Polygon Posets and the Weak Order of Coxeter Groups

KIMMO ERIKSSON
Department of Mathematics, SU, S-106 91 Stockholm, Sweden

Abstract. We explore the connection between *polygon posets*, which is a class of ranked posets with an edge-labeling which satisfies certain 'polygon properties', and the weak order of Coxeter groups. We show that every polygon poset is isomorphic to a *join ideal* in the weak order, and for Coxeter groups where no pair of generators have infinite order the converse is also true.

The class of polygon posets is seen to include the class of *generalized quotients* defined by Björner and Wachs, while itself being included in the class of *alternative* generalized quotients also considered by these authors. By studying polygon posets we are then able to answer an open question about common properties of these two classes.

1. Introduction

This paper ties together topics in algebraic combinatorics (Coxeter groups), partially ordered sets, and combinatorial games: We define, axiomatically, a family of partially ordered sets called *polygon posets*. The background to this definition is the properties of the position posets of the 'numbers game', a combinatorial game of Mozes [12]. We show how the polygon posets are embedded in the *weak order* of Coxeter groups, and how they are relevant to Coxeter group theory in various other ways.

The organization of the material is as follows.

In §2 we give a characterization of the weak order on a Coxeter group (W, S) as the unique poset that admits an S-labeling of its edges satisfying some simple conditions. The class of polygon posets is defined by dropping one of these conditions. Hence the weak order of any Coxeter group is a polygon poset, and every polygon poset is in a direct way related to a certain Coxeter group (W, S).

The weak order is known to be a $\wedge$-semilattice, though in general not a lattice. In §3 we see that this is an easy consequence of the fact that the weak order is a polygon poset.

Define a *join ideal* in the weak order to be a subset $I \subseteq W$ such that if $v \in I$ and $w \in W$ then $v \wedge w \in I$, and if $v, w \in I$ and $v \vee w$ exists then $v \vee w \in I$. We show in §4 that every polygon poset P is isomorphic to a join ideal in the weak order of (W, S). If all polygons are finite, then the converse also holds.

Björner and Wachs [2] studied two kinds of generalizations of the ordinary concept of quotient in Coxeter groups, which they called *generalized quotients* and *alternative generalized quotients*. We show in §5 that the class of generalized quotients of a Coxeter group (W, S) is contained in the class of polygon posets of (W, S), which is itself contained in the class of alternative generalized quotients of (W, S).

§6 deals with a method for constructing examples of polygon posets. As an application of this method, we construct, in §7, some examples that answers, negatively, some questions of Björner and Wachs about properties of alternative generalized quotients.

For $V \subseteq W$ we also define $W \sharp V$, the *sharp quotient*, in §8. The sharp quotients, under weak order, are shown to be polygon posets as well. We discuss the connection between the existence of a finite state automaton for recognizing the language of reduced words, and finiteness of the number of nonisomorphic sharp quotients and generalized quotients. In §9 we use the methods of the paper to give a short finiteness proof for all Coxeter groups such that no pair of generators commute.

Finally, we devote §10 to the position posets of the numbers game. We show that every finite polygon poset can be realized as a position poset, but that this is not true in the infinite case. In particular, the language defined by the legal play sequences, i.e. by the edge labelings of upward going paths from the least element in the poset, is a greedoid for the numbers game, which is false for polygon posets in general.

1.1. Preliminaries on strong convergence

The concept of *strong convergence* was introduced by Anders Björner, and has been thoroughly studied by Eriksson [10]. The setting is one-player games, that is, discrete processes where the 'player' in each position chooses among some moves, until a terminal position, where no moves are available, is reached.

Definition 1.1 A game has the strong convergence property if, given any position p where some play sequence leads to some terminal position t, all ways of playing from p will terminate in t, and in the same number of moves.

The strongly convergent games are characterized in the following theorem from [10].

Theorem 1 (Polygon Property Theorem) *A game has the strong convergence property if and only if given any position where two different moves, x and y, are legal, either there are two play sequences of the same length and beginning with x and y respectively that result in the same position, or there are two such play sequences which can be continued forever.*

The 'polygon' that is referred to in the name 'Polygon Property Theorem' is the polygon shape in the game graph (with the positions being nodes and the moves being directed edges) that the two play sequences, if finite, build up. Thus, we may equivalently define strong convergence as a property of directed graphs. Observe that posets, represented by their Hasse diagrams, are a family of directed graphs, so strong convergence also applies to posets; every poset is interpreted as a game, where the elements are the positions, and where there is a move from p to q if q covers p in the poset. Suppose that the poset has a unique least element. Strong convergence then says that if there exists a maximal element, then it is unique, and all maximal chains are of the same length.

1.2. Preliminaries on Coxeter groups

A Coxeter group (or, more precisely, a Coxeter system) (W, S) is a group W together with a distinguished set S of involutory generators, and relations $(st)^{m(s,t)} = e$ (the identity of the group), where $m(s, t) \geq 2$, for all $s, t \in S$. If st has infinite order in W, then $m(s, t) = \infty$.

Every element $w \in W$ can be represented by several different words in the alphabet S. As is customary, we let $l(w)$ denote the length of a reduced expression for w, i.e. a word of minimum length. $\langle w \rangle$ will denote any reduced word for w. We let $\langle w \rangle \langle v \rangle$ denote concatenation of words, so saying that $\langle w \rangle \langle v \rangle$ is reduced is another way of expressing that v and w satisfy $l(wv) = l(w) + l(v)$. A useful result on reduced words is the following, see e.g. Bourbaki [4].

Lemma 2 (Exchange condition) *Let* $w = s_1 \cdots s_r$ *(all* $s_i \in S$*) be a reduced word. If* $l(ws) < l(w)$ *for some* $s \in S$*, then there exists an index i for which* $w = s_1 \cdots \hat{s}_i \cdots s_r s$ *(s exchanged for s_i).*

The (right) *weak order* of a Coxeter group (W, S) is a partial order on the group elements, which we will denote by $\succeq$. It is defined by $w_2 \succeq w_1$ if $w_2 = w_1 v$ where $\langle w_1 \rangle \langle v \rangle$ is reduced. We denote the poset obtained in this way by $(W, S, \succeq)$. The S-labeled Hasse diagram of $(W, S, \succeq)$ is equivalent to the Cayley graph of the Coxeter group, with W as node set and an edge (w_1, w_2) labeled $s \in S$ if $w_2 = w_1 s$ and $l(w_2) = l(w_1) + 1$. The Hasse diagram is always directed upwards, with e at the bottom. Sometimes we will instead use the *left* weak order $\succeq_L$, which is the isomorphic partial order obtained by putting v to the left instead of to the right in the definition above.

