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Finite Flag-Transitive $A_{n-2}.L^*.L$ Geometries

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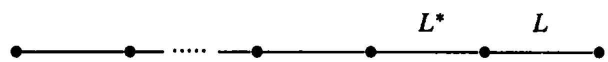
Abstract. We classify finite flag-transitive geometries of rank $n \geq 4$ belonging to the following diagram

$(A_{n-2}.L^*.L)$ 

Keywords: diagram geometry, projective space, affine grassmannian, Mathieu-Witt design

1. Introduction, definitions and notation

We will classify finite flag-transitive geometries of rank $n \geq 4$ belonging to the following diagram:

$(A_{n-2}.L^*.L)$ 

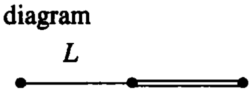
where L and L^* denote the classes of linear spaces and dual linear spaces, respectively. According to a well established habit, we only consider residually connected and firm geometries. That is, we include residual connectedness and firmness in the definition of geometry, as in [25].

The classification theorem will be stated in Section 3 (Theorem 1). The examples to be mentioned in that classification will be described in Section 2.

We leave the rank 3 case out of the scope of this paper. When $n = 3$, the diagram $A_{n-2}.L^*.L$ looks as follows:

$(L^*.L)$ 

If Γ is a geometry for this rank 3 diagram and Γ' is its shadow geometry with respect to the central node of the diagram, then Γ' satisfies the Intersection Property (IP), it belongs to the following diagram

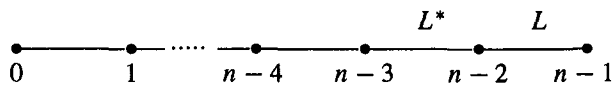
$(L.C_2)$ 

and it is thin (it admits order one) at the third node of this diagram. Furthermore, Γ' is flag-transitive if and only if Γ is flag-transitive and admits a dual automorphism. Note that Γ' is simply connected if and only if Γ is simply connected [26].

Conversely, if Γ' is a simply connected geometry belonging to $L.C_2$, thin at the last node of this diagram and satisfying (IP), then Γ' can be obtained from a geometry belonging to $L^*.L$ by taking shadows with respect to the central node of the diagram $L^*.L$ [26]. Thus, every result on $L^*.L$-geometries is equivalent to a result on $L.C_2$-geometries satisfying (IP) and thin at the last node of the $L.C_2$ diagram.

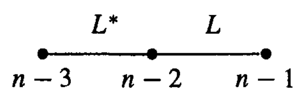
There are a lot of finite flag-transitive examples for $L^*.L$ and $L.C_2$ (see [1, 11, 13–15, 22, 28]; also [5], Section 4). A complete classification of finite flag-transitive geometries belonging to $L^*.L$ or $L.C_2$ does not seem to be in reach, at least for the moment. However, some results have been obtained in particular cases (see [1, 11, 13–15, 23, 28], for instance).

We now state a bit of notation. We take the nonnegative integers 0, 1, ..., $n-1$ as types



Elements of type 0, 1, 2, $n-1$ will be called *points*, *lines*, *planes* and *dual points* respectively (note that the word "plane" and "dual point" are synonymous when $n = 3$).

We will only consider finite geometries in our main theorem, but we could state it in the locally finite case as well. Indeed, let Γ be a geometry belonging to $A_{n-2}.L^*.L$. The $\{n-3, n-2, n-1\}$-truncation Γ' of Γ belongs to $L^*.L$



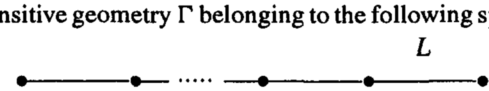
and the shadow geometry Γ'' of Γ' with respect to the central node $n-2$ of the above diagram belongs to $L.C_2$. Therefore, if Γ is locally finite (that is, it admits finite orders), then Γ'' is finite [24], hence Γ is finite, too. That is, Γ is finite if it is locally finite.

2. Examples

2.1. *Truncated projective geometries*

Let $\bar{\Gamma}$ be PG(m, q) or the m-dimensional simplex, $m \geq n$. The elements of $\bar{\Gamma}$ of dimension $< n$ form a flag-transitive geometry Γ belonging to the following special case of $A_{n-2}.L^*.L$:

$(A_{n-1}.L)$



We call Γ the *upper n-truncation* of $\bar{\Gamma}$. Clearly, $\Gamma = \bar{\Gamma}$ when $n = m$. In any case, $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\bar{\Gamma})$.

