Journal of Algebraic Combinatorics **6** (1997), 89–93


# Finite Free Resolutions and 1-Skeletons of Simplicial Complexes

NAOKI TERAI terai@cc.saga-u.ac.jp
*Department of Mathematics, Faculty of Education, Saga University, Saga 840, Japan*

TAKAYUKI HIBI hibi@math.sci.osaka-u.ac.jp
*Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka Osaka 560, Japan*

*Received June 7, 1995; Revised November 6, 1995*

**Abstract.** A technique of minimal free resolutions of Stanley–Reisner rings enables us to show the following two results: (1) The 1-skeleton of a simplicial $(d-1)$-sphere is $d$-connected, which was first proved by Barnette; (2) The comparability graph of a non-planar distributive lattice of rank $d-1$ is $d$-connected.

**Keywords:** simplicial complex, 1-skeleton, comparability graph, $d$-connected, free resolution

## 1. Introduction

A simplicial complex $\Delta$ on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ is a collection of subsets of $V$ such that (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq v$ and (ii) if $\sigma \in \Delta$ and $\tau \subset \sigma$ then $\tau \in \Delta$. Each element $\sigma$ of $\Delta$ is called a face of $\Delta$. Set $d = \max\{\sharp(\sigma);\ \sigma \in \Delta\}$ and define the dimension of $\Delta$ to be $\dim \Delta = d - 1$. Here $\sharp(\sigma)$ is the cardinality of a finite set $\sigma$.

A simplicial complex $\Delta$ of dimension $d-1$ is called a *simplicial* $(d-1)$*-sphere* if the geometric realization of $\Delta$ is homeomorphic to the $(d-1)$-sphere.

The 1-*skeleton* $\Delta^{(1)}$ of $\Delta$ is the subcomplex

$$\Delta^{(1)} = \{\sigma \in \Delta;\ \sharp(\sigma) \leq 2\}$$

of $\Delta$, which is a 1-dimensional simplicial complex (i.e., graph) on the vertex set $V$. When a simplicial complex $\Delta$ is an order complex of a finite partially ordered set $P$, the 1-skeleton of $\Delta$ is just the comparability graph $\mathrm{Com}(P)$ of $P$.

Given a subset $W$ of $V$, we write $\Delta_W$ for the subcomplex

$$\Delta_W = \{\sigma \in \Delta;\ \sigma \subset W\}$$

of $\Delta$. In particular, $\Delta_V = \Delta$ and $\Delta_\emptyset = \{\emptyset\}$.

Let $\tilde{H}_i(\Delta; k)$ denote the $i$-th reduced simplicial homology group of $\Delta$ with the coefficient field $k$. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & \text{if } i \geq 0 \\ k & \text{if } i = -1. \end{cases}$$

We fix an integer $1 \le i < v$. A 1-dimensional simplicial complex $\Delta$ on the vertex set $V$ is said to be *$i$-connected* if $\Delta_{V-W}$ is connected (i.e., $\tilde{H}_0(\Delta_{V-W}; k) = 0$) for every subset $W$ of $V$ with $\sharp(W) < i$.

The purpose of the present paper is first to give a ring-theoretical proof of a classical result that the 1-skeleton of a simplicial $(d-1)$-sphere is $d$-connected (cf. Barnette [1]), and secondly to show that the comparability graph $\mathrm{Com}(L)$ of a finite distributive lattice $L$ of rank $d-1$ is $d$-connected.

## 2. Algebraic background

We here summarize basic facts on finite free resolutions of Stanley–Reisner rings. See, e.g., [2, 4, 6, 8] for the detailed information.

Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in $v$ variables over a field $k$. Here, we identify each element $x_i$ in the vertex set $V$ with the indeterminate $x_i$ of $A$. We consider $A$ to be the graded algebra $A = \bigoplus_{n \ge 0} A_n$ with the standard grading, i.e., each $\deg x_i = 1$. Let $\mathbf{Z}$ denote the set of integers. We write $A(j)$, $j \in \mathbf{Z}$, for the graded module $A(j) = \bigoplus_{n \in \mathbf{Z}} [A(j)]_n$ over $A$ with $[A(j)]_n := A_{n+j}$. Given a simplicial complex $\Delta$ on $V$, define $I_\Delta$ to be the ideal of $A$ generated by all squarefree monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \le i_1 < i_2 < \cdots < i_r \le v$, with $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley–Reisner ring* of $\Delta$ over $k$.

