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On Young's Orthogonal Form and the Characters of the Alternating Group

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Abstract. A combinatorial method of determining the characters of the alternating group is presented. We use matrix representations, due to Thrall, that are closely related to Young's orthogonal form of representations of the symmetric group. The characters are computed directly from matrix entries of these representations and entries of the character table of the symmetric group.

Keywords: group character, alternating group, Young tableau

1. Introduction

The irreducible complex representations of the alternating group are closely related to the representations of the symmetric group. Each irreducible representation of A_n is a direct summand of the restriction of an irreducible representation of S_n; in fact, most of the irreducible representations of S_n remain irreducible upon restriction. The exceptions are those representations ρ such that $\epsilon \otimes \rho \cong \rho$, where ϵ is the sign representation. When this occurs, $\rho \downarrow A_n$ is the direct sum of two irreducible representations ρ^+ and ρ^-. Thrall constructed the isomorphism $\epsilon \otimes \rho \cong \rho$ and used it to describe matrix representations of ρ^+ and ρ^- [8]. The purpose of this paper is to present a combinatorial method for determining the characters of ρ^+ and ρ^-, using Thrall's construction. Other methods in the literature rely on algebraic arguments within the theory of characters [5].

2. Young's orthogonal form and the associator

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of n, i.e., $\lambda_1 \geq \cdots \geq \lambda_l > 0$ and $\lambda_1 + \cdots + \lambda_l = n$. The *diagram* of λ is the set $D_\lambda = \{(i, j) \mid i, j \in \mathbf{Z}, 1 \leq i \leq l, 1 \leq j \leq \lambda_i\}$. The *content* $c(i, j)$ of $(i, j) \in D_\lambda$ is defined to be $j - i$. The *conjugate* λ' of λ is the partition satisfying $(i, j) \in D_{\lambda'}$ iff $(j, i) \in D_\lambda$. If $\lambda' = \lambda$, λ is said to be *self-conjugate*. A *tableau* of shape λ is a bijection from D_λ to the set $\{1, \ldots, n\}$. More informally, we can view T as a configuration of the integers $1, \ldots, n$ with $T(i, j)$ lying at the point (i, j). A tableau T is *standard* if $T(i, j) < T(i, j + 1)$ and $T(i, j) < T(i + 1, j)$ for all i, j. The *transpose* T' of T is the tableau of shape λ' satisfying $T'(i, j) = T(j, i)$; T' is standard if and only if T is standard.

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There is a bijection between the irreducible representations of S_n and the partitions of n. The dimension of the representation corresponding to λ is equal to the number of standard tableaux of shape λ. The representations have been constructed in several ways using these tableaux [1, 2, 4–6]. The following construction is known as *Young's orthogonal form.*

Theorem 2.1 *Let V_λ be the* $\mathbf{C}$*-span of the standard tableaux of shape λ. For each standard tableau T of shape λ, define the axial distance d_T on $\{1, \ldots, n\}$ by $d_T(p,q) = c(T^{-1}(q)) - c(T^{-1}(p))$. If σ_j is the transposition $(j, j+1)$, then the linear map $\rho_\lambda\colon S_n \to GL(V_\lambda)$ defined by*

$$\rho_\lambda(\sigma_j)T = \begin{cases} 1/d_T(j,j+1)T + \sqrt{1 - 1/d_T^2(j,j+1)}\sigma_j T & \text{if } \sigma_j T \text{ is standard} \\ 1/d_T(j,j+1)T & \text{if } \sigma_j T \text{ is not standard} \end{cases}$$

extends to an irreducible representation of S_n.

Using this representation, one can easily show that $\epsilon \otimes \rho_\lambda \cong \rho_{\lambda'}$ for any partition λ. See [2] for further details on the orthogonal form and related representations.

Let λ be a self-conjugate partition. We can now establish the isomorphism between $\epsilon \otimes \rho_\lambda$ and ρ_λ by exhibiting $S \in \mathrm{GL}(V_\lambda)$ satisfying $S\rho_\lambda(g)v = \mathrm{sgn}(g)\rho_\lambda(g)Sv$ for all $g \in S_n$, $v \in V_\lambda$. Notice that, by Schur's Lemma, this defines S up to multiplication by a scalar. Also by Schur's Lemma, since S^2 commutes with ρ_λ, S^2 is a scalar. Thus, S may be chosen (in two ways) so that $S^2 = I$. Following [7] we will call such an S an *associator* for ρ_λ.