2.2. *Truncated affine grassmannians*

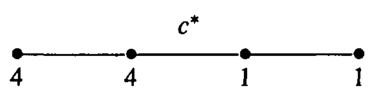
The *grassmannian* $\mathcal{G}_{n-1}$ of $\bar{\Gamma} = \mathrm{PG}(m, q)$, $m \geq n > 1$, is the point-line space with point set the set of all the $(n-2)$-dimensional subspaces of $\bar{\Gamma}$ and as lines the pencils of $(n-2)$-dimensional subspaces contained in a $(n-1)$-dimensional subspace and containing some n-dimensional subspace. (Here a -1-dimensional subspace is the empty set.) A *geometric hyperplane* of $\mathcal{G}_{n-1}$ is a proper subset of the point set of $\mathcal{G}_{n-1}$ meeting every line of $\mathcal{G}_{n-1}$ in just one or all points. The *affine grassmannian* of rank n is the geometry of maximal flags of $\bar{\Gamma}$ not containing an element of some fixed geometric hyperplane of $\mathcal{G}_{n-1}$. We refer the reader to [10, 11, 16, 27] for a general discussion of geometric hyperplanes of $\mathcal{G}_{n-1}$ and affine grassmannians. We now describe the example that will occur in our classification theorem. Given $\bar{\Gamma} = \mathrm{PG}(m, q)$, $m \geq n > 1$, let S be a subspace of $\bar{\Gamma}$ of dimension $m - n + 1$. Set H to be the set of all subspaces of dimension (type) $n-2$ of $\bar{\Gamma}$ meeting S nontrivially. Then H is a geometric hyperplane of the grassmannian $\mathcal{G}_{n-1}$. Let C be the set of flags of $\bar{\Gamma}$ of type $\{0, 1, \ldots, n-1\}$ not containing an element of H. Then C is the chamber system of a flag-transitive geometry Γ of rank n, belonging to the following special case of $A_{n-2}.L^*.L$:

$(A_{n-2}.\mathrm{Af}^*.L)$ 

Clearly, $\mathrm{Aut}(\Gamma)$ is the stabilizer of S in $\mathrm{Aut}(\bar{\Gamma})$.

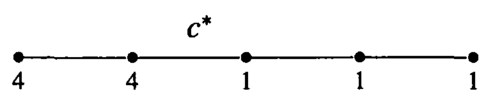
2.3. *The Witt designs for M_{23} and M_{24}*

Consider the Witt design $S(3+i, 6+i, 22+i)$ on $22+i$ points for the Mathieu group M_{22+i}, where $0 \leq i \leq 2$. The dual Witt design for M_{22+i}, with $i = 0, 1$ or 2, is the geometry of blocks (type 0) and subsets of the point set of size t with $1 \leq t \leq i+2$ (type $3+i-t$) of the Witt design $S(3+i, 6+i, 22+i)$, where incidence is symmetrized inclusion. The dual of the Witt design for M_{23} belongs to the following special case of $A_{n-2}.L^*.L$, where 4, 4, 1, 1 are the orders and c^* denotes the class of dual circular spaces:

$(A_2.c^*.A_2)$ 

The residue at a dual point of this geometry is the dual Witt design for M_{22}.

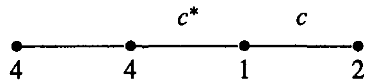
The dual of the Witt design for M_{24} has diagram and orders as follows, where the residue of a dual point is isomorphic to the dual Witt design for M_{23}:

$(A_2.c^*.A_3)$ 

If we truncate it by deleting the elements corresponding to the last node of its diagram, then we obtain a geometry for the following special case of $A_{n-2}.L^*.L$ and admitting M_{24}

as flag-transitive automorphism group:

$(A_2.c^*.c)$



3. The classification theorem

Theorem 1 *Let Γ be a finite flag-transitive geometry belonging to the diagram $A_{n-2}.L^*.L$, with $n \geq 4$:*

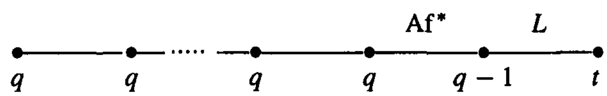
$(A_{n-2}.L^*.L)$

Then Γ is one of the examples described in the previous section.