When $k[\Delta]$ is regarded as a graded module $k[\Delta] = \bigoplus_{n \ge 0} (k[\Delta])_n$ over $A$ with the quotient grading, it has a graded finite free resolution

$$0 \longrightarrow \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{h,j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{1,j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0, \tag{1}$$

where each $\bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{i,j}}$, $1 \le i \le h$, is a graded free module of rank $0 \ne \sum_{j \in \mathbf{Z}} \beta_{i,j} < \infty$, and where every $\varphi_i$ is degree-preserving. Moreover, there exists a unique such resolution which minimizes each $\beta_{i,j}$; such a resolution is called *minimal*. If a finite free resolution (1) is minimal, then the non-negative integer $h$ is called the *homological dimension* of $k[\Delta]$ over $A$ and $\beta_{i,j} = \beta_{i,j}(k[\Delta])$ is called the $(i, j)$-th *Betti number* of $k[\Delta]$ over $A$. Furthermore, let $\beta_i = \beta_i(k[\Delta])$ denote the sum $\sum_{j \in \mathbf{Z}} \beta_{i,j}$.

Our fundamental technique in the present paper is based on the topological formula [6, Theorem (5.1)] which guarantees that

$$\beta_{i,j}(k[\Delta]) = \sum_{W \subset V,\ \sharp(W) = j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k). \tag{2}$$

Thus, in particular,

$$\beta_i(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{\sharp(W)-i-1}(\Delta_W; k).$$

**Lemma 2.1** *Let $\Delta$ be a simplicial complex on the vertex set $V$ with $\sharp(V) = v$ and $i$ an integer with $1 \le i < v$. Then the 1-skeleton $\Delta^{(1)}$ of $\Delta$ is $i$-connected if and only if $\beta_{v-i,v-i+1}(k[\Delta]) = 0$.*

**Proof:** The 1-skeleton $\Delta^{(1)}$ is $i$-connected if and only if, for every subset $W$ of $V$ with $\sharp(W) = i - 1$, we have $\tilde{H}_0(\Delta^{(1)}_{V-W}; k)$ $(= \tilde{H}_0(\Delta_{V-W}; k)) = 0$. Moreover, by virtue of Eq. (2), $\tilde{H}_0(\Delta_{V-W}; k) = 0$ for every subset $W$ of $V$ with $\sharp(W) = i - 1$ if and only if $\beta_{v-i,v-i+1}(k[\Delta]) = 0$ as desired. □

## 3. Main results

We first give a ring-theoretical proof of the following classical result which was proved by Barnette [1].

**Theorem 3.1 (Barnette [1])** *The* 1*-skeleton of a simplicial* $(d - 1)$*-sphere with* $d \geq 2$ *is* $d$*-connected.*

**Proof:** Suppose that $\Delta$ is a simplicial $(d - 1)$-sphere on the vertex set $V$ with $\sharp(V) = v$. We know that $k[\Delta]$ is Gorenstein; that is to say, $\beta_i(k[\Delta]) = 0$ for every $i > v - d$, $\beta_{v-d,j}(k[\Delta]) = 0$ if $j \neq v$ and $\beta_{v-d,v}(k[\Delta]) = 1$. Thus, in particular, we have $\beta_{i,i+1}(k[\Delta]) = 0$ for every $i \geq v - d$. Hence, by Lemma (2.1), the 1-skeleton $\Delta^{(1)}$ of $\Delta$ is $d$-connected as required. □

**Remark** The above ring-theoretical technique enables us to show the 1-skeleton of a level complex $\Delta$ (see, e.g., [3, 7]) of dimension $d - 1$ with $v$ vertices is $d$-connected if $\sharp\{\sigma \in \Delta \mid \sharp(\sigma) = d\} \neq v - d - 1$. In particular, we can see that the 1-skeleton of a Gorenstein complex $\Delta$ (see, e.g., [2, 6, 8]) of dimension $d - 1$ is $d$-connected.

