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A "Fourier Transform" for Multiplicative Functions on Non-Crossing Partitions

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Abstract. We describe the structure of the group of normalized multiplicative functions on lattices of non-crossing partitions. As an application, we give a combinatorial proof of a theorem of D. Voiculescu concerning the multiplication of free random variables.

Keywords: non-crossing partition, Moebius function, free random variables

0. Introduction

For $n \geq 1$, let $NC(n)$ denote the lattice of non-crossing partitions of $\{1, \ldots, n\}$. Paralleling the considerations of [3, Section 5.2], the notion of multiplicative function on non-crossing partitions was considered by one of us in [13]. Such a function is an element of the large incidence algebra, $\mathcal{L}$, on non-crossing partitions, i.e., it is a complex-valued function f defined on the disjoint union of the sets of intervals in various $NC(n)$'s, $n \geq 1$. The set of multiplicative functions is closed under convolution (the product operation on the large incidence algebra $\mathcal{L}$); in fact, if we also impose the normalization condition $f([0_1, 1_1]) = 1$, where $0_1 = 1_1 =$ the unique element of $NC(1)$, then the set $\mathcal{M}_1$ of multiplicative functions satisfying it is a subgroup of the group of invertible elements in $\mathcal{L}$.

In this paper we describe the structure of the group $\mathcal{M}_1$ (Theorem 1.6, Corollary 1.7). Quite surprisingly, it turns out to be possible to do this via a "transform" which converts the convolution of multiplicative functions into the multiplication of formal power series (in the same way as the convolution of functions in $L^1(\mathbf{R})$, say, is transformed into multiplication by the Fourier transform).

Our work was started as an attempt of understanding from a combinatorial point of view a theorem of Voiculescu ([18], Theorem 2.6) concerning the "distribution of the product of two free random variables". The main result of the present paper can in fact be viewed as a new, combinatorial, proof of this theorem.

The paper is divided into three sections: in the first one we review the basic definitions which we need, and state our main result; the second section contains the proof of the main result; finally, in the third section we present the cited result of Voiculescu, and explain how our work is related to it.

1. Basic definitions and the statement of the result

1.1. The lattice NC(n)

A partition π of $\{1, \ldots, n\}$ is called *non-crossing* (notion introduced in [7]) if for every $1 \leq i < j < k < l \leq n$ such that i and k are in the same block of π, and such that j and l are in the same block of π, it necessarily follows that all of i, j, k, l are in the same block of π. The set $NC(n)$ of non-crossing partitions of $\{1, \ldots, n\}$ becomes a lattice when the refinement order is considered on it (i.e., for $\pi, \sigma \in NC(n)$, $\pi \leq \sigma$ means that every block of σ is a union of blocks of π). The combinatorics of $NC(n)$ has been studied by several authors (see [12], and the list of references there); we will only review here the facts which are needed for stating our result.

1.2. The complementation map of Kreweras

is a remarkable lattice anti-isomorphism $K : NC(n) \to NC(n)$, described as follows. Let π be a non-crossing partition of $\{1, \ldots, n\}$. We view $1, \ldots, n$ as points on a circle, equidistributed and clockwisely ordered, and for each block $B = \{b_1, \ldots, b_j\}$ of π we draw the convex polygon (inscribed in the circle) with vertices $b_1, \ldots, b_j$. The quality of π of being non-crossing is reflected into the fact that the convex polygons associated to its blocks do not intersect. Now, consider on the circle the midpoints of the arcs $(1, 2), (2, 3), \ldots, (n-1, n), (n, 1)$, and denote them by $\bar{1}, \bar{2}, \ldots, \bar{n}$, respectively. We look at the non-crossing partitions σ of $\{\bar{1}, \bar{2}, \ldots, \bar{n}\}$ with the property that the convex polygons associated to the blocks of σ do not intersect the ones associated to the blocks of π (i.e., π and σ together give a non-crossing partition of $\{1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}\}$). Among the partitions σ with the named property, there is a largest one (in the refinement order), and this is, by definition, $K(\pi)$.

As a concrete example, figure 1 illustrates that $K(\{\{1, 4, 8\}, \{2, 3\}, \{5, 6\}, \{7\}\}) = \{\{1, 3\}, \{2\}, \{4, 6, 7\}, \{5\}, \{8\}\} \in NC(8)$.

It is immediate that $K^2(\pi)$ is (for every $\pi \in NC(n)$) the anti-clockwise rotation of π with $360^\circ/n$; this shows in particular that K is a bijection, also the important fact that $K(\pi)$ and $K^{-1}(\pi)$ have always the same block structure (since they differ by a rotation). It is also easy to see that $\pi \leq \rho \Rightarrow K(\pi) \geq K(\rho)$ (and the converse must also hold, since K^2 is an isomorphism of $NC(n)$).

We mention that Simion and Ullman ([12], Section 1) have shown how the definition of the complementation map K can be modified to yield an anti-automorphism Φ of $NC(n)$ which has $\Phi^2 =$ identity. Also, it was shown by Biane in [1] that K and Φ generate together the group of all skew-automorphisms (i.e., automorphisms or anti-automorphisms) of $NC(n)$, which is the dihedral group with $4n$ elements.

1.3. The canonical product decomposition of the intervals in NC(n)

Modulo a modification of the convention concerning how many one-element lattices are to be taken in the decomposition, we follow here [13], Proposition 1 in Section 3.

Given $n \geq 1$ and $\pi \leq \sigma$ in $NC(n)$, we denote by $[\pi, \sigma]$ the interval $\{\rho \mid \pi \leq \rho \leq \sigma\} \subseteq NC(n)$. We denote by $\mathcal{I}nt_n$ the set of intervals of $NC(n)$, and by $\mathcal{I}nt$ the disjoint union of the

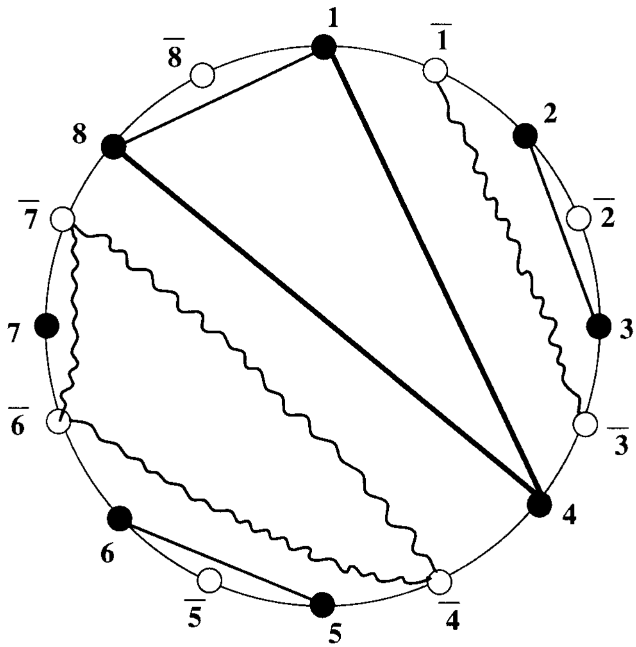

Figure 1.

$\mathcal{I}nt_n$'s, $n \geq 1$. Each interval $[\pi, \sigma] \in \mathcal{I}nt$ is carrying a lattice structure, coming from the $NC(n)$ where the interval has been taken from; of course, if the considered $[\pi, \sigma]$ happens to be $[0_n, 1_n]$, with $0_n = \{\{1\}, \{2\}, \dots, \{n\}\}$ and $1_n = \{\{1, 2, \dots, n\}\}$, the minimal and maximal elements of $NC(n)$, then the lattice $[\pi, \sigma]$ is $NC(n)$ itself.