We will obtain this theorem as a consequence of the next two results:

Proposition 2 *Let Γ be a flag-transitive geometry belonging to the diagram $A_{n-2}.Af^*.L$, with $n \geq 4$ and orders $q, q, \dots, q, q-1, t$ with $q \leq 3$ and $t < \infty$:*

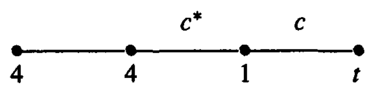
$(A_{n-2}.\mathrm{Af}^*.L)$



Then Γ is a truncated affine grassmannian as in Subsection 2.2.

Proposition 3 *Let Γ be a flag-transitive geometry of rank* 4 *with diagram and orders as follows*

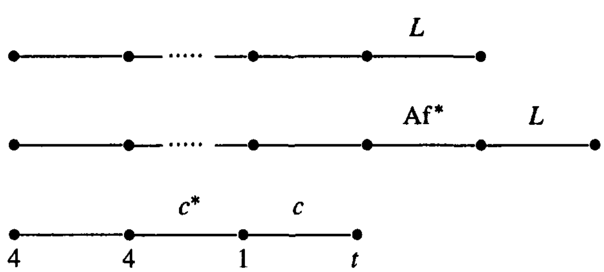
$(A_2.c^*.c)$



where $1 < t < \infty$, and the residue of a dual point is isomorphic to the dual Witt design for the group M_{22}. Then $t = 2$ and Γ is the dual of the Witt design for M_{24}, truncated as in Subsection 2.3.

Postponing the proofs of Propositions 2 and 3 to the next two sections, we now show how Theorem 1 follows from them.

Let Γ be as in the hypotheses of Theorem 1. By a result of Delandtsheer [12], the diagram of Γ is one of the following:



In the first case Γ is the upper n-truncation of PG(m, q) or of an m-dimensional symplex, for some $m \geq n$ ([2], Theorem 8; also [25], Corollary 7.11).

In the second case, if the lines of Γ are incident to more than 4 points, then Γ is an affine grassmannian by Theorem 1.2 of [11]. It is straightforward to prove that an affine grassmannian is flag-transitive only if it is obtained either as in Subsection 2.2 or from a non-degenerate symplectic form (see [10]). However, we get geometries of rank 3 in the latter case, whereas we have assumed $n \geq 4$. Thus Γ is as in Subsection 2.2. If the lines of Γ are incident to 3 or 4 points, then Proposition 2 gives us the conclusion.

Let Γ belong to the third one of the above diagrams. When $t = 1$ it is well known [20] that Γ is the dual of the Witt design for M_{23}. When $t \geq 2$ we can apply Proposition 3, thus finishing the proof of Theorem 1.

We notice that the proof of Theorem 1 relies heavily on the classification of finite simple groups and of finite 2-transitive groups. In particular, these classifications are used by Delandtsheer [12] as well as in the proof of Proposition 3. The proof of Proposition 2 however, is free of any use of these results. It relies on the results of [11] and [9].

4. Proof of Proposition 2

Let Γ be as in the hypotheses of Proposition 2, let G be a flag-transitive subgroup of Aut(Γ) and let Π be the point-line system of Γ (we have defined points and lines in Section 1). Π is a partial linear space ([11], Lemma 4.1) and G acts flag-transitively on it.

The subspaces of Π generated by two intersecting lines will be called *planar subspaces* of Π. For each element X of Γ, the points and lines of Γ incident to X form a partial linear space $\Pi(X)$. Note that $\Pi(X)$ is a subspace of Π when X has type $< n - 1$ (actually, it is a projective space). On the other hand, if X is a dual point of Γ, then $\Pi(X)$ might not be a subspace of Π.

As in [11], we start with the investigation of Π. In particular, we will determine the structure of the planar subspaces of Π.

Suppose L and M are two lines through a point p and let $\pi(L, M)$ be the planar subspace of Γ generated by them. If L and M are incident with some plane π of Γ, then $\pi(L, M) = \Pi(\pi)$, which is a projective plane because $n \geq 4$.

