\frac{(az,\, q/az; q)_\infty}{(z,\, q/z; q)_\infty}{}_4W_3(a; b; q,\, 1/b^2z) \\
&\quad + \frac{b^2-a}{b^2(1-a)}\frac{(a,\, aq/b^2; q)_\infty}{(aq/b,\, a/b; q)_\infty}\frac{(bz,\, q/bz; q)_\infty}{(z,\, q/z; q)_\infty}{}_4W_3(b^2/a; b; q,\, 1/b^2z).
\end{aligned}
$$

□

The hypergeometric series ${}_2\phi_1(z)$ (or ${}_4W_3(z)$) defines an analytic function when $|z| < 1$, which is denoted by the same symbol. By the so-called q-analogue of Barnes integrals the functions can be analytically continued to the domain of $|\arg(-z)| < 1$. The transformation formula then gives an analytic continuation of the function ${}_4W_3(z)$ for the domain of $|z| > 1$.

The coefficient functions in the transformation formula are actually quotients of the Jacobi elliptic θ function:

$$
\theta(x) = (x; q/x; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n.
$$

with which we can express our formula in a more transparent way:

$$
\begin{aligned}
&{}_4W_3(q^a; q^b; q,\, z) \\
&\quad = -q^{-a}\frac{\Gamma_q(a-b+1)\Gamma_q(b-a)}{\Gamma_q(b)\Gamma_q(1-b)}\frac{\theta(q^a z)}{\theta(z)}{}_4W_3(q^a; q^b; q,\, q^{-2b}z^{-1}) \\
&\quad\quad + \frac{q^{2b}-q^a}{q^{2b}(1-q^a)}\frac{\Gamma_q(a-b+1)\Gamma_q(a-b)}{\Gamma_q(a)\Gamma_q(a-2b+1)}\frac{\theta(q^b z)}{\theta(z)}{}_4W_3(q^{2b-a}; q^b; q,\, q^{-2b}z^{-1}).
\end{aligned}
\tag{2.4'}
$$

where $\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty}(1-q)^{1-a}$ is the q-analogue of the Gamma function.

We notice that Sears' transformation formula for nonterminating series ${}_{r+1}\phi_r$ [9] could give a relation among the very well-poised series ${}_4W_3(x)$, ${}_4W_3(x^{-1})$ and other associated series of type ${}_4\phi_3$. However the coefficient functions in the relation have delicate zeros depending on the arguments. One will need to deal with them very carefully in order to reduce the relation into our simple form.

Remark 2.5. Our formula specializes to the following:

$$
\begin{aligned}
&{}_3F_2\begin{pmatrix} a, & b, & 1+a/2 \\ & 1-b+a, & a/2 \end{pmatrix}; z \Big) \\
&\quad = -\frac{1}{a}\frac{\Gamma(1+a-b)\Gamma(b-a)}{\Gamma(b)\Gamma(1-b)}(-z)^{-a}{}_3F_2\begin{pmatrix} a, & b, & 1+a/2 \\ & 1-b+a, & a/2 \end{pmatrix}; z^{-1}\Big) \\
&\quad\quad + \frac{b^2-a}{b^2(1-a)}\frac{\Gamma(1+a-b)\Gamma(a-b)}{\Gamma(a)\Gamma(1+a-2b)}(-z)^b \\
&\quad\quad \times {}_3F_2\begin{pmatrix} 2b-a, & b, & 1+b-a/2 \\ & 1+b-a, & b-a/2 \end{pmatrix}; z^{-1}\Big)
\end{aligned}
$$

Corollary 2.6. *Let $a = q^p$, $b = t^{-1}$, $z = tx$ in* (2.4), *we have*

$$\begin{aligned} {}_4W_3(q^p; t^{-1}; q, tx) &= -{}_4W_3(q^p; t^{-1}; q, tx^{-1})x^{-p} \\ &+ \frac{1-t^2q^p}{1-q^p} \frac{(q^p, q^{p+1}t^2; q)_\infty (x, qx^{-1}; q)_\infty}{(q^{p+1}t, q^p t; q)_\infty (tx, qt^{-1}x^{-1}; q)_\infty} \\ &\times {}_4W_3(q^{-p}t^{-2}; t^{-1}; q, tx^{-1}). \end{aligned}$$