Now, each interval $[\pi, \sigma] \in \mathcal{I}nt$ can be decomposed in a natural way as a product,

$$[\pi, \sigma] \simeq [0_1, 1_1]^{k_1} \times [0_2, 1_2]^{k_2} \times \cdots \times [0_n, 1_n]^{k_n} \times \cdots, \tag{1.1}$$

where $(k_n)_{n=1}^{\infty}$ is a sequence of non-negative integers, such that $k_n = 0$ for sufficiently large n. (1.1) is a lattice-isomorphism, and can be obtained in two steps:

Step 1. If we write $\sigma = \{B_1, \dots, B_k\}$ and $\pi = \{A_{1,1}, \dots, A_{1,m_1}, \dots, A_{k,1}, \dots, A_{k,m_k}\}$ such that $B_j = A_{j,1} \cup \cdots \cup A_{j,m_j}$ for $1 \leq j \leq k$, then

$$[\pi, \sigma] \simeq \prod_{j=1}^{k} \left[\left\{A_{j,1}, \dots, A_{j,m_j}\right\},\ 1_{|B_j|}\right]; \tag{1.2}$$

on the right-hand side of (1.2), $\{A_{j,1}, \dots, A_{j,m_j}\}$ is viewed as partition of $\{1, 2, \dots, |B_j|\}$ rather than one of B_j (via the order preserving bijection between B_j and $\{1, 2, \dots, |B_j|\}$). The verification of (1.2) is immediate, and left to the reader.

Step 2. Due to (1.2), we are left to consider the decomposition of $[\pi, \sigma]$ in the case when $\sigma = 1_n$ (for some $n \geq 1$). In this case, if $K(\pi) = \{A_1, \dots, A_h\} \in NC(n)$ denotes the complementation map applied to π, we have

$$[\pi, 1_n] \simeq \left[0_{|A_1|}, 1_{|A_1|}\right] \times \cdots \times \left[0_{|A_h|}, 1_{|A_h|}\right]. \tag{1.3}$$

Indeed, using the symbol $\stackrel{\sim}{\Leftrightarrow}$ for anti-isomorphism, we have

$$[\pi, 1_n] \stackrel{\sim}{\Leftrightarrow} [0_n, K(\pi)] \quad \text{(via } K \text{ on } NC(n));$$
$$[0_n, K(\pi)] \simeq \left[0_{|A_1|}, 1_{|A_1|}\right] \times \cdots \times \left[0_{|A_h|}, 1_{|A_h|}\right] \quad \text{(by the Step 1);}$$

and $[0_{|A_1|}, 1_{|A_1|}] \times \cdots \times [0_{|A_h|}, 1_{|A_h|}]$ is anti-isomorphic to itself (via the product of the complementation maps on $NC(|A_1|), \ldots, NC(|A_h|)$).

For example, if $\pi = \{\{1, 9\}, \{2, 5\}, \{3\}, \{4\}, \{6\}, \{7, 8\}, \{10\}, \{11\}, \{12\}\}$ and $\sigma = \{\{1, 6, 9, 12\}, \{2, 4, 5\}, \{3\}, \{7, 8\}, \{10, 11\}\}$ in $NC(12)$, then Step 1 gives

$$\begin{aligned}[\pi, \sigma] \simeq {} & [\{\{1, 3\}, \{2\}, \{4\}\}, 1_4] \times [\{\{1, 3\}, \{2\}, 1_3] \times [\{\{1\}\}, 1_1] \\ & \times [\{\{1, 2\}\}, 1_2] \times [\{\{1\}, \{2\}\}, 1_2];\end{aligned} \tag{1.4}$$

and Step 2 gives

$$\begin{cases} [\{\{1, 3\}, \{2\}, \{4\}\}, 1_4] \simeq [0_2, 1_2]^2, & \text{because } K(\{\{1, 3\}, \{2\}, \{4\}\}) \\ & \qquad = \{\{1, 2\}, \{3, 4\}\} \\ [\{\{1, 3\}, \{2\}\}, 1_3] \simeq [0_1, 1_1] \times [0_2, 1_2], & \text{because } K(\{\{1, 3\}, \{2\}\}) \\ & \qquad = \{\{1, 2\}, \{3\}\} \\ [\{\{1\}\}, 1_1] \simeq [0_1, 1_1], & \text{because } K(\{\{1\}\}) = \{\{1\}\} \\ [\{\{1, 2\}\}, 1_2] \simeq [0_1, 1_1]^2, & \text{because } K(\{\{1, 2\}\}) = \{\{1\}, \{2\}\} \\ [\{\{1\}, \{2\}\}, 1_2] \simeq [0_2, 1_2], & \text{because } K(\{\{1\}, \{2\}\}) = \{\{1, 2\}\}. \end{cases} \tag{1.5}$$

Hence the canonical decomposition of $[\pi, \sigma]$ is, by (1.4) and (1.5), $[\pi, \sigma] \simeq [0_1, 1_1]^4 \times [0_2, 1_2]^4$. The specifics of working with non-crossing partitions can be seen well in the first Eq. (1.5), where we get $[0_2, 1_2]^2$, rather than $[0_3, 1_3]$, as one would expect at first glance; this is related to the fact that when connecting $\{\{1, 3\}, \{2\}, \{4\}\}$ with 1_4 by a chain in $NC(4)$, we are not allowed to start by putting together the blocks $\{2\}$ and $\{4\}$.

1.4. Multiplicative functions on non-crossing partitions

This notion is obtained by paralleling the considerations of [3], Section 5.2 (see, equivalently, Section 3.5.2 in [11]), in the context of the product decompositions observed in the previous subsection. We will be again following [13], Sections 2 and 3.

Let us recall that the *convolution* of $f, g : \mathcal{I}nt \to \mathbf{C}$ (with $\mathcal{I}nt$ the set of intervals considered in 1.3 above) is $f \star g : \mathcal{I}nt \to \mathbf{C}$ defined by:

$$(f \star g)([\pi, \sigma]) \stackrel{\text{def}}{=} \sum_{\rho \in [\pi, \sigma]} f([\pi, \rho]) g([\rho, \sigma]), \quad [\pi, \sigma] \in \mathcal{I}nt. \tag{1.6}$$

With this operation (as multiplication) and with addition and scalar multiplication defined pointwisely, the set $\mathcal{L}$ of all complex functions defined on $\mathcal{I}nt$ becomes a complex associative algebra, called the *large incidence algebra on non-crossing partitions* (compare to [3], Section 5).

Definition 1.4.1 A function $f : \mathcal{I}nt \to \mathbf{C}$ will be called *multiplicative* if whenever $[\pi, \sigma] \in \mathcal{I}nt$ has canonical product decomposition $[0_1, 1_1]^{k_1} \times [0_2, 1_2]^{k_2} \times [0_3, 1_3]^{k_3} \times \cdots$, then

$$f([\pi, \sigma]) = f([0_1, 1_1])^{k_1} f([0_2, 1_2])^{k_2} f([0_3, 1_3])^{k_3} \cdots \tag{1.7}$$

We will denote by $\mathcal{M}$ the set of all multiplicative functions $f : \mathcal{I}nt \to \mathbf{C}$, and by $\mathcal{M}_1 \subseteq \mathcal{M}$ the set of multiplicative functions f such that $f([0_1, 1_1]) = 1$.

Clearly, each sequence $(\alpha_n)_{n=1}^{\infty}$ of complex numbers determines uniquely a multiplicative function $f \in \mathcal{M}$ (defined by (1.7) and the condition that $f([0_n, 1_n]) = \alpha_n$, $n \geq 1$). Every $f \in \mathcal{M}$ can be obtained in this way, and it is in $\mathcal{M}_1$ if and only if $\alpha_1 = 1$.

It is easy to see that the convolution (1.6) of two multiplicative functions $f, g \in \mathcal{M}$ is still multiplicative[1] (see Proposition 2 in Section 3 of [13], or compare to Proposition 5.1 in [3]). If $f, g \in \mathcal{M}$ are corresponding (in the sense of the preceding paragraph) to the sequences $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$, respectively, then $f \star g$ corresponds to the sequence $(\gamma_n)_{n=1}^{\infty}$, where

$$\gamma_n = \sum_{\substack{\pi \in NC(n) \\ \pi \overset{\text{def}}{=} \{A_1,\ldots,A_h\} \\ K(\pi) \overset{\text{def}}{=} \{B_1,\ldots,B_k\}}} \alpha_{|A_1|} \cdots \alpha_{|A_h|} \beta_{|B_1|} \cdots \beta_{|B_k|}. \tag{1.8}$$

Indeed, (1.8) comes out by writing that

$$\gamma_n = (f \star g)([0_n, 1_n]) = \sum_{\pi \in NC(n)} f([0_n, \pi]) g([\pi, 1_n]),$$

and by using what we know about the canonical product decomposition of $[0_n, \pi]$ (see Step 1 in 1.3) and of $[\pi, 1_n]$ (see Step 2 in 1.3).