The *Bruhat order* on (W, S), here denoted by $\succeq_B$, can be defined by the following subword property, cf. Bourbaki [4]. Let $v = s_1 s_2 \cdots s_k$ be a reduced expression. Then $v \succeq_B u$ if and only if there is a reduced expression for u that is a subword of $s_1 s_2 \cdots s_k$, i.e. $u = s_{i_1} s_{i_2} \cdots s_{i_j}$, where $1 \leq i_1 < i_2 < \cdots < i_j \leq k$. The weak order is weaker than Bruhat order in the sense that $w \succeq v \Rightarrow w \succeq_B v$. For an expository discussion of these orderings on Coxeter groups, see [1].

A well-known feature of Coxeter groups is the existence of *minimal coset representatives*. Let $I = \{s, t\} \subset S$, and let W_I be the parabolic subgroup generated by I. Then for any given $w \in W$, the coset wW_I contains a minimal coset representative, an element v such that the expressions $\langle v \rangle (sts \cdots)_{m(s,t)}$ and $\langle v \rangle (tst \cdots)_{m(s,t)}$ are reduced, where for convenience we introduce the notation $(sts \cdots)_n$ for the alternating word $stst \cdots$ of length n.

We will also need Tits's Word Theorem. Tits introduced two admissible operations on words:

(M1) If a word contains $(sts \cdots)_{m(s,t)}$ then this part may be changed into $(tst \cdots)_{m(s,t)}$. For example, if $m(s, t) = 3$, then $rstsu$ may be changed into $rtstu$. This is called an M-simplification of the first kind.

(M2) If a word somewhere contains ss we may erase those letters. For example, $rsstu$ may be changed into rtu. This is called an M-simplification of the second kind.

Theorem 3 (Tits's Word Theorem) *A word* $s_1 s_2 \cdots s_k$ *(all* $s_i \in S$*) is reduced if and only if it cannot be shortened by any finite sequence of M-simplifications. Given two reduced*
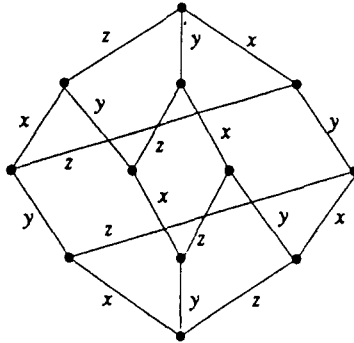
Figure 1. The Hasse diagram of $(W, S, \succeq)$ where $S = \{x, y, z\}$ and $m(x, y) = 3, m(x, z) = m(y, z) = 2$.

words for the same element in W, one can be obtained from the other by a finite sequence of M-simplifications of the first kind.

2. Polygon posets and the weak order

For a Coxeter group (W, S), let $(W, S, \succeq)$ denote the poset obtained by partially ordering W by weak order. We shall begin by giving a characterization of $(W, S, \succeq)$ as the unique poset with certain simple properties. If one looks at the example in Fig. 1, the proof should be easier to follow.

Lemma 4 *For any Coxeter group (W, S) with Coxeter relations $(xy)^{m(x,y)} = e$, there exists a poset P, unique up to isomorphism, such that:*

1. *P has a least element $\hat{0}$.*
2. *There are $|S|$ elements covering $\hat{0}$.*
3. *P admits an S-labeling of the edges of its Hasse diagram satisfying:*

   (a) *No two edges incident to the same element of P have the same label.*
   (b) *If there are two edges going upwards from $p \in P$ with labels x and y, then they are the first edges of two upward going paths from p of length $m(x, y)$ labeled alternatingly x and y. If $m(x, y) < \infty$ then these paths end in the same element, while if $m(x, y) = \infty$ the paths go on forever.*

Proof: We will give a construction of the labeled Hasse diagram of P. Let $\hat{l}(p)$ denote the length of a shortest path from $\hat{0}$ to p in the Hasse diagram. Let $P_n = \{p \in P : \hat{l}(p) \leq n\}$. We will prove that P_{n+1} can be constructed in a unique way from P_n; it is *graded* and everyone of its elements has $|S|$ incident edges in P, that is, when one counts also edges leading to elements outside P_{n+1}.

The claim is trivially true for $P_0 = \{\hat{0}\}$. Suppose it is true for P_n. We can construct every new element q of P_{n+1} by following an edge, labeled x say, upwards from an element p with $\hat{l}(p) = n$. Then $\hat{l}(q) = n + 1$ and we must now prove that for any $y \in S, y \neq x$, either the conditions (a) and (b) force the existence of an edge labeled y leading up to q from

some element r, where $\hat{l}(r) = n$, or they force the existence of an edge labeled y going upwards from q.

By (a), there is a unique alternating path $yxyx\cdots$ leading downwards from p. It must end in an element q' where both edges labeled x and y go upwards, since we know that there is one edge of each label incident to every element of P_n.

By (b), there are two upward-going alternating paths $xyxy\cdots$ and $yxyx\cdots$ of length $m(x, y)$ beginning in q'. Thus, if $\hat{l}(q) - \hat{l}(q') < m(x, y)$, then the path continues from q, so there is an edge labeled y going upwards from q. Otherwise we must have $\hat{l}(q) - \hat{l}(q') = m(x, y)$, in which case the two paths join in q, so there must be some element r where $\hat{l}(r) = \hat{l}(q) - 1$, and with an edge labeled y going upwards to q.

P_{n+1} is unambiguously constructed, it is graded, and every element has $|S|$ incident edges in P.   □

Now, $(W, S, \succeq)$ certainly has a least element, e, with $|S|$ elements covering e, namely S, and the natural S-labeling of the edges satisfies (a), since the generators are involutions, and (b) because of the special property of the minimal coset representatives. Hence we have found our unique poset.

Theorem 5 *The poset $(W, S, \succeq)$ is characterized by the properties in Lemma 4.*

From the construction of the poset P, it is easy to deduce also the following properties:

(c) *If there are two edges going downwards from $p \in P$ with labels x and y, then they are the first edges of two downward going paths from p of length $m(x, y)$ labeled alternatingly x and y. If $m(x, y) < \infty$ then these paths end in the same element, while if $m(x, y) = \infty$ this case never occurs.*

(d) *If there is an upward going path of length $m(x, y) < \infty$ from $p \in P$, alternatingly labeled x and y and beginning with x, then there is also another one, beginning with y.*

If we drop the condition (2), which says that there are $|S|$ elements covering $\hat{0}$, and instead demand that (c) and (d) hold, then we get an interesting class of posets.

Definition 2.1 A *polygon poset*, associated to a Coxeter group (W, S), is a poset P with a least element $\hat{0}$ and an S-labeling of the edges of the Hasse diagram satisfying properties (a)–(d) above.