Theorem 2.2 (Thrall) [8] *Let T_0 be a standard tableau of shape λ, where λ is self-conjugate. For any standard tableau T also of shape λ, define $sgn(T)$ to be $sgn(w)$, where $w \in S_n$ satisfies $wT = T_0$. Let $d(\lambda)$ be the length of the main diagonal of λ; i.e., $d(\lambda)$ is the largest integer j such that $(j, j) \in D_\lambda$. Let $S'\colon V_\lambda \to V_\lambda$ be defined by*

$$S'(T) = sgn(T)T'.$$

Then $S = i^{(n-d(\lambda))/2}S'(T)$ is an associator for ρ_λ.

Proof: We first establish that $S'\rho_\lambda(g) = \mathrm{sgn}(g)\rho_\lambda(g)S'$. Let $\sigma_j = (j, j+1)$, and assume that T, U are standard tableaux of shape λ satisfying $\sigma_j T = U$. The subspace spanned by T, U, T', U' is invariant under the action of both S' and $\rho_\lambda(\sigma_j)$. The equality $S'\rho_\lambda(\sigma_j) = -\rho_\lambda(\sigma_j)S'$ can easily be verified on this subspace after observing that $\mathrm{sgn}(T) = -\mathrm{sgn}(U)$, $\mathrm{sgn}(T') = -\mathrm{sgn}(U')$, $d_T(j, j+1) = -d_{T'}(j, j+1)$, and $d_U(j, j+1) = -d_{U'}(j, j+1)$. If $\sigma_j T$ is not standard, then the subspace spanned by T and T' is invariant under the action of both S' and $\rho_\lambda(\sigma_j)$. The equality $S'\rho_\lambda(\sigma_j) = -\rho_\lambda(\sigma_j)S'$ can easily be verified on subspaces of this form as well, so $S'\rho_\lambda(\sigma_j)(v) = -\rho_\lambda(\sigma_j)S'(v)$ for all $v \in V_\lambda$. Since the transpositions σ_j, $1 \le j \le n-1$, generate S_n, the result follows.

Finally, $(S')^2 = (-1)^{(n-d(\lambda))/2}I$ since

$$(S')^2T = S'(\mathrm{sgn}(T)T') = \mathrm{sgn}(T)\mathrm{sgn}(T')T,$$

and $\mathrm{sgn}(T) = (-1)^{(n-d(\lambda))/2}\mathrm{sgn}(T')$. Thus, $S^2 = I$. □

3. The difference characters of A_n

We continue to consider the case where λ is a self-conjugate partition. The purpose of this section is to decompose $\rho_\lambda \downarrow A_n$ into irreducible representations and determine their characters.

Proposition 3.1 *The eigenspaces of S are A_n-modules under the action of $\rho_\lambda(g)$, $g \in A_n$. Let V^+, V^- be the eigenspaces of S corresponding to the eigenvalues $1, -1$, respectively, and let ρ_λ^+, ρ_λ^-, be the projections of $\rho_\lambda \downarrow A_n$ into $GL(V^+)$, $GL(V^-)$, respectively. Then (assuming $n > 1$) $\rho_\lambda \downarrow A_n$ decomposes into irreducible representations as the direct sum of ρ_λ^+ and ρ_λ^-.*

Proof: For $v \in V_\lambda$, if $Sv = \alpha v$, then $S\rho_\lambda(g)v = \rho_\lambda(g)Sv = \rho_\lambda(g)\alpha v = \alpha\rho_\lambda(g)v$. Since $S^2 = I$ but (for $n > 1$) S is not a scalar, V^+ and V^- are both nontrivial, and $V^+ \oplus V^- = V_\lambda$. Let $\chi^\lambda \downarrow A_n$ be the character of $\rho_\lambda \downarrow A_n$. Since ρ_λ is an irreducible S_n-module, and A_n has index 2 in S_n, the inner product $[\chi^\lambda \downarrow A_n, \chi^\lambda \downarrow A_n]$ is at most 2. Thus, ρ_λ^+ and ρ_λ^- must be irreducible. □