It is, moreover, easy to see that $\mathcal{M}_1$ of 1.4.1 is a subgroup of the invertible elements of the large incidence algebra $\mathcal{L}$. Indeed, $\mathcal{M}_1$ is also closed under convolution, since we have $(f \star g)([0_1, 1_1]) = f([0_1, 1_1])g([0_1, 1_1]) = 1$ for every $f, g \in \mathcal{M}_1$. The unit δ of $\mathcal{L}$ is in $\mathcal{M}_1$, and corresponds to the sequence $(1, 0, 0 \cdots)$. Each $f \in \mathcal{M}_1$ has $f([\pi, \pi]) = f([0_1, 1_1])^n = 1$ for all $n \geq 1$ and $\pi \in NC(n)$, which implies that f is invertible in $\mathcal{L}$—see for instance [15], Proposition 3.6.2. In order to verify that the inverse of $f \in \mathcal{M}_1$ is still in $\mathcal{M}_1$, one can proceed as follows: starting with the sequence $\alpha_n = f([0_n, 1_n])$, $n \geq 1$, determine recursively by using (1.8) a sequence $(\beta_n)_{n=1}^{\infty}$ such that the γ_n's obtained in (1.8) are $\gamma_1 = 1$ and $\gamma_2 = \gamma_3 = \cdots = 0$; then the multiplicative function g determined by $(\beta_n)_{n=1}^{\infty}$ will have $f \star g = \delta$—hence $g = f^{-1}$.

It is interesting to remark next that

Proposition 1.4.2 *The convolution operation is commutative on* $\mathcal{M}_1$.

Proof: Let f, g be in $\mathcal{M}_1$, and let us make the notations $f([0_n, 1_n]) = \alpha_n$, $g([0_n, 1_n]) = \beta_n$, $(f \star g)([0_n, 1_n]) = \gamma_n$, $(g \star f)([0_n, 1_n]) = \gamma_n'$, $n \geq 1$. Then γ_n is expressed in terms of the α's and the β's by Eq. (1.8). Since the complementation map is bijective we can also write, by denoting $K(\pi) = \rho$ in (1.8):

$$\gamma_n = \sum_{\substack{\rho \in NC(n) \\ \rho \overset{\text{def}}{=} \{B_1, \ldots, B_k\} \\ K^{-1}(\rho) \overset{\text{def}}{=} \{A_1, \ldots, A_h\}}} \beta_{|B_1|} \cdots \beta_{|B_k|} \alpha_{|A_1|} \cdots \alpha_{|A_h|}. \tag{1.9}$$

Moreover, in the sum on the right-hand side of (1.9) we can replace "$K^{-1}(\rho)$" by "$K(\rho)$" (because, as remarked in 1.2, $K(\rho)$ and $K^{-1}(\rho)$ have the same block structure). But when this is done, the right-hand side of (1.9) becomes exactly the expression of γ_n'. We conclude that $\gamma_n = \gamma_n'$, i.e., that $(f \star g)([0_n, 1_n]) = (g \star f)([0_n, 1_n])$, for every $n \geq 1$, which implies $f \star g = g \star f$. $\square$

It should be noted that (by exactly the same argument) convolution is in fact commutative on the larger semigroup $\mathcal{M} \supset \mathcal{M}_1$. As the proof of 1.4.2 clearly shows, this phenomenon depends on the self-duality of $NC(n)$ (its analogue can't therefore hold in the framework of the lattice of all partitions of $\{1, \ldots, n\}$).

1.5. Remark: Convolution on $\mathbf{C_c}$*(R) and the Fourier transform*

Let us recall now another framework where an operation called "convolution" is studied. Let $C_c(\mathbf{R})$ denote the space of continuous compactly supported functions on the real line. For $f, g \in C_c(\mathbf{R})$, their convolution $f \star g \in C_c(\mathbf{R})$ is defined by

$$(f \star g)(t) = \int_{-\infty}^{\infty} f(s) g(t-s)\, ds, \quad t \in \mathbf{R}. \tag{1.10}$$

As it is well-known, one way of studying the convolution on $C_c(\mathbf{R})$ is via the Fourier transform. For $f \in C_c(\mathbf{R})$, its Fourier transform $\mathcal{F} f$ is defined by

$$(\mathcal{F} f)(z) = \int_{-\infty}^{\infty} e^{itz} f(t)\, dt = \sum_{n=0}^{\infty} \left(\frac{i^n}{n!} \int_{-\infty}^{\infty} t^n f(t)\, dt \right) z^n; \tag{1.11}$$

$\mathcal{F} f$ is an analytic function—but for our purposes it is more convenient to view it as a formal power series in z. The relevance of the Fourier transform for convolution is that it transforms it into the simpler operation of pointwise multiplication of power series,

$$[\mathcal{F}(f \star g)](z) = (\mathcal{F} f)(z)(\mathcal{F} g)(z), \quad f, g \in C_c(\mathbf{R}). \tag{1.12}$$

The relation between the present remark and the considerations preceding it would seem at first to be reduced to the fact that in both cases an operation called "convolution" and denoted by "$\star$" is studied. In particular, one would be inclined to find it unlikely that the analogue of (1.12) could be somehow reached in the framework of non-crossing partitions. It is quite surprising that this is in fact the case. While the deeper reasons of this phenomenon remain to be elucidated (and a more general context for a "combinatorial Fourier transform" remains to be found), let us state the main result of the paper, which is the following.

Theorem 1.6 *For every f in the group $\mathcal{M}_1$ of* 1.4.1 *we denote by φ_f the formal power series*

$$\varphi_f(z) = \sum_{n=1}^{\infty} f([0_n, 1_n]) z^n, \tag{1.13}$$

and we denote by $\varphi_f^{\langle -1 \rangle}$ the inverse of φ_f in the group of the formal power series of the form $z + \gamma_2 z^2 + \gamma_3 z^3 + \cdots$, endowed with the operation of composition; in other words, $\varphi_f^{\langle -1 \rangle}$ is the unique formal power series ψ, without constant coefficient, in a variable z, such that $\sum_{n=1}^{\infty} f([0_n, 1_n])(\psi(z))^n = z$.

If we put, for every $f \in \mathcal{M}_1$:

$$(\mathcal{F} f)(z) = \frac{1}{z} \varphi_f^{\langle -1 \rangle}(z) \tag{1.14}$$

(*formal power series in z, with constant coefficient equal to* 1), *then we have*:

$$[\mathcal{F}(f \star g)](z) = (\mathcal{F} f)(z)(\mathcal{F} g)(z), \quad f, g \in \mathcal{M}_1; \tag{1.15}$$

i.e., the "Fourier transform" defined by (1.14) *converts the convolution of multiplicative functions on non-crossing partitions into the multiplication of formal power series.*

Corollary 1.7 *The convolution group $\mathcal{M}_1$ considered in Section* 1.4 *is isomorphic to a countable direct product of copies of* **C**.

Proof: Let $\mathcal{G}$ be the multiplicative group of formal power series with constant coefficient equal to 1. It is immediate that $\mathcal{F} : \mathcal{M}_1 \to \mathcal{G}$ is a bijection, and the Theorem 1.6 ensures that it is a group isomorphism. But $\mathcal{G}$ is indeed isomorphic to a countable direct product of copies of **C** (since the formal logarithm takes it into the additive group of formal power series without constant coefficient). □

2. The proof of the result

Notation 2.1

1° For every $n \geq 1$, we will denote by $NC'(n)$ the set of non-crossing partitions of $\{1, \ldots, n\}$ which have $\{1\}$ as a one-element block. (Thus $NC'(1) = NC(1)$, while for $n \geq 2$, $NC'(n)$ is in natural bijection with $NC(n-1)$.)

2° For f, g in the group $\mathcal{M}_1$ considered in Section 1.4, we will denote by $f \overset{\vee}{\star} g \in \mathcal{M}_1$ the multiplicative function uniquely determined by

$$(f \overset{\vee}{\star} g)([0_n, 1_n]) = \sum_{\substack{\pi \in NC'(n) \\ \pi \overset{\text{def}}{=} \{A_1,\ldots,A_h\} \\ K(\pi) \overset{\text{def}}{=} \{B_1,\ldots,B_k\}}} \alpha_{|A_1|} \cdots \alpha_{|A_h|} \beta_{|B_1|} \cdots \beta_{|B_k|}, \tag{2.1}$$

where $\alpha_m \overset{\text{def}}{=} f([0_m, 1_m])$, $\beta_m \overset{\text{def}}{=} g([0_m, 1_m])$, for $m \geq 1$.