Corollary 6 $(W, S, \succeq)$ *is a polygon poset.*

The pair of paths that properties (b)–(d) are all about, should be thought of as constituting a polygon in the Hasse diagram, in which the edges are alternatingly labeled x and y. Then the *cup property* (b), the *hat property* (c), and the *half-polygon property* (d), guarantees the existence of such a polygon whenever there is a cup, a hat, a half-polygon, resp. See Fig. 2.

We will see later that every polygon poset is graded.

3. The weak order is a semilattice

Another property of polygon posets that can be derived is *strong convergence*, since it follows from the cup property and the existence of a least element, by the Polygon Property
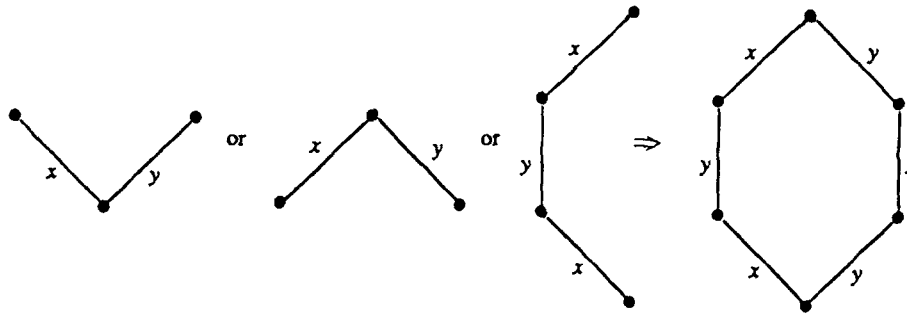
Figure 2. A cup, a hat, or a half-polygon implies a polygon of length $m(x, y)$. Here $m(x, y) = 3$.

Theorem, see §1. This means that an infinite polygon poset has no maximal elements, while a finite polygon poset has a unique maximum element, and that all its maximal chains are of equal length.

The *dual* of a poset P is the poset obtained by turning the Hasse diagram of P upside down. Note that if P has the hat property, its dual has the cup property. If P has the half-polygon property, then so does its dual. If P has a least element, the dual of P has a maximum element. Since by strong convergence any finite polygon poset has a maximum element, it is clear from the definition that the dual of any finite polygon poset is also a polygon poset.

A *lower interval* in a poset with a least element $\hat{0}$ is any interval of type $[\hat{0}, x]$. It is a trivial consequence of the definition of right weak order that any interval $[u, w]$ in $(W, S, \succeq)$ is isomorphic to the lower interval $[e, u^{-1}w]$.

Proposition 7   *Any interval in $(W, S, \succeq)$ is a polygon poset.*

Proof: Properties (a), (c), (d), and having a $\hat{0}$, are clearly inherited by lower intervals, while (b) remains to be proved. But an arbitrary lower interval $[e, w]$ is isomorphic to the dual of $[e, w^{-1}]$. Since $[e, w^{-1}]$ has the hat property (c), its dual must have the cup property (b).   □

We now point out a fundamental property of the weak order, due to Anders Björner.

Theorem 8   *The weak order on a Coxeter group is a $\wedge$-semilattice.*

Proof: We must show that any two elements w and u in the Coxeter group have a greatest lower bound, a *meet*, which we denote by $w \wedge u$. The set $[e, w] \cap [e, u]$ contains all lower bounds of w and u. The intersection $[e, w] \cap [e, u]$ inherits the properties (a)–(d) and the least element e, hence it is a polygon poset. It is of course finite, so by strong convergence it has a greatest element.   □

The above result was proved by Anders Björner in the early eighties (see his survey paper [1]), but the proof has not been published.
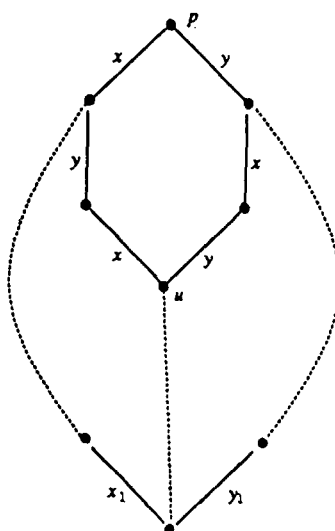
Figure 3. Sketch of the induction step. Here $m(x, y) = 3$.

4. Polygon posets are join ideals in the weak order

A result analogous to Tits's Word Theorem holds for the labels of shortest paths from $\hat{0}$ in a polygon poset. What really matters is only the half-polygon property and the hat property, as we will see below. To make the analogy transparent we introduce the terminology that a word that is obtained by reading the labels of a shortest path, i.e. a maximal chain, from $\hat{0}$ to an element p in a polygon poset is a *reduced expression* for p.

Lemma 9 *Any polygon poset P is graded, so all reduced expressions for an element p in P have the same length.*

Proof: The lower interval $[\hat{0}, p]$ inherits the hat property of P. Hence its dual has the cup property, and p becomes a least element, so it is strongly convergent by the Polygon Property Theorem. Since it is finite, all maximal chains in the dual of $[\hat{0}, p]$ are of equal length. Then this is of course true also for $[\hat{0}, p]$.   □

Now extend the M-simplifications of the first kind to operate also on these words. In the Hasse diagram, an M-simplification of the first kind means exchanging one half-polygon for the other.

Lemma 10 *If $y_1 y_2 \cdots y_k$ and $x_1 x_2 \cdots x_k$ are two reduced expressions for some element p of a polygon poset P, then they can be obtained from each other by a finite sequence of M-simplifications of the first kind.*

Proof (Sketch):   True if $k = 0$, that is, if $p = \hat{0}$. The induction should be clear from Fig. 3: Let $x = x_k$ and $y = y_k$. The hat property gives the polygon between p and u, where $\langle u \rangle$ has length $k - m(x, y)$, and the induction hypothesis gives the following chain

of transformations between reduced expressions:

$$y_1 y_2 \cdots y \to \cdots \to \langle u \rangle (\cdots yxy)_{m(x,y)} \to \langle u \rangle (\cdots xyx)_{m(x,y)} \to \cdots \to x_1 x_2 \cdots x$$

$\square$

Proposition 11 *If $s_1 s_2 \cdots s_k$ is a reduced expression for $p \in P$, then it is also reduced in the corresponding Coxeter group (W, S). All reduced expressions for $w = s_1 s_2 \cdots s_k$ in W are also reduced expressions for p in P, and vice versa. In other words, the intervals $[\hat{0}, p] \subset P$ and $[e, w] \subset (W, S, \succeq)$ are isomorphic.*

Proof: By Tits's Word Theorem, any non-reduced expression $s_1 s_2 \cdots s_k$ for $w \in W$ can be transformed to any reduced expression for w by only using M-simplifications of the first and second kind. In other words, an expression for $w \in W$ is reduced if no adjacent repeated letters arise after any sequence of M-simplifications of the first kind. Thanks to the half-polygon property of P, all words obtainable from $s_1 s_2 \cdots s_k$ by repeated half-polygon exchanges are also reduced expressions for p. By property (a), no reduced expression for p has adjacent repeated letters. Hence $s_1 s_2 \cdots s_k$ must be reduced also in the Coxeter group.