Corollary 3.2 *Let $g \in A_n$. If χ^+, χ^- are the characters of ρ_λ^+, ρ_λ^-, respectively, then $tr(S\rho_\lambda(g)) = \chi^+(g) - \chi^-(g)$.*

Proof: We have

$$S\rho_\lambda(g) = S\rho_\lambda^+(g) + S\rho_\lambda^-(g) = \rho_\lambda^+(g) - \rho_\lambda^-(g).$$

Taking traces of this equation gives $\mathrm{tr}(S\rho_\lambda(g)) = \chi^+(g) - \chi^-(g)$. □

Since the value of $\chi^\lambda \downarrow A_n = \chi^+ + \chi^-$ can be found in the character table of S_n [4], one can solve for χ^+ and χ^- if $\mathrm{tr}(S\rho_\lambda(g))$, the *difference character* of χ^+ and χ^-, is known. We need only compute $\mathrm{tr}(S\rho_\lambda(g))$ for one element in each even conjugacy class of S_n, since, by Clifford's theorem on the characters of normal subgroups, $\chi^+(g) = \chi^-(hgh^{-1})$, where $g \in A_n$, $h \in S_n - A_n$ [3]. For cycle type $(a_1, \ldots, a_m)$ (with the a_i listed in nonincreasing order), a representative permutation will be

$$\begin{aligned}(1 \cdots b_1)&(b_1 + 1 \cdots b_2) \cdots (b_{m-1} + 1 \cdots b_m)\\ &= \sigma_1\sigma_2 \cdots \sigma_{b_1-1}\sigma_{b_1+1} \cdots \sigma_{b_2-1}\sigma_{b_2+1} \cdots \sigma_{b_{m-1}-1}\sigma_{b_{m-1}+1} \cdots \sigma_{b_m-1}\end{aligned}$$

where $b_j = a_1 + \cdots + a_j$. The *cycles* of this permutation are of course the factor permutations $(1 \cdots b_1)$, $(b_1 + 1 \cdots b_2), \ldots$, but we will abuse notation at times and refer to the sets $\{1, \ldots, b_1\}$, $\{b_1 + 1, \ldots, b_2\}, \ldots$, as the cycles of the permutation as well.

Let $q = ((n - d(\lambda))/2)$. It is straightforward that

$$\mathrm{tr}(S\rho_\lambda(g)) = i^q \sum_T \mathrm{sgn}(T')\rho_\lambda(g)_{T',T} = (-i)^q \sum_T \mathrm{sgn}(T)\rho_\lambda(g)_{T',T},$$

where the sum is taken over all standard tableaux of shape λ. If g is the representative permutation of its cycle type, $\rho_\lambda(g)$ can be expressed as a product of $\rho_\lambda(\sigma_i)$ as above. Since $\rho_\lambda(\sigma_i)T$ is in the span of T and $\sigma_i T$ (which is defined to be 0 if it is not standard), $\rho_\lambda(g)T$ can be expanded so that there is at most one term corresponding to each subsequence of the σ_i's occurring in the expansion of g. It is not hard to see that distinct subsequences correspond to distinct permutations acting on T, since one can readily convert the permutation into cyclic notation. If $wT = T'$, then w is a product of q disjoint 2-cycles. This w can be expressed as the product of a subsequence of the σ_i's if and only if each pair $T(j, k)$ and $T(k, j)$ is transposed by one of the σ_i's. In other words, both of the following must be true for all $j \neq k$ such that $(j, k) \in D_\lambda$:

- $T(j, k)$ and $T(k, j)$ are in the same cycle of g.
- $|T(j, k) - T(k, j)| = 1$.

We will call such a standard tableau *transposable*. Thus, $\rho_\lambda(g)_{T',T} \neq 0$ iff T is transposable, and the expansion of $\rho_\lambda(g)T$ has a unique term in the span of T'.

We can now begin to determine $\mathrm{tr}(S\rho_\lambda(g))$. Assume throughout that g is the representative permutation of its cycle type.