We would like to call the operation $\overset{\vee}{\star}$ of 2.1.2° by the name of "pinched-convolution"; this comes from the fact that the summation formula defining $(f \overset{\vee}{\star} g)([0_n, 1_n])$ is obtained from the one defining $(f \star g)([0_n, 1_n])$ (see Eq. (1.8) above) by "pinching out" the terms in $NC(n) \backslash NC'(n)$. The reason for introducing $\overset{\vee}{\star}$ is that considerations involving it will turn out to simplify quite a lot the proof of Theorem 1.6.

Unlike the convolution operation on $\mathcal{M}_1$, one cannot expect that $\overset{\vee}{\star}$ is commutative, however there is a nice "symmetrization lemma" that holds.

Lemma 2.2 *For $f, g \in \mathcal{M}_1$ we have*

$$\varphi_{f \overset{\vee}{\star} g}(z) \varphi_{g \overset{\vee}{\star} f}(z) = z \varphi_{f \star g}(z) \tag{2.2}$$

(*where φ_h for $h \in \mathcal{M}_1$ is defined as in* (1.13) *of Theorem* 1.6).

Proof: Fix a positive integer n. The coefficients of z^{n+1} on the two sides of (2.2) are

$$\sum_{j=1}^{n} (f \overset{\vee}{\star} g)([0_j, 1_j])(g \overset{\vee}{\star} f)([0_{n+1-j}, 1_{n+1-j}])$$

$$= \sum_{j=1}^{n} \sum_{\substack{\pi \in NC'(j) \\ \rho \in NC'(n+1-j)}} f([0_j, \pi]) g([\pi, 1_j]) g([0_{n+1-j}, \rho]) f([\rho, 1_{n+1-j}]), \tag{2.3}$$

and respectively

$$(f \star g)([0_n, 1_n]) = \sum_{\sigma \in NC(n)} f([0_n, \sigma]) g([\sigma, 1_n]). \tag{2.4}$$

What we need is hence the equality of the sums appearing in (2.3) and the right-hand side of (2.4). It turns out that more is true: there exists a natural bijection between the index sets of the sums in (2.3) and (2.4),

$$\bigcup_{\substack{1 \leq j \leq n \\ \text{(disjoint)}}} NC'(j) \times NC'(n+1-j) \rightarrow NC(n) \tag{2.5}$$

such that if $(\pi, \rho) \in NC'(j) \times NC'(n+1-j)$ corresponds by (2.5) to $\sigma \in NC(n)$, then the term indexed by (π, ρ) in the sum (2.3) equals the term indexed by σ in the sum (2.4)—or more precisely:

$$\begin{cases} f([0_n, \sigma]) = f([0_j, \pi])\, f([\rho, 1_{n+1-j}]) \\ g([\sigma, 1_n]) = g([\pi, 1_j])g([0_{n+1-j}, \rho]). \end{cases} \tag{2.6}$$

The description of the bijection (2.5) goes as follows: start with $1 \le j \le n$, $\pi \in NC'(j)$, $\rho \in NC'(n+1-j)$; denote by $\check{\pi} \in NC(j-1)$ the partition obtained by deleting the one-element block $\{1\}$ of π, and consider on the other hand $K(\rho) \in NC(n+1-j)$ (the complementation map applied to ρ). Then $\sigma \in NC(n)$ which corresponds by (2.5) to (π, ρ) is obtained by simply juxtaposing $\check{\pi}$ and $K(\rho)$, in this order. (For example: if $n = 6$, $j = 3$, $\pi = \{\{1\}, \{2, 3\}\}$, $\rho = \{\{1\}, \{2, 4\}, \{3\}\}$, then $\sigma = \{\{1, 2\}, \{3, 6\}, \{4, 5\}\}$.)

It is easy to verify that the map (2.5), as defined in the preceding paragraph, is indeed a bijection. Its inverse is described as follows: start with $\sigma \in NC(n)$, and denote by j the smallest element of the block of σ containing n. Then each of $\{1, \dots, j-1\}$ and $\{j, \dots, n\}$ is a union of blocks of σ, thus σ is obtained as the juxtaposition of two non-crossing partitions $\sigma_1 \in NC(j-1)$ and $\sigma_2 \in NC(n+1-j)$. We let $\pi \in NC'(j)$ be the partition obtained by adding a one-element block to the left of σ_1, and we put $\rho = K^{-1}(\sigma_2) \in NC'(n+1-j)$ ($K^{-1}(\sigma_2)$ has $\{1\}$ as a one-element block—this is implied by the fact that 1 and $n+1-j$ are in the same block of σ_2). Then the pair (π, ρ) obtained in this way is the pre-image of σ by the map (2.5).

From the explicit descriptions made in the preceding two paragraphs, it is clear that (when σ corresponds to (π, ρ)—i.e., is the juxtaposition of $\check{\pi}$ and $K(\rho)$, as above):

$$\begin{aligned} f([0_n, \sigma]) &= f([0_{j-1}, \check{\pi}])\, f([0_{n+1-j}, K(\rho)]) \\ &= f([0_j, \pi])\, f([\rho, 1_{n+1-j}]), \end{aligned} \tag{2.7}$$

i.e., the first relation (2.6) takes indeed place. (In (2.7), we have $f([0_j, \pi]) = f([0_{j-1}, \check{\pi}])$ due to the hypothesis that $f([0_1, 1_1]) = 1$, and the equality $f([0_{n+1-j}, K(\rho)]) = f([\rho, 1_{n+1-j}])$ follows from the Step 2 of 1.3.)

In order to verify the second relation (2.6), one “applies the complementation map” to the bijection (2.5). More precisely (as the reader can check without difficulty on a circular picture), the following happens: if $\sigma \in NC(n)$ corresponds by (2.5) to $(\pi, \rho) \in NC'(j) \times NC'(n+1-j)$, then $K^{-1}(\sigma)$ is the juxtaposition of $K(\pi)$ and $\check{\rho}$. (For instance, in the concrete example given above, with $n = 6$ and $j = 3$: $K^{-1}(\sigma) = \{\{1, 3\}, \{2\}, \{4, 6\}, \{5\}\}$, while $K(\pi) = \check{\rho} = \{\{1, 3\}, \{2\}\}$.) But then

$$\begin{aligned} g([\sigma, 1_n]) &= g([0_n, K(\sigma)]) && \text{(by Step 2 in 1.3)} \\ &= g([0_n, K^{-1}(\sigma)]) && \text{(because } K^{-1}(\sigma) \text{ is a rotation of } K(\sigma)) \\ &= g([0_j, K(\pi)])g([0_{n-j}, \check{\rho}]) \end{aligned}$$

(because $K^{-1}(\sigma)$ is the juxtaposition of $K(\pi)$ and $\check{\rho}$)

$$= g([\pi, 1_j])g([0_{n+1-j}, \rho])$$

(by the same argument as for (2.7)). □

The next Proposition 2.3 is based on the enumeration of non-crossing partitions in $NC(n)$ according to their block containing $1 \in \{1, \dots, n\}$. We mention that an important particular case of this proposition (when g of Eq. (2.8) is the ζ function on non-crossing partitions) was previously done in [13] (Theorem in Section 3), and is the combinatorial equivalent of a result of Voiculescu ([17], Theorem 2.9).

Proposition 2.3 *For every* $f, g \in \mathcal{M}_1$ *we have*

$$\varphi_f \circ \varphi_{f \overset{\vee}{\star} g} = \varphi_{f \star g} \tag{2.8}$$

(*where* φ_h *for* $h \in \mathcal{M}_1$ *is defined as in* (1.13) *of Theorem* 1.6, *and* "$\circ$" *denotes the formal composition of series*).

Proof: We will need the following Lemma, the simple proof of which is left to the reader.