Since all reduced expressions for w in W are obtained from $s_1 s_2 \cdots s_k$ by repeated M-simplifications of the first kind, again the half-polygon property of P guarantees that all these expressions are also reduced expressions for p.

The converse follows from the lemma above. $\square$

Now recall the definition of a *join ideal* in a $\wedge$-semilattice P, given in §1. I is a join ideal in P if it is an order ideal in P, that is, $p \preceq q \in P \Rightarrow p \in P$, and I further satisfies $p, q \in I \Rightarrow p \vee q \in I$ if p and q have a least upper bound. Recall the easy fact that in a $\wedge$-semilattice, if p and q have any common upper bound at all, they must have a least upper bound.

Theorem 12 (a) *Any polygon poset P is isomorphic to a join ideal in $(W, S, \succeq)$.*

(b) *If (W, S) has no pair $s, t \in S$ where st has infinite order, then every join ideal in $(W, S, \succeq)$ is a polygon poset.*

Proof: (a) We want to define a map

$$\phi \colon P \to W$$

that is an isomorphism onto its image. For an element $p \in P$ with a reduced expression $s_1 s_2 \cdots s_k$, let $\phi(p) = s_1 s_2 \cdots s_k$, computed as a product in W. The proposition above implies that the image $\phi(P)$ is an order ideal I in $(W, S, \succeq)$ and that ϕ is an isomorphism from P to I. Due to this isomorhism, I is also a polygon poset.

Suppose $\phi(p)$ and $\phi(q)$ have a least upper bound $\phi(p) \vee \phi(q) = u \in W$. Then we must show that u belongs to I as well. Since $[e, u]$ is a polygon poset and I is a polygon poset containing e, the intersection $I \cap [e, u] = I'$ is a polygon poset. I' is finite, hence has a unique maximum element $v \preceq u$, and I' contains both $\phi(p)$ and $\phi(q)$, so v is also an upper bound of these elements. This is possible only if $v = u$, so u belongs to $I \cap [e, u]$ and hence to I.

(b) Every join ideal I in W clearly inherits the hat and half-polygon properties of $(W, S, \succeq)$. It remains to be proved that I has the cup property. A cup in I would consist of three elements: w, ws and wt, where $s, t \in S$, such that $l(ws) = l(wt) = l(w) + 1$. Since $m(s, t) < \infty$, there exists an upper bound $w(sts\cdots)_{m(s,t)}$ (the top of the polygon in $(W, S, \succeq)$). Hence there exists a least upper bound $ws \vee wt$ in $(W, S, \succeq)$, which therefore also is in the join ideal I. The interval $[e, ws \vee wt] \subset I$ is a polygon poset that contains w, ws and wt, so the cup property of this interval gives the cup property in I. □

Note that the condition in part (b), saying that no $m(x, y) = \infty$, cannot be omitted. Consider for example the infinite dihedral group W generated by $S = \{s, t\}$ where $m(s, t) = \infty$. Then $[e, s, t]$ is a join ideal (since $s \vee t$ does not exist) which is not a polygon poset.

Corollary 13 *The concepts of 'lower interval in the weak order' and 'finite polygon poset' are equivalent.*

Proof: We know from before that lower intervals in the weak order are finite polygon posets. The theorem says that polygon posets are join ideals, in particular they are order ideals, in the weak order. Finite polygon posets have a unique maximal element. A finite order ideal with a unique maximal element is a lower interval. □

Corollary 14 *Any infinite polygon poset is a complete $\wedge$-semilattice with no maximal elements. Any finite polygon poset is a lattice.*

Proof: We already know that the weak order is a complete $\wedge$-semilattice. Clearly every order ideal inherits this property; in particular every polygon poset. By strong convergence, a finite polygon poset has a unique maximal element, hence is a lattice, while an infinite polygon poset has no maximal elements. □

5. Polygon posets and generalized quotients

In this section we will discuss the connections with the work of Björner and Wachs [2] on generalized quotients in Coxeter groups. They define, for an arbitrary subset $V \subseteq W$, the *generalized quotient*

$$W/V = \{w \in W : v \in V \Rightarrow l(wv) = l(w) + l(v)\}.$$

A subset U of W has the *chain property* under Bruhat order if, whenever $u, w \in U$ and $u \preceq_B w$, there exists a chain $u = u_0 \preceq_B u_1 \preceq_B \cdots \preceq_B u_k = w$ in U such that $l(u_i) = l(u) + i$, for $1 \leq i \leq k$. A subset U of W is *directed* under Bruhat order if every pair of elements in U has a common upper bound in U.

Björner and Wachs showed that W/V is directed and has the chain property under Bruhat order. (In fact, they proved the stronger assertion that Bruhat order on W/V is CL-shellable.)

Under left weak order, W/V was shown to be a complete $\wedge$-semilattice with no maximal elements if it is infinite, and with a unique maximal element if finite. As we saw in Corollary 14, this is true also for polygon posets. This similarity is no coincidence—indeed, every generalized quotient is a polygon poset order ideal in the left weak order $\succeq_L$ on (W, S).

Proposition 15 W/V *under left weak order is a polygon poset.*

Proof: It is obvious from the definition of generalized quotients that W/V is an order ideal in $(W, S, \succeq_L)$. Thereby, it inherits the hat and half-polygon properties. If $s\langle w\rangle\langle v\rangle$ and $t\langle w\rangle\langle v\rangle$ are both reduced, then $(sts\cdots)_{m(s,t)}\langle w\rangle\langle v\rangle$ is reduced by the cup property of $(W, S, \succeq_L)$. Hence W/V has the cup property under $\succeq_L$. $\square$

Björner and Wachs also considered an alternative definition of generalized quotient. In a Coxeter group (W, S), let T be the set of conjugates of S, that is $T = \{wsw^{-1}: w \in W, s \in S\}$. For any subset A of T, define $W^A = \{w \in W: t \in A \Rightarrow wt \succeq_B w\}$. According to Björner and Wachs, every W/V is also an *alternative generalized quotient* W^A for some $A \subseteq T$. The sets W^A can be characterized as the *convex* order ideals in W under left weak order, where a subset U of a poset P is said to be *convex* if for all $u, w \in U$, every minimum length path from u to w in the Hasse diagram of P is in U. We will now show that every polygon poset is isomorphic to some $(W^A, \succeq_L)$.