Corollary 2.7. (*The case of Jack functions*) *Let $q = t^\alpha$, and $t \to 1$ we have*

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} p, & -1/\alpha, & 1+p/2 \\ & 1+p+1/\alpha, & p/2 \end{matrix}; x\right) \\ &= -x^{-p}{}_3F_2\left(\begin{matrix} p, & -1/\alpha, & 1+p/2 \\ & 1+p+\alpha, & p/2 \end{matrix}; x^{-1}\right) \\ &+ \frac{p+2/\alpha}{p} \frac{\Gamma(p+1+1/\alpha)\Gamma(p+1/\alpha)}{\Gamma(p)\Gamma(p+1+2/\alpha)}(-z)^{-1/\alpha} \\ &\cdot {}_3F_2\left(\begin{matrix} -p-2/\alpha, & -1/\alpha, & 1-p/2-1/\alpha \\ & 1-p-1/\alpha, & -1/2-p/2 \end{matrix}; x^{-1}\right). \end{aligned}$$

Remark 2.8. (The case of Hall-Littlewood functions) When $q = 0$, the transformation formula reduces to the following trivial identity:

$$\frac{1-x}{1-tx} = -x^{-p}\frac{1-x^{-1}}{1-tx^{-1}} + \frac{1-x}{1-tx}\left(1 + \frac{t-x^{-1}}{1-tx^{-1}}x^{-p}\right).$$

Corollary 2.9. *We assume that $t = q^{1/\alpha}$, then the transformation formula implies that*

$$\begin{aligned} {}_4W_3(q^p; q^{-1/\alpha}; q, q^{1/\alpha}x) &= -x^{-p}{}_4W_3(q^p; q^{-1/\alpha}; q, q^{1/\alpha}x^{-1}) \\ &+ \frac{1-q^{p+2/\alpha}}{1-q^p} \frac{\Gamma_q(p+1+1/\alpha)\Gamma_q(p+1/\alpha)}{\Gamma_q(p)\Gamma_q(p+1+2/\alpha)} \\ &\cdot \frac{(x, qx^{-1}; q)_\infty}{(q^{1/\alpha}x, q^{1-1/\alpha}x^{-1}; q)_\infty} \\ &\times {}_4W_3(q^{-p-2/\alpha}; q^{-1/\alpha}; q, q^{1/\alpha}x^{-1}) \end{aligned}$$

where $\Gamma_q(a)$ is the q-analogue of the Gamma function defined after (2.4′).

In deriving of the above special cases we have repeatedly used the q-binomial theorem:

$${}_1\phi_0(a; -; z) = \sum_{n=0}^{\infty} \frac{(a;q)_\infty}{(q;q)_\infty} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

The following result is quoted from [Ex. 2.2, 3]. We furnish a proof here for completeness.

Proposition 2.10. *For* $max(|z|, |aq|) < 1$, *one has*

$$
{}_4\phi_3\left(\begin{matrix} a, & a^{1/2}q, & -a^{1/2}q, & b \\ & a^{1/2}, & -a^{1/2}, & aq/b \end{matrix}; q, z\right) = \frac{(aq, bz; q)_\infty}{(z, aq/b; q)_\infty} {}_2\phi_1(b^{-1}; z; bqz; q, aq).
$$