Lemma *Let n be a positive integer, and let B be a subset of* $\{1, \dots, n\}$ *such that* $B \ni 1$. *Denote by* $NC_B(n)$ *the set of non-crossing partitions of* $\{1, \dots, n\}$ *that are having B as a block. Then we have a natural bijection*

$$NC_B(n) \to \prod_{p=1}^{m} NC'(j_{p+1} - j_p), \tag{2.9}$$

where $1 = j_1 < j_2 < \cdots < j_m$ *is the list of elements of B and* $j_{m+1} \overset{\text{def}}{=} n + 1$, *and where the notation* NC' *is as introduced in* 1° *of* 2.1. *The bijection* (2.9) *associates to* $\pi \in NC_B(n)$ *the m-tuple* $(\pi_p)_{1 \le p \le m}$, *where* π_p *is the restriction of* π *to the interval* $\{j_p, j_p + 1, \dots, j_{p+1} - 1\}$. (*Note that* $\{j_p\}$ *is a one-element block of* π_p, *while the rest of* π_p, $\overset{\vee}{\pi}_p = \pi_p \setminus \{\{j_p\}\}$, *is a union of blocks of* π.) *Moreover, if* $\pi \to (\pi_p)_{1 \le p \le m}$ *as above, then* $K(\pi)$ *is the juxtaposition of* $K(\pi_1), \dots, K(\pi_m)$, *in this order* (*where* $K =$ *the complementation map, as above*).

Now, let f, g be as in the statement of the proposition. We fix a positive integer n, and we write:

$$\begin{aligned}(f \star g)([0_n, 1_n]) &= \sum_{\pi \in NC(n)} f([0_n, \pi])g([\pi, 1_n]) \\ &= \sum_{1 \in B \subseteq \{1,\dots,n\}} \left(\sum_{\pi \in NC_B(n)} f([0_n, \pi])g([\pi, 1_n]) \right).\end{aligned} \tag{2.10}$$

Let us also fix a $B = \{j_1, \dots, j_m\}$, with $1 = j_1 < \cdots < j_m \le n$, and let us make, in the sum indexed by $NC_B(n)$ which appears in (2.10) the "change of variable" provided by the bijection (2.9). Instead of "$\sum_{\pi \in NC_B(n)}$" we will thus have "$\sum_{\pi_1 \in NC'(j_2 - j_1), \dots, \pi_m \in NC'(j_{m+1} - j_m)}$";

moreover—by taking into account how π is related to $(\pi_p)_{1\le p\le m}$, and how $K(\pi)$ is related to $(K(\pi_p))_{1\le p\le m}$—we will replace:

(a) $$f([0_n,\pi]) = f([0_m,1_m])\cdot\prod_{p=1}^{m} f\big([0_{j_{p+1}-j_p},\pi_p]\big)$$

($f([0_m,1_m])$ comes from the block B of π; then the other blocks of π are given by the union of the $\overset{\vee}{\pi}_p$'s, where $\overset{\vee}{\pi}_p=\pi_p\backslash\{\{j_p\}\}$, $1\le p\le m$, and this brings up the product $\prod_{p=1}^{m} f([0_{j_{p+1}-j_p-1},\overset{\vee}{\pi}_p]) = \prod_{p=1}^{m} f([0_{j_{p+1}-j_p},\pi_p])$); and

(b) $$\begin{aligned} g([\pi,1_n]) = g([0_n,K(\pi)]) &= \prod_{p=1}^{m} g\big([0_{j_{p+1}-j_p},K(\pi_p)]\big) \\ &= \prod_{p=1}^{m} g\big([\pi_p,1_{j_{p+1}-j_p}]\big). \end{aligned}$$

Therefore, we obtain:

$$\begin{aligned} &\sum_{\pi\in NC_B(n)} f([0_n,\pi])g([\pi,1_n]) \\ &\quad= \sum_{\substack{\pi_1\in NC'(j_2-j_1),\dots,\\ \pi_m\in NC'(j_{m+1}-j_m)}} f([0_m,1_m])\cdot\prod_{p=1}^{m} f\big([0_{j_{p+1}-j_p},\pi_p]\big)\cdot\prod_{p=1}^{m} g\big([\pi_p,1_{j_{p+1}-j_p}]\big) \\ &\quad= f([0_m,1_m])\cdot\prod_{p=1}^{m}\left(\sum_{\pi_p\in NC'(j_{p+1}-j_p)} f\big([0_{j_{p+1}-j_p},\pi_p]\big)g\big([\pi_p,1_{j_{p+1}-j_p}]\big)\right) \\ &\quad= f([0_m,1_m])\cdot\prod_{p=1}^{m}(f\overset{\vee}{\star}g)\big([0_{j_{p+1}-j_p},1_{j_{p+1}-j_p}]\big). \end{aligned} \tag{2.11}$$

Next, we let B run in the set of subsets of $\{1,\dots,n\}$ which contain 1, and replace (2.11) in (2.10). It is convenient to make in (2.11) the substitution $j_2-j_1=i_1,\dots,j_{m+1}-j_m=i_m$; B is completely determined by $(m;i_1,\dots,i_m)$, and when B runs in $\{B\subseteq\{1,\dots,n\}\mid B\ni 1\}$, the corresponding $(m;i_1,\dots,i_m)$ runs in $\{(m;i_1,\dots,i_m)\mid 1\le m\le n;\ i_1,\dots,i_m\ge 1;\ i_1+\cdots+i_m=n\}$. So, what we obtain in the continuation of (2.10) is

$$(f\star g)([0_n,1_n]) = \sum_{m=1}^{n}\sum_{\substack{i_1,\dots,i_m\ge 1\\ \text{such that}\\ i_1+\cdots+i_m=n}} f([0_m,1_m])\cdot\prod_{p=1}^{m}(f\overset{\vee}{\star}g)\big([0_{i_p},1_{i_p}]\big). \tag{2.12}$$

It is clear that the right-hand side of (2.12) is the coefficient of z^n in the series

$$\sum_{m=1}^{\infty} f([0_m, 1_m])\left(\sum_{i=1}^{\infty}(f \overset{\vee}{\star} g)([0_i, 1_i])z^i\right)^m = (\varphi_f \circ \varphi_{f \overset{\vee}{\star} g})(z);$$

since the left-hand side of (2.12) is the coefficient of z^n in $\varphi_{f\star g}(z)$, the proof of (2.8) is hence completed. □

2.4. *Proof of the Theorem 1.6*

Once the Eqs. (2.2) and (2.8) are established, we are only left to perform a short algebraic manipulation. Let f and g be multiplicative functions in the group $\mathcal{M}_1$ (considered in the Theorem). We start from the relation (2.2) obtained in the Lemma 2.2, and compose it on the right with $\varphi_{f\star g}^{\langle -1\rangle}$ (where φ_h for $h \in \mathcal{M}_1$ has the significance introduced in 1.6). Denoting the power series z (in the variable z) by id, what we get is:

$$\left(\varphi_{f \overset{\vee}{\star} g} \circ \varphi_{f\star g}^{\langle -1\rangle}\right) \cdot \left(\varphi_{g \overset{\vee}{\star} f} \circ \varphi_{f\star g}^{\langle -1\rangle}\right) = \left(id \circ \varphi_{f\star g}^{\langle -1\rangle}\right) \cdot \left(\varphi_{f\star g} \circ \varphi_{f\star g}^{\langle -1\rangle}\right). \tag{2.13}$$

We have $id \circ \varphi_{f\star g}^{\langle -1\rangle} = \varphi_{f\star g}^{\langle -1\rangle}$, $\varphi_{f\star g} \circ \varphi_{f\star g}^{\langle -1\rangle} = id$ (because id is the unit element for composition, while $\varphi_{f\star g}^{\langle -1\rangle}$ is the inverse of $\varphi_{f\star g}$ under the same operation); hence the right-hand side of (2.13) is $id \cdot \varphi_{f\star g}^{\langle -1\rangle}$ (the series $z\varphi_{f\star g}^{\langle -1\rangle}(z)$ in the variable z).

On the other hand we have

$$\varphi_{f \overset{\vee}{\star} g} \circ \varphi_{f\star g}^{\langle -1\rangle} = \varphi_f^{\langle -1\rangle}; \tag{2.14}$$

this follows from the Eq. (2.8) of Proposition 2.3, by composing it on the left with $\varphi_f^{\langle -1\rangle}$ and on the right with $\varphi_{f\star g}^{\langle -1\rangle}$. By switching the roles of f and g in (2.14), and taking into account that $f \star g = g \star f$, we also get that $\varphi_{g \overset{\vee}{\star} f} \circ \varphi_{f\star g}^{\langle -1\rangle} = \varphi_g^{\langle -1\rangle}$; hence the left-hand side of (2.13) is $\varphi_f^{\langle -1\rangle}\varphi_g^{\langle -1\rangle}$.