Theorem 16 *Every polygon poset P is isomorphic to a convex subset of $(W, S, \succeq_L)$, the Coxeter group (W, S) under weak order.*

Proof: Identify P with the isomorphic join ideal in $(W, S, \succeq_L)$. We must show that every reduced expression for wu^{-1} gives a path from u to w in P. First note that there is at least some path between u and w, since they are connected via e. Since P is a polygon poset, one can easily deduce that all paths that can be constructed from some initial path in P by repeated M-simplifications are also in P. The M-simplifications of the second kind correspond to eliminating from the path edges traveled back and forth. The M-simplifications of the first kind means exchanging one half-polygon for the other. If the first half-polygon contains either a hat or a cup, then the exchange is possible thanks to the hat resp. cup properties of P. Otherwise the half-polygons are 'vertical' so to speak, so the half-polygon exchange can be done in P thanks to the half-polygon property.

By Tits's Word Theorem, all reduced expressions are obtained in this way. Thus P is convex. $\square$

Corollary 17 *The class of generalized quotients is contained in the class of polygon posets, which is in turn contained in the class of alternative generalized quotients:*

$$\{W/V\} \subset \{polygon\ posets\ in\ W\} \subset \{W^A\}.$$

Both inclusions are in fact strict. Strictness of the first one follows from the fact, as will be discussed in the coming section, that every W/V is a greedoid, which is not true for polygon posets. For the second one, consider the subset $U = \{e, x, y\}$ of the Coxeter group (W, S) where $S = \{x, y\}$ and $m(x, y) = 3$. Then U is a convex subset of the weak order, and hence a W^A, but U is not a polygon poset, since it does not have the cup property.

6. Construction of polygon posets

We shall now acquire a means to construct polygon posets.

Definition 6.1   Let $G_{(W,S)}$ be the set of all order ideals in $(W, S, \succeq_L)$ that are polygon posets. For any order ideal U in $(W, S, \succeq_L)$, define a closure operator σ by

$$\sigma(U) = \bigcap_{U \subseteq V \in G_{(W,S)}} V.$$

The intersection of polygon subposets with the same least element is a polygon poset, so we have $\sigma(U) \in G_{(W,S)}$. $\sigma(U)$ is in a natural sense the polygon poset generated by U.

Let $\check{G}_{(W,S)}$ be the set of all subsets of $(W, S, \succeq_L)$ that have the cup property. Since having the cup property is a much weaker condition than being a polygon poset, we have $G_{(W,S)} \subset \check{G}_{(W,S)}$. For any order ideal U in $(W, S, \succeq_L)$, define

$$\check{\sigma}(U) = \bigcap_{U \subseteq V \in \check{G}_{(W,S)}} V,$$

the smallest subset of $(W, S, \succeq_L)$ having the cup property and containing U. Obviously, $\check{\sigma}(U) \subseteq \sigma(U)$.

Lemma 18   *Suppose (W, S) is a Coxeter group with $x, y, z \in S$, where $m(x, y) \geq 3$, $m(x, z) \geq 3$ and $m(y, z) \geq 3$. If U is an order ideal in $(W, S, \succeq_L)$ containing $\{z, xy\}$, then $\sigma(U)$ is infinite.*

Proof: In $\check{\sigma}(U)$ we can complete polygons in x, y, z as shown in Fig. 4. It is clear from the figure that we get a repeating pattern, so the completing process will go on forever. Hence $\check{\sigma}(U)$, and thereby $\sigma(U)$, is infinite.   □

Definition 6.2   In the following we will say that a Coxeter group is *superhexagonal* if $m(x, y) \geq 3$ for all pairs of generators x, y.
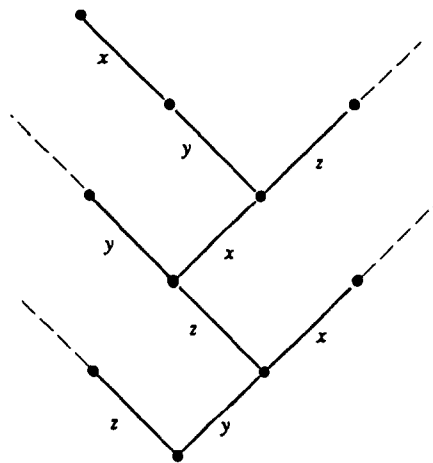

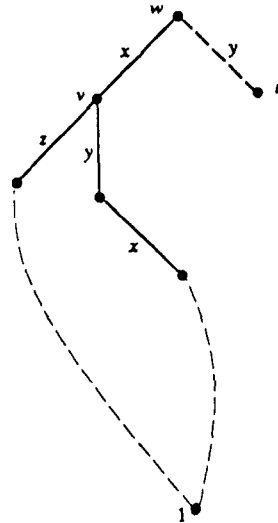*Figure 4.*   Repeating pattern in $\check{\sigma}(U)$.

Figure 5. Impossible situation!

Theorem 19 *Suppose* (W, S) *is a superhexagonal Coxeter group. If* U *is an order ideal in* $(W, S, \succeq_L)$ *then* $\sigma(U) = \breve{\sigma}(U)$.

Proof: Let $V = \breve{\sigma}(U)$. V has the cup property, so we only have to show that V is an order ideal, since every order ideal in the weak order has the hat property and the half-polygon property. Suppose that V is not an order ideal. Then there must be some shortest element $w \in V$ with a reduced expression (remember that this is *left* weak order) $w = yu$, where $u \notin V$. The minimality of $\breve{\sigma}(U)$ guarantees that all elements can be reached from U by repeated completion of upward polygons, like in Fig. 4. Therefore, there is also a reduced expression $w = xv$ with $v \in V$, and the x edge must have been added when completing some polygon, say an xz-polygon, where $z \neq y$. See Fig. 5 for a sketch of the situation. In particular there must be some reduced expression for v ending with z. By choice of w, all reduced expressions for v must be in V. By the hat property of $(W, S, \succeq_L)$, one such must end with yx. But then $[e, v^{-1}]$, which is the dual of $[e, v]$, would be a finite polygon poset containing $\{z, xy\}$, thereby contradicting the lemma above, which says that such a polygon poset must be infinite. $\qquad\qquad \Box$

As a first application of this construction method, we shall show that (the language induced by) a polygon poset need not be a greedoid.

Definition 6.3 A language $\mathcal{L}$ is a *greedoid* if it is left-hereditary, which means that

$$\alpha\gamma \in \mathcal{L} \Rightarrow \alpha \in \mathcal{L}$$

and if $\mathcal{L}$ further satisfies the following exchange condition (stronger exchange conditions may sometimes apply):

$$\alpha, \beta \in \mathcal{L}, |\beta| > |\alpha| \Rightarrow \exists x \in \beta : \alpha x \in \mathcal{L}$$
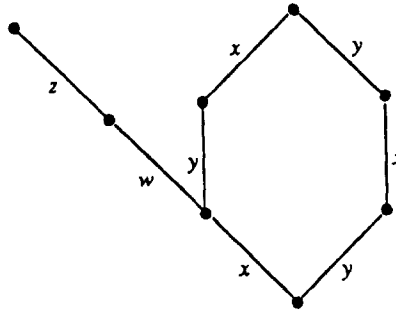
Figure 6. The polygon poset generated will not be a greedoid.