Lemma 3.3 *If g has more than $d(\lambda)$ cycles of odd length, then $tr(S\rho_\lambda(g)) = 0$.*

Proof: If T is transposable, then each cycle of g of odd length must contain one entry on the main diagonal of T. This is impossible if T has shape λ and the number of cycles of odd length exceeds $d(\lambda)$. □

Lemma 3.4 *If the number of cycles of g is less than $d(\lambda)$, then $tr(S\rho_\lambda(g)) = 0$.*

Proof: If the number of cycles of g is less than $d(\lambda)$, then there exists a cycle of g that contains more than one entry on the main diagonal of λ. We construct an involution on the transposable tableaux. Given a standard tableau T of shape λ, choose b minimal so that there exists $a < b$ with a and b in the same cycle of g and both on the main diagonal of T. Let $T^* = \sigma_{a+1}\sigma_{a+3}\cdots\sigma_{b-4}\sigma_{b-2}T$. This is an involution, since T and T^* are equal on the main diagonal and $\sigma_{a+1}\sigma_{a+3}\cdots\sigma_{b-4}\sigma_{b-2}$ has order 2. In the computation of $\rho_\lambda(g)_{T',T}$ and $\rho_\lambda(g)_{(T^*)',T^*}$, the only factors that differ are those corresponding to $\sigma_a, \sigma_{a+2}, \ldots, \sigma_{b-3}, \sigma_{b-1}$; for each of these the difference is a factor of -1. Thus, $\rho_\lambda(g)_{T',T} = (-1)^{(b-a+1)/2}\rho_\lambda(g)_{(T^*)',T^*}$. Since $\mathrm{sgn}(T) = (-1)^{(b-a-1)/2}\mathrm{sgn}(T^*)$, and $b - a$ is odd, $\mathrm{sgn}(T)\rho_\lambda(g)_{T',T} + \mathrm{sgn}(T^*)\rho_\lambda(g)_{(T^*)',T^*} = 0$. Summing over all transposable tableaux, $\mathrm{tr}(S\rho_\lambda(g)) = 0$. □

Lemma 3.5 *If g has a cycle of even length, then $tr(S\rho_\lambda(g)) = 0$.*

Proof: We construct an involution similar to that of the previous lemma. Let $(c \cdots c + 2k - 1)$ be a cycle of g and let T be a transposable tableau of shape λ. If this cycle intersects the main diagonal of T, it does so in at least two places (since T is transposable). In this case, let a, b be the two smallest numbers in this intersection. If the intersection is empty, let $a = c - 1$, $b = c + 2k$. In either case, define T^* to be $\sigma_{a+1}\sigma_{a+3}\cdots\sigma_{b-4}\sigma_{b-2}T$. As in the proof of the previous lemma, this is an involution. Also as before, $\rho_\lambda(g)_{T',T} = (-1)^{(b-a+1)/2}\rho_\lambda(g)_{(T^*)',T^*}$, and $\mathrm{sgn}(T) = (-1)^{(b-a-1)/2}\mathrm{sgn}(T^*)$, so $\mathrm{sgn}(T)\rho_\lambda(g)_{T',T} + \mathrm{sgn}(T^*)\rho_\lambda(g)_{(T^*)',T^*} = 0$, and $\mathrm{tr}(S\rho_\lambda(g)) = 0$. □

Remark Lemma 3.5 can also be established by purely group-theoretic means, by showing that, given the hypotheses of the lemma, the conjugacy classes of g in S_n and A_n are the same, and, thus, $\chi^+(g) = \chi^-(g)$.

The only remaining cycle types are those consisting of $d(\lambda)$ cycles of odd length. In analyzing this case, it will help to use the concepts of *hook* and *hooklength*. For $(i, j) \in D_\lambda$, the corresponding hook H_{ij} is the set of points in D_λ of the form (i, k), $j \leq k$ or (k, j), $i \leq k$. The hooklength h_{ij} is the cardinality of H_{ij}. A hook of a tableau T is the image under T of a hook of its diagram.

Also needed will be the following result about posets. A *linear extension* of a poset P with $|P| = m$ is a bijection $\tau\colon P \to \{1, \ldots, m\}$ such that $i \leq_P j$ implies $\tau(i) \leq \tau(j)$.

Lemma 3.6 *Let P be a disconnected poset of size m. For each element $p \in P$, choose an indeterminate x_p. Then*

$$\sum_{\tau\in L(P)} \prod_{i=1}^{m-1} \frac{1}{x_{\tau^{-1}(i+1)} - x_{\tau^{-1}(i)}} = 0,$$

where $L(P)$ denotes the set of linear extensions of P.