So we have obtained:

$$\varphi_f^{\langle -1\rangle}(z)\varphi_g^{\langle -1\rangle}(z) = z\varphi_{f\star g}^{\langle -1\rangle}(z), \tag{2.15}$$

and dividing in (2.15) by z^2 yields the desired relation $\mathcal{F}(f \star g) = (\mathcal{F} f)(\mathcal{F}g)$. □

Remark 2.5 There are also other applications of the Eqs. (2.2) and (2.8) that may be of interest (besides the above proof, which was the main goal of the section). As an example, we show here how (2.8) can be used to obtain a formula for the convolution with the Moebius function on non-crossing partitions.

So, let μ denote the Moebius function on non-crossing partitions, i.e., the inverse in the large incidence algebra on non-crossing partitions, $\mathcal{L}$, of the function $\zeta : \mathcal{I}nt \to \mathbf{C}$ identically equal to 1. Since ζ is, clearly, the multiplicative function on non-crossing partitions determined by the sequence $(1, 1, 1, \ldots, 1, \ldots)$, the considerations preceding Proposition 1.4.2 above show that μ is also a multiplicative function. The sequence of numbers determining μ is found to be the one of the signed Catalan numbers,

$$\mu([0_n, 1_n]) = \frac{(-1)^{n-1}(2n-2)!}{(n-1)!n!}, \quad n \geq 1 \tag{2.16}$$

(see [7], Section 7). For the general theory of the Moebius function on posets see [10], or [15] Chapter 3.

As it was realized in [13], the convolution with μ plays an important role in the combinatorial approach to the theory of free random variables; this will be confirmed by the development presented in the next section (see the discussion in 3.2, 3.3). The formula which we derive in the next paragraph (Eq. (2.18)) is equivalent to the result in the Theorem in Section 3 of [13].

Let h be a multiplicative function in $\mathcal{M}_1$, and let us put $g \overset{\text{def}}{=} h \star \mu$ (the relation defining g is, of course, equivalent to $h = g \star \zeta$). By comparing the Eqs. (1.8) and (2.1), and by taking into account that ζ is identically 1, we see that $(g \overset{\vee}{\star} \zeta)([0_n, 1_n]) = (g \star \zeta)([0_{n-1}, 1_{n-1}]) = h([0_{n-1}, 1_{n-1}])$ for every $n \geq 2$; this implies the equality:

$$\varphi_{g \overset{\vee}{\star} \zeta}(z) \;=\; z(1 + \varphi_h(z)). \tag{2.17}$$

Now, from (2.8) we get that $\varphi_{g \overset{\vee}{\star} \zeta} = \varphi_g^{\langle -1 \rangle} \circ \varphi_{g \star \zeta} = \varphi_g^{\langle -1 \rangle} \circ \varphi_h$; hence, if we compose with $\varphi_h^{\langle -1 \rangle}$ on the right, the left-hand side of (2.17) becomes just $\varphi_g^{\langle -1 \rangle}$ (or, in other words: $\varphi_{h \star \mu}^{\langle -1 \rangle}$). On the other hand, composing the right-hand side of (2.17) with $\varphi_h^{\langle -1 \rangle}$ brings us to $\varphi_h^{\langle -1 \rangle}(z) \cdot (1 + z)$ (argument similar to the one used in (2.13)), therefore the equality which is obtained reads:

$$\varphi_{h \star \mu}^{\langle -1 \rangle}(z) \;=\; (1 + z)\varphi_h^{\langle -1 \rangle}(z), \quad \text{for every } h \in \mathcal{M}_1. \tag{2.18}$$

The above mentioned Theorem in Section 3 of [13] states that if (for some $h \in \mathcal{M}_1$) we set $A(z) \overset{\text{def}}{=} 1 + \varphi_{h \star \mu}(z)$, $B(z) \overset{\text{def}}{=} 1 + \varphi_h(z)$, then A and B satisfy the equation $A(zB(z)) = B(z)$. This is equivalent to (2.18), in a formulation which avoids considering inverses under composition (in order to check the equivalence, one only needs to compose with φ_h on the right in (2.18)).

3. The connection with the *S*-transform of Voiculescu

The present work was started as an attempt of understanding, from a combinatorial point of view, a theorem of Voiculescu [18] concerning the *moments of the product of two free random variables*. In this section we will review the mentioned result of Voiculescu, and present its connection with the Theorem 1.6 above.

We have to start with a few basic definitions related to free random variables; our presentation here will deal only with combinatorial aspects of this notion (for more details, see for instance the monograph [19]).

3.1. Review of freeness and of the S-transform

Definition 3.1.1 Let $\mathcal{A}$ be a complex algebra with unit I, and let $\tau : \mathcal{A} \to \mathbf{C}$ be a linear functional, normalized by $\tau(I) = 1$. For $a \in \mathcal{A}$, the numbers in the sequence $(\tau(a^n))_{n=0}^{\infty}$ will be called the *moments* of a with respect to τ. For $a, b \in \mathcal{A}$, the value of τ at monomials in a and b (e.g., $\tau(a^2bab^3a^5)$) will be called *mixed moments* of a and b (with respect to τ). Note that $\mathcal{A}$ is not assumed to be commutative—thus for instance the mixed moment $\tau(abab)$ is in general not the same thing as $\tau(a^2b^2)$.

The terminology in 3.1.1 is inspired from the situation when $\mathcal{A}$ is an algebra of random variables on a probability space $(\Omega, \mathcal{F}, P)$ (e.g., $\mathcal{A} = L^\infty(\Omega, \mathcal{F}, P)$), and the functional τ is the integral, $\tau(a) = \int_\Omega a(\omega)\, dP(\omega)$, for $a \in \mathcal{A}$. Of course, the framework in 3.1.1 is leaving aside the measure-theoretic facet of the situation, while on the other hand it is gaining a more complicated algebraic structure from the fact that $\mathcal{A}$ isn't necessarily commutative. Even with these differences, it is useful to think of $\mathcal{A}$ in 3.1.1 as of "an algebra of random variables" (and this is why the elements of $\mathcal{A}$ are sometimes referred to as "non-commutative random variables"). Following this line of thought, the concept of freeness in the next definition comes as a non-commutative analogue of the classical notion of independence for random variables.

Definition 3.1.2 Let $\mathcal{A}$ be a complex algebra with unit I, and let $\tau : \mathcal{A} \to \mathbf{C}$ be a linear functional, normalized by $\tau(I) = 1$. Consider two elements $a, b \in \mathcal{A}$, and denote their moments by $\tau(a^n) = \alpha_n$, $\tau(b^n) = \beta_n$, $n \geq 1$. Let us call *alternating product* based on a and b a (non-void) product of factors from $(a^n - \alpha_n I)_{n=1}^{\infty} \cup (b^n - \beta_n I)_{n=1}^{\infty}$, such that: for every factor coming from $(a^n - \alpha_n I)_{n=1}^{\infty}$, its immediate neighbors in the product are from $(b^n - \beta_n I)_{n=1}^{\infty}$, and vice-versa—the immediate neighbors of every factor coming from $(b^n - \beta_n I)_{n=1}^{\infty}$ are from $(a^n - \alpha_n I)_{n=1}^{\infty}$. (For instance, $(a^3 - \alpha_3 I)(b - \beta_1 I)(a^4 - \alpha_4 I)$ and $(b^2 - \beta_2 I)(a^5 - \alpha_5 I)(b - \beta_1 I)(a^2 - \alpha_2 I)$ are examples of alternating products.) The elements $a, b \in \mathcal{A}$ are called *free* with respect to τ if $\tau(p) = 0$ for every alternating product p based on a and b.

Remark 3.1.3 It is important to note that (in the above notations): if $a, b \in \mathcal{A}$ are free with respect to τ, then the mixed moments of a and b (in the sense of 3.1.1) can be calculated in terms of the individual moments $\tau(a^n) = \alpha_n$ and $\tau(b^n) = \beta_n$, $n \geq 1$. For the sake of keeping the notations simple, we will only show how the calculation goes in a particular case; it will be clear, however, that the same method would work for an arbitrary mixed moment.