Suppose L and M are not coplanar in Γ. By an inductive argument one can prove that there is a dual point H of Γ incident with both L and M (compare [11], proof of Proposition 4.2). $\Pi(H)$ is a dual affine space ([2], 9.2) and L, M generate a dual affine plane π in it. Clearly, $\pi \subseteq \pi(L, M)$.

Assume $\pi \neq \pi(L, M)$. That is, some of the lines of $\pi(L, M)$ are missing in π. Anyhow, $\pi(L, M)$ is either a projective plane or a projective plane minus a line ([11], Proposition 4.2). Assume the latter. Then the stabilizer of $\pi(L, M)$ in G cannot be transitive on the set of points of $\pi(L, M)$, as it cannot map any point of the removed line onto any point outside that line. On the other hand, the stabilizer $G_{H,\pi}$ of π in the stabilizer G_H of H in G acts flag-transitively on π and stabilizes $\pi(L, M)$, as $\pi(L, M)$ is the unique planar subspace of Π containing π. We have reached a contradiction. Hence $\pi(L, M)$ is a projective plane.

Thus, if L and M are non-coplanar intersecting lines of Γ, then $\pi(L, M)$ is either a dual affine plane or a projective plane. Let L, M and L', M' be two pairs of non-coplanar intersecting lines such that $\pi(L, M)$ is projective but $\pi(L', M')$ is dual affine, if possible. As G is flag-transitive, we may assume that $L = L'$ and that L, M, M' pass through the same point p. As we have remarked above, there are dual points H, H' incident with L and M and with L and M', respectively. By the flag-transitivity of G we can also assume that $H = H'$. As $n \geq 4$, the dual of $\Pi(H)$ is isomorphic to $\mathrm{AG}(n-1, q)$. Hence the (dual of the) residue $\Gamma_{p,H}$ of the flag (p, H) of Γ is isomorphic to $\mathrm{AG}(n-2, q)$ and the lines L, M, M' are hyperplanes of this affine geometry. As none of M or M' is coplanar with L in Γ, both M and M' are parallel to L when viewed as hyperplanes of $\Gamma_{p,H} \simeq \mathrm{AG}(n-2, q)$. Clearly, M, M' and L are pairwise distinct. Hence $q > 2$. That is, $q = 3$, as $q \leq 3$ by assumption.

Let $[L]$ be the parallelism class of L in the affine geometry $\Gamma_{p,H}$. As $\pi(L, M)$ is projective whereas $\pi(L', M')$ is dual affine, the stabilizer $G_{p,L,H}$ in G of the flag (p, L, H) does not act transitively on $[L] - \{L\}$. Exploiting a theorem of Higman [19] and recalling that $q = 3$, we see that the above cannot happen. Therefore, either all planar subspaces generated by two intersecting non-coplanar lines are dual affine planes, or all of them are projective planes.

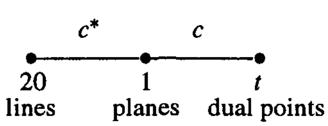
Assume that all planar subspaces generated by two intersecting non-coplanar lines are dual affine planes. Then Π is the space of points and lines of $\mathrm{PG}(m, q)$ (for some $m \geq n$) missing some $(m - n + 1)$-dimensional subspace S and Γ is as in Subsection 2.2 (see [11] and [9], Theorem 1.1).

Let all planar subspaces generated by two intersecting lines be projective, if possible. Then Π is the point-line system of $\mathrm{PG}(m, q)$ for some $m > n$ and G acts flag-transitively on it. We have $G \geq \mathrm{PSL}_{m+1}(q)$ (see [4], 2.2.1). Hence G acts transitively on the pairs of intersecting lines. Therefore, any two intersecting lines are coplanar. We will now obtain a contradiction by induction on n. Let $n = 4$, let H be a dual point of Γ and a, b distinct points of H not collinear in $\Pi(H)$. As $\Pi(H)$ is a dual affine space, there is some point c collinear with both a and b in $\Pi(H)$. Let L, M be the lines of $\Pi(H)$ through a and c and through c and b, respectively. These lines are not coplanar in the residue Γ_H of H. However, they are coplanar in Γ because any two intersecting lines of Γ are coplanar in Γ. Let π be the plane of Γ incident with both L and M. By Lemma 3.2 of [11], π is incident with H, an impossibility. Let $n > 4$. A contradiction can be obtained as above, except that the inductive hypothesis should be used now instead of Lemma 3.2 of [11]. By induction, the residue of c is an affine grassmann geometry as described in Subsection 2.2. It is easily seen that the analogue of Lemma 3.2 of [11] holds in such a geometry, with planes replaced by dual points in that statement.