Proof: First assume that P is the disjoint union of two linearly ordered components. Let υ be an arbitrary linear extension of P and let

$$f = \left(\prod_{j<k}\left(x_{\upsilon^{-1}(j)} - x_{\upsilon^{-1}(k)}\right)\right) \cdot \sum_{\tau\in L(P)} \prod_{i=1}^{m-1} \frac{1}{x_{\tau^{-1}(i+1)} - x_{\tau^{-1}(i)}}.$$

Then, if $f \neq 0$, f is a homogeneous polynomial of degree $n(n-1)/2 - (m-1)$. If p and q belong to the same component but are not adjacent in the ordering, then $(x_p - x_q)$ divides f, since $|\tau(p) - \tau(q)| \geq 2$ for $\tau \in L(P)$. If p and q belong to different components, then for any $\tau \in L(P)$ with $\tau(p) - \tau(q) = 1$ we can find $\tau' \in L(P)$ with $\tau'(q) - \tau'(p) = 1$ by letting $\tau'(p) = \tau(q)$, $\tau'(q) = \tau(p)$, and $\tau' = \tau$ otherwise. The contributions of τ and τ' to f are identical except that the variables x_p and x_q have been switched. Thus, their sum is divisible by $x_p - x_q$, and in this case as well $(x_p - x_q)$ divides f. So $n(n-1)/2 - (m-2)$ irreducible factors of f have been identified, and f must be 0.

In the general case, if P is the disjoint union of posets P_1 and P_2, a linear extension of P induces linear extensions of P_1 and P_2. If we consider all of the extensions of P that induce a particular pair of extensions of P_1 and P_2, we have the situation of the preceding paragraph, and the sum over these extensions is 0. Thus, the entire sum is 0. □

The sum in Lemma 3.6 also appears in [2], where Greene uses it to derive the Murnaghan-Nakayama Rule for evaluating the characters of S_n. He proves the above identity in the case of disconnected planar posets as well as finding an identity for the sum when P is a connected planar poset.

Theorem 3.7 *If g does not have cycle type $(h_{11}, h_{22}, \ldots, h_{d(\lambda),d(\lambda)})$, then $tr(Sp_\lambda(g)) = 0$.*

Proof: We assume that g consists of $d(\lambda)$ cycles of odd length but does not have cycle type $(h_{11}, h_{22}, \ldots, h_{d(\lambda),d(\lambda)})$. Among the transposable tableaux of shape λ, each main diagonal entry belongs to a distinct cycle of g, by a counting argument similar to that in the proof of Lemma 3.3. Let T be such a tableau. At least one cycle of g must intersect more than one of the hooks H_{ii} of T. Let g_i be the cycle of g containing $T(i, i)$, and let $g_T = g_j$, where j is minimal so that g_j intersects more than one of the hooks H_{ii}. Let λ_T be the skew-diagram $T^{-1}(g_T)$. An equivalence relation on the transposable tableaux can now be defined as follows: $T \sim U$ iff

- $\lambda_T = \lambda_U$ (implying $g_T = g_U$).
- For all $(i, j) \in \lambda_T = \lambda_U$, $T(i, j) > T(j, i)$ iff $U(i, j) > U(j, i)$.
- $T = U$ on $\lambda - \lambda_T$.

We claim that $\text{sgn}(T) = \text{sgn}(U)$ if $T \sim U$. First, assume the main diagonals of T and U are the same. Then a and $a + 1$ occupy positions symmetric across the main diagonal of T if and only if the same is true for U. So U can be obtained from T by a sequence of pairs of transpositions of the form $(a, b)(a + 1, b + 1)$, which preserve the required order relations.

Now assume that the main diagonals of T and U are different. Since $\lambda_T = \lambda_U$ contains only one point on the main diagonal, there is a unique j such that $T(j, j) \neq U(j, j)$. Assume that $a < b$, where $T(j, j) = a$ and $U(j, j) = b$. Since T and U are transposable, $b-a$ is even. In U, $b-2$ and $b-1$ must occupy positions symmetric across the main diagonal; by replacing b with $b - 2$, $b - 2$ with $b - 1$, and $b - 1$ with b, we obtain a new tableau U' with $b - 2$ on its main diagonal. Clearly, $\text{sgn}(U) = \text{sgn}(U')$, and $|U'(i, j) - U'(j, i)| = 1$ for $i \neq j$. Also, $U'(i, j) < U'(j, i)$ if and only if $U(i, j) < U(j, i)$. We can proceed in a similar manner, acting on U' with a product of 3-cycles, until we have produced a tableau with a on its main diagonal. We can then argue as in the previous paragraph, so $\text{sgn}(T) = \text{sgn}(U)$.