Let us assume, for instance, that our goal is to calculate $\tau(abab)$ (knowing that a and b are free). We start from the equality

$$\tau((a - \alpha_1 I)(b - \beta_1 I)(a - \alpha_1 I)(b - \beta_1 I)) = 0 \tag{3.1}$$

(given by 3.1.2), and we expand the product $(a-\alpha_1 I)(b-\beta_1 I)(a-\alpha_1 I)(b-\beta_1 I)$ as a sum of 16 terms, arriving to

$$\begin{aligned}\tau(abab) = & \alpha_1\tau(bab) + \beta_1\tau(a^2b) + \alpha_1\tau(ab^2) + \beta_1\tau(aba)\\ & -\alpha_1\beta_1\tau(ab) - \alpha_1^2\tau(b^2) - \alpha_1\beta_1\tau(ba)\\ & -\beta_1\alpha_1\tau(ab) - \beta_1^2\tau(a^2) - \alpha_1\beta_1\tau(ab)\\ & +\alpha_1\beta_1\alpha_1\tau(b) + \alpha_1\beta_1^2\tau(a) + \alpha_1^2\beta_1\tau(b) + \beta_1\alpha_1\beta_1\tau(a)\\ & -\alpha_1\beta_1\alpha_1\beta_1\tau(I) \qquad (3.2)\\ = & \alpha_1\tau(bab) + \beta_1\tau(a^2b) + \alpha_1\tau(ab^2) + \beta_1\tau(aba)\\ & -\alpha_1\beta_1[3\tau(ab) + \tau(ba)]\\ & +3\alpha_1^2\beta_1^2 - \alpha_1^2\beta_2 - \alpha_2\beta_1^2 \qquad (3.3)\end{aligned}$$

((3.3) is obtained from (3.2) by replacing $\tau(I)=1,\ \tau(a)=\alpha_1,\ \tau(a^2)=\alpha_2,\ \tau(b)=\beta_1,$ $\tau(b^2)=\beta_2$, and by collecting terms). Thus, the expansion of (3.1) has reduced the calculation of $\tau(abab)$ to the one of some other mixed moments, of strictly smaller degree. By continuing the same procedure (i.e., replacing $\tau(bab)$ from the expansion of $\tau((b-\beta_1 I)(a-\alpha_1 I)(b-\beta_1 I))=0$, replacing $\tau(a^2b)$ from the expansion of $\tau((a^2-\alpha_2 I)(b-\beta_1 I))=0$, etc) it is clear that we must come in the end to an expression of $\tau(abab)$ only in terms of the α's and the β's. If one effectively does all the calculations, it will turn out that (after most of the terms finish by canceling out) the final expression for $\tau(abab)$ is

$$\tau(abab) \ = \ \alpha_2\beta_1^2 + \alpha_1^2\beta_2 - \alpha_1^2\beta_1^2. \qquad (3.4)$$

The result of Voiculescu [18] that we want to discuss is addressing the following

Problem 3.1.4 Given $\mathcal{A}$, τ as in 3.1.1, 3.1.2, and given two elements $a, b \in \mathcal{A}$ that are free with respect to τ. Describe the sequence of moments $(\tau(c^n))_{n=0}^{\infty}$ of the product $c = ab \in \mathcal{A}$ in terms of the sequences of moments $(\tau(a^n))_{n=0}^{\infty}$ and $(\tau(b^n))_{n=0}^{\infty}$ of a and b.

Of course, $\tau(c^n) = \tau(\underbrace{abab\cdots ab}_{2n \text{ factors}})$ is really a mixed moment of a and b with respect to τ, hence (as remarked in 3.1.3), we know for sure that it can be expressed in terms of the individual moments $\tau(a^n)$ and $\tau(b^n)$, $n \geq 1$. For instance, for $n = 1$ it is immediate that

$$\tau(c) \ = \ \tau(ab) \ = \ \tau(a)\tau(b), \qquad (3.5)$$

while for $n = 2$ the example calculated in 3.1.3 shows that

$$\tau(c^2) \ = \ \tau(a^2)\tau(b)^2 + \tau(a)^2\tau(b^2) - \tau(a)^2\tau(b)^2; \qquad (3.6)$$

but unfortunately, when n increases the method presented in 3.1.3 soon becomes difficult to use (even for mere theoretical purposes), because of the very large number of mixed moments involved in the calculation. What the problem in 3.1.4 really asks for is a more direct way of obtaining the moments of c.

In [18, Theorem 2.6], a solution for the problem in 3.1.4 is given in the case when $\tau(a) \neq 0 \neq \tau(b)$. Without much loss of generality, we will assume here that $\tau(a) = \tau(b) = 1$ (the case "$\tau(a) \neq 0 \neq \tau(b)$" is reduced to this by replacing a with $\frac{1}{\tau(a)}a$ and b with $\frac{1}{\tau(b)}b$).

Theorem 3.1.5 ([18], Theorem 2.6) *Let $\mathcal{A}$ be a complex algebra with unit I, and let $\tau : \mathcal{A} \to \mathbf{C}$ be a linear functional, normalized by $\tau(I) = 1$. To an element $a \in \mathcal{A}$ having $\tau(a) = 1$ we attach a formal power series S_a ("the S-transform of a with respect to τ") in the following way: if χ denotes the inverse under composition of the series $\sum_{n=1}^{\infty} \tau(a^n)z^n$, then*

$$S_a(z) \;=\; \chi(z)z^{-1}(1+z). \tag{3.7}$$

Now, consider two elements $a, b \in \mathcal{A}$ that are free with respect to τ, and such that $\tau(a) = \tau(b) = 1$, and denote $ab = c$ (note that we also have $\tau(c) = 1$, by (3.5)). Then

$$S_c(z) \;=\; S_a(z)S_b(z). \tag{3.8}$$

In other words, the moments of the product $c = ab$ are calculated as follows: we take the S-transforms S_a and S_b after the recipe in (3.7), we multiply them together to obtain S_c, and then the series $\sum_{n=1}^{\infty} \tau(c^n)z^n$ is found as the inverse under composition for $zS_c(z)(1+z)^{-1}$.

It should be mentioned that the proof given to Theorem 3.1.5 in [18] is not easy, and goes by studying the exponential map of a certain infinite dimensional Abelian Lie group $(\Sigma, \boxed{\times})$, which is in some sense the universal object containing the information about multiplication of free random variables. Our goal in what follows is to present an alternative proof which is entirely combinatorial, based on Theorem 1.6 above.

3.2. The line of the combinatorial proof of Theorem 3.1.5

that we will present is described as follows. Consider—and fix for the rest of the section—a complex algebra $\mathcal{A}$ with unit I, and a linear functional $\tau : \mathcal{A} \to \mathbf{C}$, normalized by $\tau(I) = 1$. For every $a \in \mathcal{A}$ we denote by h_a the unique multiplicative function on non-crossing partitions (in the sense of Section 1.4) which has:

$$h_a([0_n, 1_n]) = \tau(a^n), \quad n \geq 1; \tag{3.9}$$

moreover, we will use the notation

$$f_a \stackrel{\text{def}}{=} h_a \star \mu, \quad a \in \mathcal{A}, \tag{3.10}$$

where h_a is as in (3.9) and μ is the Moebius function on non-crossing partitions, as in Remark 2.5. We will show that:

1° if $a, b \in \mathcal{A}$ are free with respect to τ, then

$$f_{ab} \;=\; f_a \star f_b; \tag{3.11}$$

2° for every $a \in \mathcal{A}$ such that $\tau(a) = 1$, we have

$$S_a \;=\; \mathcal{F}(f_a) \tag{3.12}$$

(where S_a is as in (3.7) of 3.1.5, and $\mathcal{F}(\cdot)$ is the Fourier transform considered in the Theorem 1.6; note that, due to the condition $\tau(a) = 1$, we have $h_a \in \mathcal{M}_1$, hence $f_a = h_a \star \mu$ is indeed in the domain of $\mathcal{F}$).