See [3] for greedoid theory. Every polygon poset induces a language consisting of the words given by the labels of upward-going paths from $\hat{0}$. For every generalized quotient, Björner and Wachs showed that this language is a greedoid. This is not true for polygon posets in general, as we shall show by constructing a counterexample. See Fig. 6. Let $(W, S, \succeq)$ be the right weak order of a superhexagonal Coxeter group with generators x, y, z, w. (We use right order here for compatibility with the greedoid conditions). Let U be the order ideal generated by $\{yx, xwz\}$. Now generate a polygon poset $V = \sigma(U) = \breve{\sigma}(U)$. Then $|xwz| > |yx|$, but neither of x, w and z can be used to extend yx in V, so the exchange condition of greedoids is not satisfied.

Since the language is a greedoid for all generalized quotients, by Björner and Wachs, we can conclude that the set of generalized quotients is a proper subset of the set of polygon posets.

7. The Bruhat order on polygon posets

Björner and Wachs asked whether the chain property and directedness under Bruhat order of W/V were true also for W^A in general. We will now answer this question by showing that not even polygon posets have these properties in general.

7.1. Counterexample to the chain property

Let (W, S) be a superhexagonal Coxeter group with generators $S = \{x, y, z, w\}$. There exists a polygon poset V contained in $(W, S, \succeq_L)$ that does not have the chain property under Bruhat order. We will now give a construction of it.

Let U be the order ideal in $(W, S, \succeq_L)$ generated by $\{zy, zwyx\}$. Let $V = \sigma(U) = \breve{\sigma}(U)$. We have $zwyx \succeq_B zy$, but any chain must include either zwy or zyx, and clearly neither of them is in $\breve{\sigma}(U)$. See Fig. 7.

7.2. Counterexample to directedness

Let (W, S) be a superhexagonal Coxeter group with generators $S = \{x, y, z, w, v\}$. There exists a polygon poset V contained in $(W, S, \succeq_L)$ that is not directed under Bruhat order.
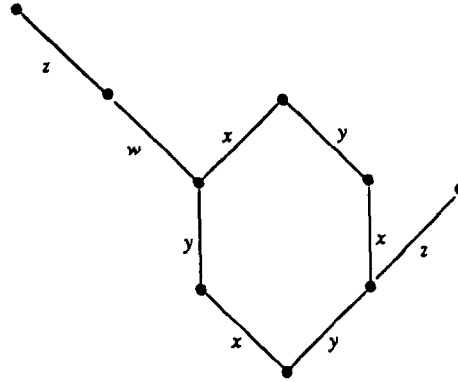
Figure 7. Neither *zwy* nor *zyx* will appear when completing cups to polygons in this poset.
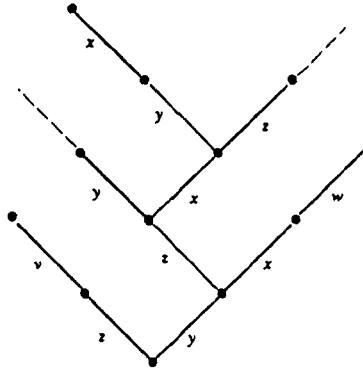

Figure 8. No word with both *v* and *w* will appear when completing polygons in this poset.

Let U be the order ideal generated by $\{vz, wxy\}$ in $(W, S, \succeq_L)$. Let $V = \sigma(U) = \breve{\sigma}(U)$. See Fig. 8. As in Lemma 18, we get an infinitely high 'wall' in the middle of V, generated by z and xy. When completing polygons, we may draw all polygons involving v on the left side of the wall, and all polygons involving w on the right side of the wall. Thus, this wall will separate all words containing w from all words containing v, so no word will contain both w and v. Thus there can be no common upper bound in V to vz and wxy in Bruhat order.

8. Sharp quotients

This section contains another example of a class of polygon posets.

Definition 8.1   For $V \subseteq W$, let

$$W \natural V \stackrel{\text{def}}{=} \{w \in W : v \in V \Leftrightarrow l(wv) = l(w) + l(v)\}.$$

$W \sharp V$ is called a *sharp quotient*. The difference from the definition of W/V is the '$\Leftrightarrow$' instead of '$\Rightarrow$', so $W \sharp V \subseteq W/V$.

In order to show that the sharp quotient $W \sharp V$ under left weak order is a polygon poset, we must have the following lemma, which is an easy consequence of the Exchange Condition.

Lemma 20 *If* $s_k \cdots s_1$ *and* $s_r \cdots s_1 s$ $(k > r)$ *are reduced expressions, but* $s_k \cdots s_1 s$ *is not reduced, then* $s_k \cdots s_1 = s_k \cdots \hat{s}_i \cdots s_1 s$ *for some index* $i > r$.

Proof: By the Exchange Condition, $s_k \cdots s_1 = s_k \cdots \hat{s}_i \cdots s_1 s$ for some index i. If we cancel identical factors from the left, we get $s_i \cdots s_1 = s_{i-1} \cdots s_1 s$, so $s_i \cdots s_1 s$ is not reduced. Thus we must have $i > r$. □

Proposition 21 *Sharp quotients are closed under meet in the left weak order, i.e. if* $u, w \in W \sharp V$ *then* $u \wedge w \in W \sharp V$.

Proof: Take $w, u \in W \sharp V$. Let α be a lower bound of w and u in $(W, S, \succeq_L)$, such that $\alpha \notin W \sharp V$. We must show that $\alpha \neq w \wedge u$, that is, that α is not the *greatest* lower bound.

First note that $w, u \in W \sharp V$ implies that

$$l(wv) = l(w) + l(v) \Leftrightarrow l(uv) = l(u) + l(v),$$

Since α is a lower bound of w and u, we have in particular that $w \succeq_L \alpha$, which implies that

$$l(wv) = l(w) + l(v) \Rightarrow l(\alpha v) = l(\alpha) + l(v),$$

but not the other way around, since $\alpha \notin W \sharp V$. There must be some shortest v such that $l(\alpha v) = l(\alpha) + l(v)$ while $l(wv) < l(w) + l(v)$, in which case of course also $l(uv) < l(u) + l(v)$. Split $\langle v \rangle$ as $\langle \beta \rangle s$, where $s \in S$. Then $\langle w \rangle \langle \beta \rangle$, $\langle u \rangle \langle \beta \rangle$ and $\langle \alpha \rangle \langle \beta \rangle s$ are reduced while $\langle w \rangle \langle \beta \rangle s$ and $\langle u \rangle \langle \beta \rangle s$ are not reduced.