Proof: Using q-binomial theorem it follows that

$$
\begin{aligned}
{}_4W_3(a; b; q, z) &= \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (aq/b;q)_n} \frac{1-q^{2n}a}{1-a} z^n \\
&= \frac{(aq;q)_\infty}{(aq/b;q)_\infty} \sum_{n=0}^{\infty} \frac{(b;q)_n}{(q;q)_n} \frac{(aq^{n+1}/b;q)_\infty}{(aq^n;q)_\infty} (1-q^{2n}a) z^n \\
&= \frac{(aq;q)_\infty}{(aq/b;q)_\infty} \sum_{m,n=0}^{\infty} \frac{(b;q)_n}{(q;q)_n} \frac{(q/b;q)_m}{(q;q)_m} (aq^n)^m (1-q^{2n}a) z^n \\
&= \frac{(aq;q)_\infty}{(aq/b;q)_\infty} \sum_{m=0}^{\infty} \frac{(q/b;q)_m}{(q;q)_m} a^m \sum_{n=0}^{\infty} \frac{(b;q)_n}{(q;q)_n} (q^m z)^n (1-q^{2n}a) \\
&= \frac{(aq;q)_\infty}{(aq/b;q)_\infty} \sum_{m=0}^{\infty} \frac{(q/b;q)_m}{(q;q)_m} a^m \left(\frac{(q^m bz;q)_\infty}{(q^m z;q)_\infty} - a \frac{(q^{m+2}bz;q)_\infty}{(q^{m+2}z;q)_\infty} \right) \\
&= \frac{(aq, bz;q)_\infty}{(aq/b, z;q)_\infty} \sum_{m=0}^{\infty} \frac{(q/b;q)_m}{(q;q)_m} a^m \left(\frac{(z;q)_m}{(bz;q)_m} - a \frac{(q^2 z;q)_m}{(q^2 bz;q)_m} \right) \\
&= \frac{(aq, bz;q)_\infty}{(aq/b, z;q)_\infty} \left(\sum_{m=0}^{\infty} \frac{(a/b, z;q)_m}{(q, bz;q)_m} a^m \right. \\
&\qquad \left. - \sum_{m=0}^{\infty} \frac{(q/b, z;q)_{m+1}}{(q, bz;q)_{m+1}} a^{m+1} \frac{(1-q^{m+1}z)(1-q^m)}{(1-q^{m+1}bz)(1-q^{m+1}/b)} \right) \\
&= \frac{(aq, bz;q)_\infty}{(aq/b, z;q)_\infty} \sum_{m=0}^{\infty} \frac{(q/b, z;q)_m}{(q, bz;q)_m} \frac{(1-b)(z-1/b)}{(1-q^m bz)(1-q^m/b)} (aq)^m \\
&= \frac{(aq, bz;q)_\infty}{(aq/b, z;q)_\infty} {}_2\phi_1(1/b, z; qbz; q, aq).
\end{aligned}
$$

□

Proposition 2.11. *For* $max(|z|, |aq|) < 1$, *we have*

$$
{}_4\phi_3\left(\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b \\ & a^{1/2}, & -a^{1/2}, & aq/b \end{matrix}; q, z\right) = (1-bz) {}_2\phi_1(aq, bq; aq/b; q, z).
$$

Proof: This is a consequence of (2.9) and the Heine's transformation formula for ${}_2\phi_1$ series [3]:

$$ {}_2\phi_1(a, b; c; q, z) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1(c/b, z; az; q, b) $$

where $|z| < 1$ and $|b| < 1$. □

Using the formula (2.11) we can derive a summation formula for the very well-poised series ${}_4\phi_3$.

Proposition 2.12. *For* $|b^2 q| > 1$ *one has*

$$ {}_4\phi_3\left(\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b \\ & a^{1/2}, & -a^{1/2}, & aq/b \end{matrix}; q, 1/qb^2\right) = (1 - 1/qb)\frac{(q/b, aq/b^2; q)_\infty}{(aq/b, 1/b^2q; q)_\infty}. $$

Proof: This is obtained from our transformation formula (2.10) by applying Heine's q-analogue of Gauss' summation formula [3]:

$$ {}_2\phi_1(a, b; c; q, c/ab) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}. \tag{2.13} $$

□

Applying Watson's transformation to the series ${}_2\phi_1$ in (2.11) we obtain another transformation relation for the very well-poised series ${}_4\phi_3$.

Proposition 2.14. *For* $|x/b| < 1$, $|aq| < 1$, *and c and* a/b *are not integral powers of* q, *we have*

$$ \begin{aligned} &{}_4W_3(a; b; q, x/b) \\ &\quad= \frac{(bq, 1/b; q)_\infty}{(aq/b, b/a; q)_\infty}\frac{(aqx/b, b/ax; q)_\infty}{(x/b, bq/x; q)_\infty}\frac{(1-x)}{(1-1/qx)} {}_4W_3(a; b; q, 1/bqx) \\ &\qquad+ \frac{(aq, q/b^2; q)_\infty}{(aq/b, a/b; q)_\infty}\frac{(qx, 1/x; q)_\infty}{(x/b, qb/x; q)_\infty}\frac{(1-x)}{(1-1/qx)} {}_4W_3(b^2/a; b; q, 1/bqx). \end{aligned} $$