We now turn to the study on comparability graphs of finite distributive lattices. Every partially ordered set ("poset" for short) is finite. A *poset ideal* in a poset $P$ is a subset $I \subset P$ such that $\alpha \in I$, $\beta \in P$ and $\beta \leq \alpha$ together imply $\beta \in I$. A *clutter* is a poset in which no two elements are comparable. A *chain* of a poset $P$ is a totally ordered subset of $P$. The *length* of a chain $C$ is $\ell(C) := \sharp(C) - 1$. The *rank* of a poset $P$ is defined to be $\text{rank}(P) := \max\{\ell(C);\ C \text{ is a chain of } P\}$. Given a poset $P$, we write $\Delta(P)$ for the set of all chains of $P$. Then $\Delta(P)$ is a simplicial complex on the vertex set $P$, which is called the *order complex* of $P$. The *comparability graph* $\text{Com}(P)$ of a poset $P$ is the 1-skeleton $\Delta^{(1)}(P)$ of the order complex $\Delta(P)$. When $x \leq y$ in a poset $P$, we define the closed interval $[x, y]$ to be the subposet $\{z \in P;\ x \leq z \leq y\}$ of $P$.

A *lattice* is a poset $L$ such that any two elements $\alpha$ and $\beta$ of $L$ have a greatest lower bound $\alpha \wedge \beta$ and a least upper bound $\alpha \vee \beta$. Let $\hat{0}$ (resp. $\hat{1}$) denote the unique minimal (resp. maximal) element of a lattice $L$. A lattice $L$ is called *distributive* if the equalities $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ and $\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$ hold for all $\alpha, \beta, \gamma \in L$. Every closed interval of a distributive lattice is again a distributive lattice. A fundamental structure theorem for (finite) distributive lattices (see, e.g., [9, p. 106]) guarantees that, for every finite distributive lattice $L$, there exists a unique poset $P$ such that $L = J(P)$, where $J(P)$ is the poset which consists of all poset ideals of $P$, ordered by inclusion. We say that a distributive lattice $L = J(P)$ is *planar* if $P$ contains no three-element clutter. A *boolean lattice* is a distributive lattice $L = J(P)$ such that $P$ is a clutter.

A chain $\mathcal{C}: \hat{0} = \alpha_0 < \alpha_1 < \cdots < \alpha_{s-1} < \alpha_s = \hat{1}$ of a distributive lattice $L$ is called *essential* if each closed interval $[\alpha_i, \alpha_{i+1}]$ is a boolean lattice. In particular, all maximal chains of $L$ is essential. Moreover, the chain $\hat{0} < \hat{1}$ of $L$ is essential if and only if $L$ is a boolean lattice. An essential chain $\mathcal{C}: \hat{0} = \alpha_0 < \alpha_1 < \cdots < \alpha_{s-1} < \alpha_s = \hat{1}$ is called *fundamental* if, for each $1 \leq i < s$, the subchain $\mathcal{C} - \{\alpha_i\}$ is not essential. The following Lemma (3.2) is discussed in [5].

**Lemma 3.2 ([5])** *Let $L$ be a distributive lattice of rank $d-1$ with $\sharp(L) = v$ and $\Delta = \Delta(L)$ its order complex. Then the $(v-d, v-d+i)$-th Betti number $\beta_{v-d,v-d+i}(k[\Delta])$ is equal to the number of fundamental chains of $L$ of length $d-i-1$.*

We are now in the position to give the second result of the present paper.