Let $w = s_k s_{k-1} \cdots s_1 \langle \alpha \rangle$ be a reduced expression for w. Then for some i between 1 and k, by the lemma, we have $w\beta = s_k s_{k-1} \cdots s_1 \langle \alpha \beta \rangle = s_k \cdots \hat{s}_i \cdots s_1 \langle \alpha \beta \rangle s$, so $w\beta$ is an upper bound for $\alpha \beta s$ and $\alpha \beta$. Let $\langle \gamma \rangle \langle \alpha \beta \rangle$ be a reduced expression for $\alpha \beta s \vee \alpha \beta$. Then $l(\gamma) \geq 1$. We have $w\beta \succeq_L \gamma \alpha \beta$, so $w \succeq_L \gamma \alpha$. By a similar argument we have also $u \succeq_L \gamma \alpha$. Hence $\gamma \alpha$ is a lower bound of u and w. Since $l(\gamma) \geq 1$ and $\gamma \alpha \succeq_L \alpha$, α cannot be the greatest lower bound of u and w. □

Theorem 22 $(W \sharp V, \succeq_L)$ *is a polygon poset.*

Proof: From the proposition above we deduce that $(W \sharp V, \succeq_L)$ has a least element, as well as the hat property, in the same way as the cup property followed from join ideals being closed under join in Theorem 12. Obviously it has the half-polygon property. The cup property follows in the same way as for generalized quotients, in Proposition 15. □

9. Connections with FSA recognizing the language of reduced words

It has recently been proved that there exists a finite state automaton that recognizes the language $\mathcal{L}$ of reduced words of a given Coxeter group (W, S). For different approaches, see Brink and Howlett [5], Davis and Shapiro [6], H. Eriksson [7], Headley [11]. In this section we shall see how this result can be interpreted in terms of the concepts that were discussed in the previous sections: sharp quotients and generalized quotients. We will also apply the polygon poset construction technique of §6 to give a new very short proof of the existence of an FSA when (W, S) is superhexagonal.

An automaton recognizing reduced words can be modeled as a directed graph with the states as vertices, and with an edge $(\overrightarrow{u, v})$ labeled x if reading x in state u causes a transition to state v.

Let $\mathcal{A}$ be a *minimal* automaton that recognizes $\mathcal{L}$. For any $w \in W$, define

$$C_w = \{v \in W : \langle w \rangle \langle v \rangle \text{ reduced}\}.$$

Thus C_w is the set of possible continuations of a reduced expression for w. Let S_w be the state in $\mathcal{A}$ that is reached after reading a reduced expression for w. Clearly, we can have $S_v = S_w$ only if $C_v = C_w$, and since $\mathcal{A}$ is minimal it will have $S_v = S_w$ if and only if $C_v = C_w$. Since $\mathcal{A}$ is finite, there can only be finitely many different continuation sets. If W is infinite, there must thus be many elements with the same continuation set. Define a relation $\sim$ on W by $v \sim w$ if $C_v = C_w$. Then $\sim$ is an equivalence relation, so we can define the quotient $W/\sim$. Obviously $W/\sim$ can be identified with the set of states of $\mathcal{A}$.

The elements of $W/\sim$ are equivalence classes of the type

$$\{w \in W : C_w = C_v\} = \{w \in W : \langle w \rangle \langle u \rangle \text{ reduced} \Leftrightarrow u \in C_v\} = W \sharp C_v,$$

that is, the sharp quotients. Observe also that $W \sharp V \neq \emptyset$ implies that $V = C_v$ for some $v \in W$.

By the existence of a *finite* automaton, we have the following result.

Proposition 23 *There are only finitely many different subsets of W that can arise as sharp quotients.*

Note in particular that the sharp quotients induce a partitioning of $(W, S, \succeq_L)$ into a finite number of disjoint polygon posets.

Note further that C_w is equal to a 'right weak order generalized quotient' $W/_R\{w\}$, defined in analogy with the ordinary generalized quotients by

$$W/_R V \overset{\text{def}}{=} \{u \in W : v \in V \Rightarrow l(vu) = l(v) + l(u)\}.$$

We know that there are only finitely many different C_w. Since

$$W/_R V = \bigcap_{w \in V} W/_R\{w\},$$

there will only be finitely many non-isomorphic generalized quotients in right weak order. However, this carries over bijectively to left weak order, since $W/\{w\}$ is the set of inverse elements to $W/_R\{w^{-1}\}$:

Proposition 24   *There are only finitely many different subsets of W that can arise as generalized quotients.*

9.1.   FSA for superhexagonal Coxeter groups

Now, let (W, S) be a superhexagonal Coxeter group. We shall use our techniques to show in a quick way that the language $\mathcal{L}$ of reduced words is recognized by a finite state automaton. As pointed out in above, this is equivalent to showing that there are only finitely many different continuation sets $C_w = \{v \in W : l(wv) = l(w) + l(v)\}$ when w runs through all of W, since the states in a minimal automaton recognizing $\mathcal{L}$ must correspond one-to-one with the different continuation sets. Define C_w^k to be the k:th level of C_w, that is,

$$C_w^k = \{v \in C_w : l(v) = k\}$$

and let $N = \max\{m(s, t) : s, t \in S, m(s, t) < \infty\}$, the length of the longest finite polygon. We shall prove that C_w is uniquely determined if $\bigcup_{k=1}^N C_w^k$ is known. Since there are only finitely many possible candidates to $\bigcup_{k=1}^N C_w^k$, there can be only finitely many different C_w.

It is enough to prove that we can construct C_w^{n+1} given $\bigcup_{k=1}^n C_w^k$, where $n \geq N$, as the desired result then follows by induction. Thus we must prove that for any $v \in C_w^n$ and any generator s we can determine if $vs \in C_w^{n+1}$, i.e. if $\langle w \rangle \langle v \rangle s$ is reduced.

If $\langle v \rangle s$ is not reduced then $\langle w \rangle \langle v \rangle s$ cannot be reduced, so in that case we are done. Otherwise a reduced expression for v must end with some $t \neq s$.

Let r be the greatest integer such that there is a reduced expression $\langle u \rangle (\cdots tst)_r$ for v. If $m(s, t) = \infty$ then $\langle w \rangle \langle v \rangle s$ must be reduced, since there would otherwise be a polygon hat to an infinite polygon, which is absurd. Thus we can assume that $r \leq m(s, t) \leq N$. However, the case when $r = m(s, t)$ is already covered since then $\langle v \rangle s$ is not reduced.