Remark 2.15. Notice that the transformation formula can be also expressed in terms of the Jacobi theta functions and q-Gamma functions. The second ${}_4W_3$ can be written in a neat form by the q-Euler's transform [3] in our case:

$$ \begin{aligned} &{}_2\phi_1(b^2q/a, bq; bq/a; q, 1/b^2qx) \\ &\quad= \frac{(1/z; q)_\infty}{(1/b^2qz; q)_\infty} {}_2\phi_1(1/a, 1/b; bq/a; q, 1/z). \end{aligned} $$

3. Transition functions

In Section 1 we mentioned that the symmetric function $X_{-\lambda_1}\cdots X_{-\lambda_k}.1$ is not equal to the Macdonald function $Q_\lambda(q, t)$ when $|\lambda| = k \geq 2$. However they are related by a hypergeometric function.

We have obtained the following various formulas for the two-row Macdonald functions:

$$\begin{aligned} Q_{r,s} &= {}_4\phi_3\left(\begin{matrix} q^{r-s}, & q^{1+(r-s)/2}, & -q^{1+(r-s)/2}, & t^{-1} \\ & q^{(r-s)/2}, & -q^{(r-s)/2}, & q^{r-s+1}t \end{matrix}; q, tR\right) Q_r Q_s \\ &= \frac{1}{1-q^{r-s}}[{}_2\phi_1(q^{r-s}, t^{-1}; q^{r-s+1}t; q, tR) \\ &\qquad - q^{r-s}{}_2\phi_1(q^{r-s}, t^{-1}; q^{r-s+1}t; q, tq^2R)]Q_rQ_s \\ &= (1-R){}_2\phi_1(q^{r-s+1}, qt^{-1}; q^{r-s+1}t; q, tR)Q_rQ_s. \end{aligned} \tag{3.1}$$

Now we add another formula for the $Q_{r,s}$ in terms of the vertex operators introduced in Section 1.

Theorem 3.2. *For $r \geq s \geq 0$,*

$$\begin{aligned} Q_{r,s} &= \frac{(q^{r-s+1}; q)_\infty}{(q^{r-s+1}t; q)_\infty}{}_2\phi_1(t, tR; qR; q, q^{r-s+1})X_{-r}X_{-s}.1 \\ &= \frac{(tR; q)_\infty}{(qR; q)_\infty}{}_2\phi_1(qt^{-1}, q^{r-s+1}; q^{r-s+1}t; q, tR)X_{-r}X_{-s}.1 \end{aligned}$$

where $R = R_{12}$, the raising operator acting on the X_nX_m directly.

Proof: From the raising operator formula (1.6) it follows that

$$\begin{aligned} Q_{r,s} &= {}_4W_3(q^{r-s}; t^{-1}; q, tR)Q_rQ_s \\ &= \frac{(q^{r-s}, R; q)_\infty}{(tR, q^{r-s+1}t; q)_\infty}{}_2\phi_1(t, tR; qR; q, q^{r-s+1})Q_rQ_s \\ &= \frac{(q^{r-s+1}; q)_\infty}{(q^{r-s+1}t; q)_\infty}{}_2\phi_1(t, tR; qR; q, q^{r-s+1})\frac{(R; q)_\infty}{(tR; q)_\infty}Q_rQ_s \\ &= \frac{(q^{r-s+1}; q)_\infty}{(q^{r-s+1}t; q)_\infty}{}_2\phi_1(t, tR; qR; q, q^{r-s+1})X_{-r}X_{-s}.1 \end{aligned}$$

The other relation is obtained using (2.11). □

Let $q = t^\alpha$, $t \to 1$ in the second relation we derive the following

Corollary 3.3. *In the case of Jack symmetric functions we have*

$$Q_{r,s}(\alpha) = (1-R)^{1-1/\alpha}{}_2F_1\left(\begin{matrix} r-s+1, & 1-1/\alpha \\ & r-s+1/\alpha \end{matrix}; R\right) X_{-r}(\alpha)X_{-s}(\alpha).1$$

Example 3.4. When $t = q^2$ (q-zonal symmetric functions), we have

$$Q_{r,s} = (1-qR)^{-1}\left(1 - \frac{1-q^{r-s+1}}{1-q^{r-s+3}}qR\right) X_{-r}X_{-s}.1$$
$$(1-q^{r-s+3})Q_{r,s} = \left(1 + q^{r-s+2}\frac{R-q}{1-qR}\right) X_{-r}X_{-s}.1$$