For an equivalence class B of the given relation, define a poset P_B as follows. Let $T \in B$. As a set, P_B consists of all points $(i, j) \in \lambda_T$ such that $i = j$ or $T(i, j) > T(j, i)$. We say that $(i, j) \leq (i', j')$ in P_B iff $i \leq i'$ and $j \leq j'$, or $i \leq j'$ and $j \leq i'$. Label the elements of P_B (in any fashion) as $p_1, \ldots, p_m$; if $p_k = (i, j)$ and $i \neq j$, let $p_k^t = (j, i)$ (which is not in P_B). We claim that there is a bijection from $L(P_B)$, the set of linear extensions of B, to B. If τ is a linear extension of P_B, a unique tableau T is determined by the conditions

- $T(\tau^{-1}(i)) < T(\tau^{-1}(j))$ for all $i < j$,
- $T(p_i^t) = T(p_i) - 1$ for all i for which p_i^t is defined, and
- T agrees with the tableaux in B on $\lambda - \lambda_T$.

Since $T(p_i^t)$ and $T(p_i)$ are consecutive integers for all i, the relations inherited from P_B are enough to guarantee that T is standard, so $T \in B$.

Conversely, if $T \in B$, the integers $T(p_i)$ can be put in increasing order $T(p_{j_1}), \ldots, T(p_{j_m})$, and we can define $\tau: P_B \to \{1, \ldots, m\}$ by $\tau(p_{j_i}) = i$. Since T is standard and transposable, τ preserves the relations of P_B, so τ is a linear extension. The two maps we have defined are clearly inverses, so the bijection is established.

The only factors in the computation of $\rho_\lambda(g)_{T',T}$ that differ as T varies over B are those corresponding to the transpositions $\sigma_{T(p_{j_i})}$. The product of these factors for $T \in B$ corresponding to the linear extension τ is

$$\frac{1}{(c(\tau^{-1}(2)) - c(\tau^{-1}(1)))\cdots(c(\tau^{-1}(m)) - c(\tau^{-1}(m-1)))}$$

Setting $x_{p_j} = c(p_j)$, Lemma 3.6 can be applied if P_B is disconnected. Assume that P_B contains (k, k). Now P_B cannot intersect a hook $H_{k'k'}$ with $k' > k$, since, for any p_j in the intersection, we would have $T(k, k) < T(k', k') \leq T(p_j)$ for all $T \in B$. Thus, P_B would contain a second main diagonal point, namely (k', k'), and this is a contradiction. Since P_B is not contained entirely within H_{kk}, it must intersect some $H_{k'k'}$ with $k' < k$. We claim that the intersection of P_B with H_{kk} is not connected to the rest of P_B. Otherwise, we could choose $(i, j), (k, l) \in \lambda_T$ $(T \in B)$ such that $(i, j) \in H_{ii}$, $(k, l) \in H_{kk}$, $i < k$, and $j \leq l$. If $i \leq i' \leq k$ and $j \leq j' \leq l$, then $T(i, j) < T(i', j') < T(k, l)$ for $T \in B$, so $(i', j') \in \lambda_T$. Thus, λ_T contains at least $2(l - k) + 3$ points: it contains all points on the hook H_{kk} from (l, k) to (k, k) to (k, l), and also must contain the points $(k-1, l)$ and $(l, k-1)$. However, the cycle g_{k-1}, which contains $T(k-1, k-1)$ and lies entirely within $H_{k-1,k-1}$, would then have to be contained in the $2(l-k)+1$ points on $H_{k-1,k-1}$ from $(l-1, k-1)$ to $(k-1, k-1)$ to $(k-1, l-1)$. Since the lengths of the cycles are nonincreasing, this is a contradiction.