5. Proof of Proposition 3

Let Γ be as in the hypotheses of Proposition 3 and let $G \leq \mathrm{Aut}(\Gamma)$ be flag-transitive on Γ. Let Γ' be the geometry of lines, planes and dual points of Γ. Note that Γ' has diagram and orders as follows:

$(c^*.c)$



As in the previous section, we say that two lines are coplanar if they are incident to a common plane. We use the same convention for dual points: two dual points are *coplanar* if there is a plane incident with both of them. Our first goal is to prove that the Intersection Property (IP) (see [25]) holds in Γ'. For, then we can apply a result of Sprague [28] to identify the elements of Γ' with subsets of size 20, 21 and 22 in a set S of size $22 + t$. Moreover, the action of G on S will be shown to be a $t+1$-fold extension of $\mathrm{PSL}_3(4)$ acting naturally on a 21 point subset of S, from which it easily follows that $t = 2$, $G \simeq M_{24}$ and Γ is the truncation of the dual Witt design for M_{24}.

As Γ' belongs to $c^*.c$, (IP) in it amounts to the following three properties, see Chapter 6 of [25], and also [3] or [28]:

(1) any two distinct coplanar lines are incident to at most one common plane;
(2) given two distinct lines A, B, a plane α and a dual point U, if both α and U are incident to both A and B, then α is incident to U;
(3) given two distinct lines A, B and two distinct dual points U, V, if both U and V are incident to both A and B, then A and B are coplanar.

The following are the dual properties:

(1*) any two distinct dual points are incident to at most one common plane;
(2*) given two distinct dual points U, V, a plane α and a line A, if both α and A are incident to both U and V, then α is incident to A;
(3*) given two distinct dual points U, V and two distinct lines A, B, if both A and B are incident to both U and V, then U and V are coplanar.

As a matter of fact, these are exactly the properties used by Sprague in [28].

5.1. Properties (1), (1), (2) and (2*) hold in* Γ'

Lemma 4 *Both* (1) *and* (1*) *hold in* Γ'.

Proof (Compare the proof of Lemma 4.1 of [11]): Let A, B be distinct lines. If α, β are planes incident to both A and B, in the residue of A we find a dual point U incident to both α and β. In the residue of U we see that $\alpha = \beta$. Hence (1) holds in Γ'. The dual property (1*) can be proved by a dual argument. □

Lemma 5 *Property* (2) *holds in* Γ'.

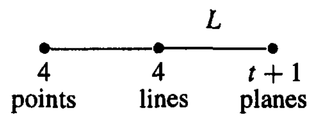
Proof: Let (2) fail to hold, if possible. Then there are distinct lines A and B, a plane α and a dual point U such that both α and U are incident to both A and B but α is not incident to U. In the residue of α in Γ we find a point a incident to both A and B. The residue Γ_U of U in Γ is isomorphic to the Witt design obtained from the Steiner system $S(3, 6, 22)$ for M_{22} and the lines A and B appear in $S(3, 6, 22)$ as pairs of points (planes of Γ) in the same block a. These two pairs of points of $S(3, 6, 22)$ are disjoint, because the plane α, which is incident to both A and B, is not incident to U, and it is the unique plane incident to both A and B, by (1).

The stabilizer G_U of U in G acts as M_{22} or Aut(M_{22}) on $\Gamma_U \cong S(3, 6, 22)$ (see [20]). Hence it is transitive on the set of ordered pairs of disjoint pairs of points of $S(3, 6, 22)$ in the same block of $S(3, 6, 22)$. That is, G_U is transitive on the set of ordered pairs of intersecting lines of Γ_U non-coplanar in Γ_U. As (A, B) is such a pair and A, B are coplanar in Γ, any two distinct intersecting lines of Γ_U are coplanar in Γ. By flag-transitivity, the same holds in the residue of every dual point: if two intersecting lines are incident to some common dual point, then they are coplanar.