**Theorem 3.3** *Suppose that a finite distributive lattice $L$ of rank $d-1$ is non-planar. Then the comparability graph $Com(L)$ of $L$ is $d$-connected.*

**Proof:** Let $P = \{p_1, p_2, \ldots, p_{d-1}\}$ denote a poset with $L = J(P)$ and $\mathcal{M}: \hat{0} = \alpha_0 < \alpha_1 < \cdots < \alpha_{d-2} < \alpha_{d-1} = \hat{1}$ an arbitrary maximal chain of $L$. We may assume that each $\alpha_i$ is the poset ideal $\{p_1, p_2, \ldots, p_i\}$ of $P$. Since $L$ is non-planar, there exists a three-element clutter, say, $\{p_l, p_m, p_n\}$ with $1 \leq l < m < n \leq d-1$. Hence, for some $l \leq i < m$, $p_i$ and $p_{i+1}$ are incomparable in $P$, and for some $m \leq j < n$, $p_j$ and $p_{j+1}$ are incomparable in $P$. Let $l \leq i < m$ (resp. $m \leq j < n$) denote the least (resp. greatest) integer $i$ (resp. $j$) with the above property. Then $\beta = \{p_1, \ldots, p_{i-1}, p_{i+1}\}$ and $\gamma = \{p_1, \ldots, p_{j-1}, p_{j+1}\}$ both are poset ideals of $P$. Moreover, $\alpha_{i-1} < \beta < \alpha_{i+1}$ in $L$ with $\beta \neq \alpha_i$ and $\alpha_{j-1} < \gamma < \alpha_{j+1}$ in $L$ with $\gamma \neq \alpha_j$. Thus the closed intervals $[\alpha_{i-1}, \alpha_{i+1}]$ and $[\alpha_{j-1}, \alpha_{j+1}]$ both are boolean. Hence, if $i+1 \leq j-1$, then the chain $\mathcal{M} - \{\alpha_i, \alpha_j\}$ is essential. On the other hand, if $i+1 > j-1$, i.e., $i = m-1$ and $j = m$, then $p_l < p_{l+1} < \cdots < p_{m-1}$ and $p_{m+1} < p_{m+2} < \cdots < p_n$ in $P$; thus $\{p_{m-1}, p_m, p_{m+1}\}$ is a clutter of $P$. Hence the closed interval $[\alpha_{m-2}, \alpha_{m+1}]$ of $L$ is boolean, and the chain $\mathcal{M} - \{\alpha_{m-1}, \alpha_m\}$ is essential. Consequently, there exists no fundamental chain of $L$ of length $d-2$. Thus, by Lemma (3.2), $\beta_{v-d,v-d+1}(k[\Delta(L)]) = 0$. Hence, by Lemma (2.1) again, the comparability graph $\text{Com}(L) = \Delta^{(1)}(L)$ of $L$ is $d$-connected as desired. □

**Remark** Easily seen from the above proof, for a planar distributive lattice $L$ of rank $d-1$ which is not a chain, the following conditions are equivalent.

(1) The comparability graph $\text{Com}(L)$ of $L$ is $d$-connected.
(2) There exists no element $\alpha \in L$ such that both $[\hat{0}, \alpha]$ and $[\alpha, \hat{1}]$ are chains.

## References

1. D. Barnette, "Graph theorems for manifolds," *Israel J. Math.* **16** (1973), 63–72.
2. W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge/New York/Sydney, 1993.

3. T. Hibi, "Level rings and algebras with straightening laws," *J. Algebra* **117** (1988), 343–362.
4. T. Hibi, *Algebraic Combinatorics on Convex Polytopes*, Carslaw Publications, Glebe, N.S.W., Australia, 1992.
5. T. Hibi, "Face number inequalities for matroid complexes and Cohen-Macaulay types of Stanley-Reisner rings of distributive lattices," *Pacific J. Math.* **154** (1992), 253–264.
6. M. Hochster, "Cohen-Macaulay rings, combinatorics, and simplicial complexes," in *Ring Theory II*, Lect. Notes in Pure and Appl. Math., No. 26, B.R. McDonald and R. Morris (Eds.), pp. 171–223. Dekker, New York, 1977.
7. R.P. Stanley, "Cohen-Macaulay complexes," in *Higher Combinatorics*, M. Aigner (Ed.), pp. 51–62. Reidel, Dordrecht/ Boston, 1977.
8. R.P. Stanley, *Combinatorics and Commutative Algebra*, Birkhäuser, Boston/Basel/Stuttgart, 1983.
9. R.P. Stanley, *Enumerative Combinatorics, Volume I*, Wadsworth & Brooks/Cole, Monterey, Calif., 1986.