Now, by Lemma 3.6,

$$\sum_{\tau \in L(P_B)} \frac{1}{\left(x_{\tau^{-1}(2)} - x_{\tau^{-1}(1)}\right)\cdots\left(x_{\tau^{-1}(m)} - x_{\tau^{-1}(m-1)}\right)} = 0.$$

Since $\mathrm{sgn}(T)$ is constant for $T \in B$, the contributions of the tableaux in B to $\mathrm{tr}(S\rho_\lambda(g))$ sum to 0. Thus, $\mathrm{tr}(S\rho_\lambda(g)) = 0$. □

The remaining case is covered by the following.

Theorem 3.8 *Assume that g has cycle-type $(h_{11}, h_{22}, \ldots, h_{d(\lambda),d(\lambda)})$, and let $k_i = (h_{ii} - 1)/2$. Then $tr(S\rho_\lambda(g)) = \pm i^{k_1+\cdots+k_{d(\lambda)}}\sqrt{h_{11}\cdots h_{d(\lambda),d(\lambda)}}$, the sign depending on the choice of S.*

Proof: First, assume that λ is contained in the hook H_{11}. The proof of this case will be by induction. If $k_1 = 0$, then $\mathrm{tr}(S\rho_\lambda(g)) = 1$. If T is a transposable tableau consisting of a single hook of length $2l - 1$, then T is a subtableau of exactly two transposable tableaux T_1 and T_2 consisting of single hooks of length $2l + 1$. We can assume that $T_1(l+1, 1) = T_2(1, l+1) = 2l$, and $T_1(1, l+1) = T_2(l+1, 1) = 2l + 1$. We can also assume that $\operatorname{sgn}(T_1) = \operatorname{sgn}(T)$. If $T(1,l) = 2l-1$, then we do the following calculation (in which ρ_λ and g are ambiguous but can be determined by their context):

$$\begin{aligned}
&\operatorname{sgn}(T_1)\rho_\lambda(g)_{T_1',T_1} + \operatorname{sgn}(T_2)\rho_\lambda(g)_{T_2',T_2} \\
&= \sqrt{1-\frac{1}{4l^2}}\cdot \operatorname{sgn}(T)\rho_\lambda(g)_{T',T} + \frac{1}{2l-1}\cdot\sqrt{1-\frac{1}{4l^2}}\cdot\operatorname{sgn}(T)\rho_\lambda(g)_{T',T} \\
&= \sqrt{\frac{2l+1}{2l-1}}\operatorname{sgn}(T)\rho_\lambda(g)_{T',T}.
\end{aligned}$$

If $T(l,1) = 2l-1$, then

$$\begin{aligned}
&\operatorname{sgn}(T_1)\rho_\lambda(g)_{T_1',T_1} + \operatorname{sgn}(T_2)\rho_\lambda(g)_{T_2',T_2} \\
&= \frac{1}{2l-1}\cdot\sqrt{1-\frac{1}{4l^2}}\operatorname{sgn}(T)\rho_\lambda(g)_{T',T} + \sqrt{1-\frac{1}{4l^2}}\operatorname{sgn}(T)\rho_\lambda(g)_{T',T} \\
&= \sqrt{\frac{2l+1}{2l-1}}\operatorname{sgn}(T)\rho_\lambda(g)_{T',T}.
\end{aligned}$$

Since $-i\sqrt{(2l+1)/(2l-1)}\cdot i^{l-1}\sqrt{2l-1} = -i^l\sqrt{2l+1}$, the result follows.

Now consider an arbitrary self-conjugate λ. If T is transposable and has shape λ, g can be written as a product of transpositions σ_j with j and $j+1$ in the same hook H_{kk}. Each such σ_j corresponds to a factor in the computation of $\rho_\lambda(g)_{T',T}$. Thus, the factors can be grouped by hooks, and $\operatorname{tr}(S\rho_\lambda(g))$ is (up to a sign change) simply the product $(i^{k_1}\sqrt{h_{11}})\cdots(i^{k_{d(\lambda)}}\sqrt{h_{d(\lambda),d(\lambda)}})$. □

This is the most difficult step in the construction of the character table of A_n from that of S_n, since the other characters can be found by restriction. It should be noted that the methods of this paper could be used in conjunction with the orthogonality relations for characters, rendering some of the calculations unnecessary. In particular, the relations can be used to deduce Theorem 3.7 from Theorem 3.8, and vice versa.

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