Let now x be a point and X, Y any two lines incident to x. We will prove that X and Y are coplanar. If not, let $k > 1$ be the minimal number of planes and of lines needed to connect X to Y in Γ_x. That is, there are planes $\alpha_1, \alpha_2, \ldots, \alpha_k$ and lines $X_0, X_1, X_2, \ldots, X_k$ with $X_0 = X$, $X_k = Y$, x incident to X_i for all $i \leq k$ and α_i incident to both X_{i-1} and X_i for every $i = 1, 2, \ldots, k$, and k is minimal with this property. In the residue of X_1 there is a dual point V incident to both α_1 and α_2. Thus, X_0 and X_2 are intersecting lines both incident to the dual point V. Hence they are coplanar in Γ. However, this contradicts the minimality of k. Therefore any two intersecting lines are coplanar.

Let now Γ'' be the geometry of points, lines and planes of Γ. As any two intersecting lines of Γ are coplanar and (1) holds in Γ' (Lemma 4), the geometry Γ'' has diagram and orders as follows:

$(A_2.L)$



Hence Γ'' is a truncation of PG$(m, 4)$ for some $m \geq 3$ ([2], Theorem 8) and G acts flag-transitively on it. Therefore $\mathrm{PSL}_{m+1}(4) \leq G \leq P\Gamma L_{m+1}(4)$ (see [19] and [4], 2.2.1).

Let now (p, L) be a point-line flag and let π, π' be distinct planes incident to L. Let W be the (unique) dual point incident to both π and π'. The stabilizer $G_{p,L,\pi,\pi'}$ of p, L, π and π' fixes W and the 3-space Π of PG$(m, 4) = \Gamma''$ spanned by $\pi \cup \pi'$. That is, $G_{p,L,\pi,\pi'} = G_{p,L,\pi,\pi',\Pi} = G_{p,L,\pi,\pi',W} = G_{p,L,\pi,W}$ (we have the last equality because Γ is thin at the third node of the diagram).

The stabilizer G_W of W acts as M_{22} or Aut(M_{22}) on Γ_W (see [20]). Hence $G_{p,L,\pi,\pi'}$ $(= G_{p,L,\pi,W})$ acts on the projective plane $\Gamma_{\pi,W}$ as a Borel subgroup of $\mathrm{PSL}_3(4)$ or PSL_3 $(4).2_2$. On the other hand, $G_{p,L,\pi,\pi'}$ is the stabilizer of p, L, π, π' and Π in $G \geq \mathrm{PSL}_{m+1}(q)$. The stabilizer of π and Π in $\mathrm{PSL}_{m+1}(q)$ acts as $\mathrm{PGL}_3(4) = \mathrm{PSL}_3(4).3$ on the plane π. Hence the group induced by $G_{p,L,\pi,\pi'}$ on π contains a Borel subgroup of $\mathrm{PGL}_3(4)$, which is not a subgroup of a Borel subgroup of either of $\mathrm{PSL}_3(4)$ or $\mathrm{PSL}_3(4).2$. This contradiction finishes the proof. □

Lemma 6 *Property* (2*) *holds in* Γ'.

Proof: Let U_1, U_2 be distinct dual points and let L and π be a line and a plane incident to both U_1 and U_2. Let L and π be not incident, if possible. We can choose planes π_1, π_2 in Γ_L incident to U_1 and U_2, respectively. For $i = 1, 2$, there is a line L_i in the residue of

U_i incident to both π and π_i. In Γ_L we find a dual point V incident to both π_1 and π_2. As (2) holds in Γ' (Lemma 5), either $L_1 = L_2$ or V is incident to π.

Let $L_1 = L_2$. As L and π are not incident, $L \neq L_1 = L_2$. Hence $\pi_1 = \pi_2 = \pi$ by (1) and (1*) in Γ' (Lemma 4). This forces L to be incident to π, contrary to our assumptions. Therefore $L_1 \neq L_2$. Hence V is incident to π. By (1*) in Γ' one of the following occurs:

(i) $\pi = \pi_1 = \pi_2$ and $V \neq U_1, U_2$;
(ii) either $V = U_1$ and $\pi = \pi_2$ or $V = U_2$ and $\pi = \pi_1$.

In any case, L and π are incident: again a contradiction. Therefore (2*) holds in Γ'. □

5.2. *Kernels*

Given a flag F of Γ', G_F is the stabilizer of F in G. The elementwise stabilizer in G_F of the residue Γ'_F of F in Γ' will be denoted by K_F. Then $\bar{G}_F = G_F/K_F$ is the action of G_F on Γ'_F. Given a plane π, Γ_π^- (respectively, Γ_π^+) will denote the set of lines (dual points) incident to π. Henceforth (L, π, U) is a chamber of Γ'.