In the case of zonal symmetric functions one has

$$Q_{r,s}\left(\frac{1}{2}\right) = \frac{1 - \frac{r-s+1}{r-s+3}R}{1-R} X_{-r}\left(\frac{1}{2}\right) X_{-s}\left(\frac{1}{2}\right).1.$$

We now wish to extend the definition of $Q_{r,s}$ to any pair of integers. In general we define

$$Q_{r,s} = {}_4W_3(q^{r-s}; t^{-1}; q, tR)Q_rQ_s. \tag{3.5}$$

From now on we let $t = q^k$, k an integer. In this situation we have a duality for two row Macdonald functions.

Proposition 3.6. *For integers k, p with $k \geq 0$, we have*

$$\begin{aligned}{}_4W_3(q^p; q^{-k}; q, tx) &= -x^{-p}{}_4W_3(q^p; q^{-k}; q, tx^{-1}) \\ &\quad + q^{\binom{k}{2}}(-x)^k \frac{(q^{p+1}; q)_{k-1}}{(q^{p+k+1}; q)_{k-1}} {}_4W_3(q^{-p-2k}; q^{-k}; q, tx^{-1}).\end{aligned}$$

For $t = q^{-k}$ and $k \geq 0$, we have from Corollary 2.9 again

Proposition 3.7.

$$\begin{aligned}{}_4W_3(q^p; q^k; q, tx) &= -x^{-p}{}_4W_3(q^p; q^k; q, tx^{-1}) \\ &\quad + q^{\binom{k+1}{2}}(-x)^{-k} \frac{(q^{p-2k}; q)_{k+1}}{(q^{p-k}; q)_{k+1}} {}_4W_3(q^{2k-p}; q^k; q, tx^{-1}).\end{aligned}$$

Remark 3.8. The two cases can be combined by using the q-shifted factorial of negative integer:

$$(a;q)_{-n} = \frac{1}{(aq^{-n};q)_n} = \frac{1}{(1-aq^{-1})\cdots(1-aq^{-n})}, \qquad n \in \mathbb{N}$$

Then either of the above two propositions is true for any integer k.

One of the applications of the transformation formula (2.4) for the very well-poised q-series is the following duality relations for Macdonald functions.

Theorem 3.9. *For any nonnegative integers r, s, k we have*

$$2Q_{r,s} = (-1)^k q^{\binom{k+1}{2}} \frac{(q^{r-s-2k};q)_{k+1}}{(q^{r-s-k};q)_{k+1}} Q_{s+k,\, r-k}, \quad (t = q^{-k})$$
$$2Q_{r,s} = (-1)^k q^{\binom{k}{2}} \frac{(q^{r-s+1};q)_{k-1}}{(q^{r-s+k+1};q)_{k-1}} Q_{s-k,\, r+k}, \quad (t = q^{k}).$$

Proof: Using the simple property of the raising operators:

$$R(Q_rQ_s) = Q_{r+1}Q_{s-1} = R^{-1}(Q_sQ_r),$$

it follows that

$$\begin{aligned}
Q_{r,s} &= {}_4W_3(q^{r-s};q^{-k};q,\, tR)Q_rQ_s \\
&= -\,{}_4W_3(q^{r-s};q^{-k};q,\, tR^{-1})R^{s-r}Q_rQ_s \\
&\quad + (-1)^k q^{\binom{k}{2}} \frac{(q^{r-s+1};q)_{k-1}}{(q^{r-s+k+1};q)_{k-1}} {}_4W_3(q^{s-r-2k};q^{-k};q,\, tR^{-1})R^kQ_rQ_s \\
&= -\,{}_4W_3(q^{r-s};q^{-k};q,\, tR^{-1})Q_sQ_r \\
&\quad + (-1)^k q^{\binom{k}{2}} \frac{(q^{r-s+1};q)_{k-1}}{(q^{r-s+k+1};q)_{k-1}} {}_4W_3(q^{s-r-2k};q^{-k};q,\, tR^{-1})Q_{r+k}Q_{s-k} \\
&= -\,{}_4W_3(q^{r-s};q^{-k};q,\, tR)Q_rQ_s \\
&\quad + (-1)^k q^{\binom{k}{2}} \frac{(q^{r-s+1};q)_{k-1}}{(q^{r-s+k+1};q)_{k-1}} {}_4W_3(q^{s-r-2k};q^{-k};q,\, tR)Q_{s-k}Q_{r+k} \\
&= -\,Q_{r,s} + (-1)^k q^{\binom{k}{2}} \frac{(q^{r-s+1};q)_{k-1}}{(q^{r-s+k+1};q)_{k-1}} Q_{s-k,\, r+k}.
\end{aligned}$$