Case 1. Suppose that $r = m(s, t) - 1$. If both us and ut are elements in C_w^{n-r+1} then, by the cup property, $vs = u(sts\cdots)_{m(s,t)} \in C_w^{n+1}$.
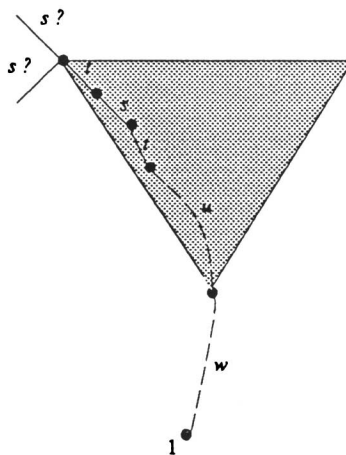

Figure 9.   Where does the edge labeled s go? The shaded part is $\cup_{k=1}^n C_w^k$, translated by w.

Otherwise, if us, say, is not in C_w^{n-r+1} then $\langle w \rangle \langle u \rangle s$ is not reduced, so there must be some reduced expression $\langle \beta \rangle s$ for wu. Hence wv can be expressed $\langle \beta \rangle (sts \cdots)_{m(s,t)}$, so $\langle w \rangle \langle v \rangle s$ is not reduced. Thus $vs \notin C_w^{n+1}$.

Case 2. $r \leq m(s, t) - 2$. Then $l(u) = n - r \geq N - r \geq 2$, so a reduced expression for u must end with some z, different from s and t. Thus by Lemma 18 (used as in Theorem 19) it is impossible that $\langle w \rangle \langle u \rangle$ end with ts or st, hence $\langle w \rangle \langle v \rangle = \langle wv \rangle (\cdots tst)_r$ does not end with $(\cdots tst)_{r+2}$, and thereby it cannot end with $(\cdots tst)_{m(s,t)}$. Thus $\langle w \rangle \langle v \rangle s$ is reduced, so $vs \in C_w^{n+1}$.

10. Position posets and the numbers game

Finally we comment on the connection to the numbers game of Mozes [12]. This is a 1-player game played on graph $G = (V, E)$, with real numbers placed on the nodes. The rules are that any node s with a negative number x_s may be fired. This means that to the number on every neighbor t of the fired node s, one adds the number $k_{st}x_s$, where $k_{st} > 0$ is the weight of edge $(s, t) \in E$. Finally the sign of x_s is reversed. Figure 10 shows the possible ways to play the game from a certain position on a little graph, where all edge weights are equal to 1.

The game goes on until all numbers are nonnegative. If the position is represented by the vector (in $\mathbb{R}^{|V|}$) of node numbers, the moves are linear transformations on this vector.

For certain choices of the edge weights, this game is strongly convergent. In these cases, the language of all legal play sequences (over the alphabet of nodes) is a subset of the language of reduced words in an associated Coxeter group (W, S), where S can be identified with the set V of nodes of the graph. Conversely, to every Coxeter group can
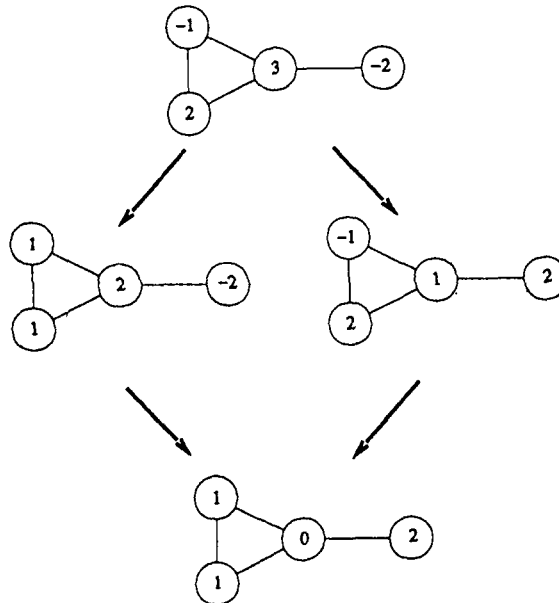

Figure 10. How the numbers game is played.

be associated a game of this kind. If some play sequence $s_1 s_2 \cdots s_n$ is legal, then every reduced expression of the element $s_1 s_2 \cdots s_n$ in the group (W, S) is a legal play sequence, and such play sequences are equivalent (denoted by $\equiv$) in the sense that they have the same effect on the position. This can be restated as follows: if $\alpha \equiv \beta$, then any start position p for which α and β are legal play sequences will result in $\alpha(p) = \beta(p)$. See Eriksson [9, 10] for all details.

If the positions do not recur during play, then the set $P(p)$ of positions that are reachable from a given start position p can be partially ordered by letting $p_2 \succeq p_1$ if p_2 can be reached by playing from p_1. This *position poset*, with edge labeling given by the fired nodes, is *almost always* a polygon poset, in the following sense. We know that the cup property and the half-polygon property hold [9, 10]. A *hat* in $P(p)$ signifies two play sequences such that $\alpha(p) = \beta(p)$, and if $\alpha \equiv \beta$, then the hat property of $(W, S, \succeq)$ provides the polygon under the hat in $P(p)$. Thus, if for some p the position poset $P(p)$ does not have the hat property, it is because for some play sequences $\alpha \not\equiv \beta$ we have $\alpha(p) = \beta(p)$ for this particular position p. This is equivalent to $\alpha^{-1}\beta(p) = p$. If p is regarded as a vector in $\mathbb{R}^{|V|}$ and α, β as matrices, then this implies that p is an eigenvector with eigenvalue 1 to the matrix $\alpha^{-1}\beta$, which is not the identity matrix. Hence, the eigenspace corresponding to eigenvalue 1 has dimension less than $|V|$, so it has measure zero relative to $\mathbb{R}^{|V|}$.

Theorem 25 *To each finite polygon poset Q there is a numbers game graph G and a start position p, such that the position poset $P(p)$ is isomorphic to Q.*

Proof: We know that Q is isomorphic to some interval $[e, w]$ in $(W, S, \succeq)$. Let G be the graph of a game associated with this Coxeter group W. Let q be the position where every node of G has number -1. Play from q according to $\langle w^{-1} \rangle$. Let p' be the resulting position.

Now, let $p = -p'$, i.e. all numbers in p have the reverse sign of corresponding numbers in p'. Clearly we can now play the moves of $\langle w^{-1} \rangle$ in reverse order, i.e. according to $\langle w \rangle$, and this must result in the position $-q$ where all numbers are 1, so $-q$ is terminal. Thus $P(p)$ is isomorphic to $[e, w]$, and hence to Q. $\qquad\square$

For infinite polygon posets the situation is quite different. In [9] was shown that the language of legal play in a numbers game is always a greedoid, but as we saw in §6, infinite polygon posets are not greedoids in general. Thus the class of polygon posets is larger than the class of positin posets.

However, Björner and Wachs [2] proved that all generalized quotients are greedoids as well. It is an open question whether the position posets have also the other properties special to generalized quotients; the chain property, CL-shellability and directedness under Bruhat order.

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