Lemma 7 $K_\pi = K_L \cap K_U = 1$.

Proof: Since every line, respectively, dual point of Γ_π is also in Γ_U, respectively, Γ_L, the group K_π contains K_U, respectively, K_L. Hence $K_\pi \geq K_U \cap K_L$. On the other hand, since Γ_L is a circle, K_π is contained in K_L. Similarly $K_\pi \leq K_U$. Hence $K_\pi = K_L \cap K_U$. But then also $K_{\pi'} = K_L \cap K_U = K_\pi$ for every plane π' incident with both L and U. Thus if two planes π and π' are incident with the same line and dual point, the kernels K_π and $K_{\pi'}$ are the same. By connectedness of Γ all kernels $K_{\pi''}$ for any plane π'' are equal to K_π and K_π has to be trivial. □

Let N_π^+, N_π^- be the kernels of the actions of G_π on Γ_π^+ and Γ_π^-, respectively.

Lemma 8 $\mathrm{PSL}_3(4) \trianglelefteq N_\pi^+ \trianglelefteq \mathrm{PSL}_3(4){:}2$ *acting naturally on the* 21 *lines in* Γ_π^-.

Proof: We have $N_\pi^+ \cap K_U \leq K_\pi = 1$ (Lemma 7). On the other hand, $N_\pi^+ \cap G_L = K_L \leq G_U$. As $K_L \cap K_U = 1$ (Lemma 7), we can recognize K_L inside $\bar{G}_U$, which is either M_{22} or $\mathrm{Aut}(M_{22})$ [20]. K_L is contained in the stabilizer of two points of the Steiner system $S(3, 6, 22)$. Hence $K_L \leq 2^4{:}\mathrm{PSL}_2(4).\varepsilon$, with $\varepsilon = 1$ or 2. On the other hand, $K_U \cap N_\pi^+ = 1$ (see above). Thus, N_π^+ can also be recognized inside $\bar{G}_U$. As $N_\pi^+ \trianglelefteq G_{\pi,U}$ and $G_{\pi,U}/K_U$ appears as the stabilizer of a point of $S(3, 6, 22)$ in $\bar{G}_U$, either N_π^+ is trivial or $N_\pi^+ = \mathrm{PSL}_3(4).\varepsilon$ ($\varepsilon = 1$ or 2).

In the latter case we are done. Assume $N_\pi^+ = 1$, if possible. The group $K_L = 1$, since $K_L = N_\pi^+ \cap G_U = 1$. That is, G_L acts faithfully on the $t + 2$ planes incident to L. That action is 2-transitive, by flag-transitivity.

Furthermore, the stabilizer G_C of the flag $C = (L, U)$ equals $K_U.(2^n : S_5)$, with $n = 4$ or 5 (we see this in G_U) and G_C appears in G_L as the stabilizer of a pair of planes incident to L. Moreover, G_C acts as S_5 on the 5 points of L. Hence G_L also acts as S_5 on the 5 points of L. Hence, we have found:

(a) G_L is a 2-transitive groups of degree $t+2$;
(b) G_L admits a quotient isomorphic to S_5;
(c) the stabilizer in G_L of a pair of points admits a quotient isomorphic to $2^n : S_5$, with $n = 4$ or 5.

We can check the list of 2-transitive permutation groups (see [6]; also [8, 17, 18, 21]), searching for a group $X = G_L$ with the above properties.

Clearly, X cannot be of simple type. Indeed, if X is of simple type, then the factor group of X mentioned in (b) should be recoverd as a section of the outer automorphism group $\mathrm{Out}(S(X))$ of the socle $S(X)$ of X because $A_5 = S_5'$ is simple. However, as a consequence of the classification of the finite simple groups Schreier's conjecture holds, and $\mathrm{Out}(S(X))$ is solvable. But as A_5 is simple this leads to a contradiction. Therefore X is not of simple type.

Hence X should be of affine type. Doubly transitive permutation groups of affine type have been classified by Hering [17, 18] (see also [21], Appendix I). Let X be such a group. Then X has degree p^m for some prime p and some positive integer m and it contains a normal subgroup $N \cong p^m$, acting regularly on the p^m objects on which X acts. A point-stabilizer X_0 in X is a subgroup of $\Gamma L(q, d)$ for some divisor d of m and with $q = p^k$, $k = m/d$.