The other identity ($t = q^{-k}$) is proved similarly. □

Remark 3.10. Our formula will also reveal a duality relation for the two-parameter Kostka matrix. Macdonald conjectured that the entries of these Kostka matrices are polynomials in q, t with positive integral coefficients. The truth of the conjecture in the two-row case was proved in [2], and the polynomialness was done independently in [10].

Corollary 3.11. *For any nonnegative integers r, s, k we have*

$$2Q_{r,s}(-1/k) = (-1)^k \frac{(r-s-2k)_{k+1}}{(r-s-k)_{k+1}} Q_{s+k,\, r-k}(-1/k),$$
$$2Q_{r,s}(1/k) = (-1)^k \frac{(r-s+1)_{k-1}}{(r-s+k+1)_{k-1}} Q_{s-k,\, r+k}(1/k).$$

We record some of the special cases in the following.

If $t^2 = q^{-p}$, then by Heine's q-analogue of Gauss' summation formula (2.12) we derive that

$$
{}_2\phi_1\left(\begin{matrix} tx, & t \\ & qx \end{matrix}; q, q/t^2\right) = \frac{(q/t, qx/t; q)_\infty}{(qx, q/t^2; q)_\infty} = \frac{(\pm q^{1+p/2}, \pm q^{1+p/2}x; q)_\infty}{(qx, q^{1+p}; q)_\infty} \tag{3.12}
$$

Thus,

$$
{}_4W_3(q^p; t^{-1}; q, tx) = \frac{(q^{p+1}, x, q/t, qx/t; q)_\infty}{(tx, q^{p+1}t, qx, q/t^2; q)_\infty} = \frac{(1-x)}{(tx; q)_{p+1}}. \tag{3.13}
$$

If $t = -q^{-p}$, then Bailey-Daum summation formula [3] implies that

$$
{}_2\phi_1\left(\begin{matrix} tx, & t \\ & qx \end{matrix}; q, q^{p+1}\right) = \frac{(-q; q)_\infty(-q^{1-p}x, -q^{p+2}x; q^2)_\infty}{(qx, q^{p+1}; q)_\infty}. \tag{3.14}
$$

Therefore

$$
{}_4W_3(q^p; t^{-1}; q, tx) = (1-x)\frac{(-q^{1-p}x, -q^{p+2}x; q^2)_\infty}{(tx; q)_\infty}. \tag{3.15}
$$

References

1. Frenkel I., Lepowsky J., and Meurman A., *Vertex operator algebras and the Monster*, Academic Press, New York, 1988.
2. Garsia A.M. and Haiman M., *A graded representation model for Macdonald's polynomials*, UCSD preprint, 1992.
3. Gasper G. and Rahman M., *Basic hypergeometric series*, Cambridge University Press, Cambridge, 1990.
4. Jing N.H., "Vertex operators, symmetric functions, and the spin group Γ_n," *J. Algebra* **138** 1991, 340–398.
5. Jing N.H., "Vertex operators and Hall-Littlewood symmetric functions," *Adv. in Math.* **87** (1991), 226–248.
6. Jing N.H. and Józefiak T., "A formula for two row Macdonald functions," *Duke Math. J.* **67** (1992), 377–385.
7. Macdonald I.G., *Symmetric functions and Hall polynomials*, Oxford University, Oxford, 1979.
8. Macdonald I.G., *A new class of symmetric functions* 20e Actes Séminaire Lotharingien (L. Cerlienco and D. Foata, eds.), vol. 20, Publ. Inst. Rech. Math. Avancée, Strasboug, (1988), 131–171.
9. Sears D.B., "Transformations of basic hypergeometric functions of any order," *Proc. London Math. Soc., Ser.* **2** (3) **53** (1951), 181–191.
10. Stembridge J.R., "Some particular entries of the two-parameter Kostka matrix," *Proc. Amer. Math. Soc.*, to appear.