The following are the cases that can occur:

(1) $d = 1$;
(2) $SL_d(q) \trianglelefteq X_0$;
(3) $Sp_d(q) \trianglelefteq X_0 \leq \mathrm{Aut}(Sp_d(q))$ (d even);
(4) $G_2(q)' \leq X_0 \leq \mathrm{Aut}(G_2(q)')$, $p = 2$ and $d = 6$;
(5) $d = 2$, $p = 5, 7, 11$ or 23 and X_0 normalizes a subgroup of $GL_2(p)$ isomorphic with Q_8;
(6) $p = 3$, $m = 4$, X_0 has a normal subgroup R isomorphic with the central product of Q_8 and D_8 and $X_0/R \leq S_5$;
(7) $m = 2$, $p = 9, 11, 19, 29$ or 59 and $SL_2(5) \trianglelefteq X_0$;
(8) $m = 4$, $p = 2$ and $X_0 \cong A_6$ or A_7;
(9) $m = 6$, $p = 3$ and $X_0 \cong SL_2(13)$.

Cases (1)–(4) can be ruled out by an argument similar to the one used for the simple case.

None of (5), (7) and (9) can occur. Indeed, by (c) $|2^4 : S_5|$ divides $|X_0|$, whereas this divisibility condition fails to hold in each of (5), (7), (8) and (9). In case (6), the order of a two-points stabilizer divides $2^4.3^6.13$. This does not fit with (c).

Therefore there are no 2-transitive groups satisfying (a), (b) and (c). Hence $N_\pi^+ \neq 1$. □

5.3. A description of Γ'

Lemma 9 *Property* (3) *holds in* Γ'.

Proof: Let A and B be distinct lines and U and V be distinct dual points as in the hypotheses of (3). By Lemma 8, $K_A = N_\pi^+ \cap G_A = 2^4 : \mathrm{PSL}_2(4).\varepsilon$ ($\varepsilon = 1$ or 2). The orbit of B under the action of K_A can be seen inside Γ_U. It contains a line $B' \neq B$ coplanar with B. Let α be the plane incident to B and B'. Clearly, α is incident to U, as we have found it in Γ_U. On the other hand, (2) holds in Γ' (Lemma 7) and, applying this property to B, B', α and V, we see that α is also incident to V, because B' is incident to V (indeed it belongs to the orbit of B under the action of K_A, which fixes V). We can now apply (2*) (Lemma 6) to A, α, U and V, thus obtaining that A is incident to α, too. Therefore α is incident to both A and B, which then proves (3). □

Corollary 10 *The Intersection Property (IP) holds in* Γ'.

(Easy, by Lemmas 7 and 9.)

Corollary 11 Γ' *is the system of all subsets of size* 20, 21 *and* 22 *of a set* S *of size* $22 + t$ *with incidence being symmetrized inclusion.*

Proof: Γ' is a truncation of some (degenerate) projective geometry Π, by Corollary 10 and a theorem of Sprague [28]. Since it has orders 20, 1, t, Π is the $21 + t$-dimensional simplex (with $22 + t$ vertices). □

5.4. End of the proof

Let S be a set of size $22 + t$ and Γ' the system of all subsets of size 20, 21 and 22 of S. For every plane π of Γ', i.e., a subset of size 21, the group N_π^+ is isomorphic to $\mathrm{PSL}_3(4)$ or $\mathrm{PSL}_3(4) : 2$ and acts naturally on the 21 points of π, see Lemma 8. As N_π^+ stabilizes all dual points of Γ' it also fixes all points of $S \setminus \pi$. Now suppose V is a subset of size at least 21 of S. Then the subgroup $G(V) := \langle N_\pi^+ \mid \pi \text{ is a plane in } V \rangle$ of G is transitive on the points of V and fixes all points outside V. Hence, the action of $G(S)$ on S is a $t + 1$-fold transitive extension of the action of $G(\pi) = N_\pi^+$ for any plane π of Γ'. It is well known that N_π^+ admits a 3-fold transitive extension isomorphic to M_{24} but no 4-fold transitive extension. Proposition 3 follows immediately.

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