The proof of Theorem 3.1.5 is immediately obtained from the assertions in (3.11) and (3.12), since for every $a, b \in \mathcal{A}$ that are free with respect to τ and with $\tau(a) = \tau(b) = 1$, we will have:

$$S_{ab} \overset{(3.12)}{=} \mathcal{F}(f_{ab}) \overset{(3.11)}{=} \mathcal{F}(f_a \star f_b) \overset{(1.15)}{=} \mathcal{F}(f_a)\mathcal{F}(f_b) \overset{(3.12)}{=} S_a S_b.$$

The proofs of the assertions in (3.11) and (3.12) will be made in the Sections 3.5 and 3.6, respectively.

Remark 3.3 We take the occasion to point out here that the multiplicative function f_a considered in 3.2 is an important object in the theory of "free probability". The generating function

$$R_a(z) \overset{\text{def}}{=} \frac{1}{z}\varphi_{f_a}(z) = \sum_{n=1}^{\infty} f_a([0_n, 1_n])z^{n-1} \tag{3.13}$$

was first considered, via an approach involving Toeplitz operators, in the work [16, 17] of Voiculescu, and is called "the R-transform of the distribution of a". The combinatorial facet of f_a (which appears in 3.2) was discovered by Speicher in [13], where the numbers $(f_a([0_n, 1_n]))_{n=1}^{\infty}$ are studied as "the free cumulants of a with respect to the functional τ". The important property of the R-transform which is proved in [16, 17, 13] is that

$$R_{a+b}(z) = R_a(z) + R_b(z) \tag{3.14}$$

whenever a and b are free with respect to τ. In other words, the R-transform can be used for the problem of addition of free elements in the same way in which the S-transform is used to handle the problem of multiplication. The "analytical" formula for the R-transform proved in [17, 13] can be written as: $A(zB(z)) = B(z)$, where $A(z) \overset{\text{def}}{=} 1 + zR_a(z)$ and $B(z) = \sum_{n=0}^{\infty} \tau(a^n)z^n$ (see the Theorem in Section 3 of [13], or the equivalent derivation in the Remark 2.5 of the present paper); of course, Eq. (3.10) is itself providing a "lattice-theoretic" approach to the R-transform.

We now go to the proofs of the assertions made in (3.11) and (3.12). For (3.11) we will need a particular case of a result in [13], which shows how to express mixed moments of a pair of free elements a, b as summations over non-crossing partitions. This is stated as follows:

Lemma 3.4 *For every $n \geq 1$, let us denote by $NC_{\text{sep}}(2n)$ the set of non-crossing partitions $\pi \in NC(2n)$ that separate the even numbers from the odd ones, i.e., which have the property that each block of π is contained either in $\{1, 3, 5, \ldots, 2n-1\}$ or in $\{2, 4, 6, \ldots, 2n\}$. For $\pi \in NC_{\text{sep}}(2n)$, writing $\pi = \{\underbrace{A_1, \ldots, A_h}_{\text{odd}}, \underbrace{B_1, \ldots, B_k}_{\text{even}}\}$ will mean that the blocks*

$A_1, \dots, A_h$ of π have union $\{1, 3, 5, \dots, 2n-1\}$, *while the blocks $B_1, \dots, B_k$ have union* $\{2, 4, 6, \dots, 2n\}$.

Now, in the notations of 3.2, *consider two elements $a, b \in \mathcal{A}$, free with respect to τ. Let f_a, f_b be the multiplicative functions defined as in* (3.10). *Then for every $n \geq 1$ we have*

$$\tau(\underbrace{abab\cdots ab}_{2n \text{ factors}}) = \sum_{\substack{\pi \in NC_{\text{sep}}(2n) \\ \pi = \{\underbrace{A_1,\dots,A_h}_{\text{odd}}, \underbrace{B_1,\dots,B_k}_{\text{even}}\}}} \prod_{i=1}^{h} f_a\big([0_{|A_i|}, 1_{|A_i|}]\big) \prod_{j=1}^{k} f_b\big([0_{|B_j|}, 1_{|B_j|}]\big). \tag{3.15}$$

For the proof of the Lemma 3.4 we refer the reader to [13] (see Proposition 2 in Section 4, and the three un-numbered equations preceding it, in the cited paper).

3.5. The proof of the Assertion (3.11)

We will show that $h_{ab} = f_a \star h_b$ (where the notations are as in 3.2); this will imply (3.11) by taking convolution with μ on the right:

$$f_{ab} = h_{ab} \star \mu = f_a \star h_b \star \mu = f_a \star f_b.$$

Now, for every $n \geq 1$, $h_{ab}([0_n, 1_n]) \overset{(3.9)}{=} \tau((ab)^n)$ is exactly the quantity appearing in (3.15). We will transform the right-hand side of (3.15) by using the following remark (borrowed from [14], Section 3.4). Let $\pi = \{\underbrace{A_1, \dots, A_h}_{\text{odd}}, \underbrace{B_1, \dots, B_k}_{\text{even}}\}$ be a partition in $NC_{\text{sep}}(2n)$, and let us denote by π' and π'' respectively the two partitions in $NC(n)$ that are identified with $\{A_1, \dots, A_h\}$ and $\{B_1, \dots, B_k\}$ via the order-preserving bijections $\{1, 3, \dots, 2n-1\} \to \{1, \dots, n\}$ and $\{2, 4, \dots, 2n\} \to \{1, \dots, n\}$; then, by the very definition of the complementation map K (reviewed in Section 1.2), we have that $\pi'' \leq K(\pi')$. Conversely, it is clear that every pair $\pi', \pi'' \in NC(n)$, satisfying $\pi'' \leq K(\pi')$, comes from a $\pi \in NC_{\text{sep}}(2n)$ in this way. Note moreover that if $\pi', \pi'' \in NC(n)$ with $\pi'' \leq K(\pi')$ are corresponding to $\pi \in NC_{\text{sep}}(2n)$, then the products $\prod_{i=1}^{h} f_a([0_{|A_i|}, 1_{|A_i|}])$ and $\prod_{j=1}^{k} f_b([0_{|B_j|}, 1_{|B_j|}])$ appearing in (3.15) are nothing but $f_a([0_n, \pi'])$ and $f_b([0_n, \pi''])$, respectively. This argument shows that we have:

$$h_{ab}([0_n, 1_n]) = \tau(\underbrace{abab\cdots ab}_{2n \text{ factors}}) = \sum_{\substack{\pi', \pi'' \in NC(n) \\ \text{such that } \pi' \leq K(\pi'')}} f_a([0_n, \pi'])\, f_b([0_n, \pi'']), \tag{3.16}$$

and we can continue (3.16) with:

$$= \sum_{\pi' \in NC(n)} f_a([0_n, \pi']) \left(\sum_{0 \leq \pi'' \leq K(\pi')} f_b([0_n, \pi'']) \right)$$
$$= \sum_{\pi' \in NC(n)} f_a([0_n, \pi'])(f_b \star \zeta)([0_n, K(\pi')])$$

$$= \sum_{\pi' \in NC(n)} f_a([0_n, \pi'])h_b([0_n, K(\pi')]) \quad \text{(because } f_b \star \zeta = h_b)$$
$$= \sum_{\pi' \in NC(n)} f_a([0_n, \pi'])h_b([\pi', 1_n)]) \quad \text{(by Step 2 in Section 1.3)}$$
$$= (f_a \star h_b)([0_n, 1_n]).$$

We have thus obtained that h_{ab} and $f_a \star h_b$ agree on an arbitrary interval $[0_n, 1_n]$, $n \geq 1$, and this concludes the proof. □

3.6. The proof of the Assertion (3.12)

We use the notations introduced in 3.1.5 and 3.2. Since the multiplicative function h_a has $h_a([0_n, 1_n]) = \tau(a^n)$, $n \geq 1$, the series $\sum_{n=1}^{\infty} \tau(a^n)z^n$ appearing in 3.1.5 is just $\varphi_{h_a}(z)$ (where φ_h, $h \in \mathcal{M}_1$, is as in Theorem 1.6). Thus the S-transform of a is $S_a(z) = \varphi_{h_a}^{\langle -1 \rangle}(z) \cdot z^{-1}(1+z)$; on the other hand $[\mathcal{F}(f_a)](z) \overset{\text{def}}{=} z^{-1}\varphi_{f_a}^{\langle -1 \rangle}(z)$ (Theorem 1.6), so what we need to show is

$$(1+z)\varphi_{h_a}^{\langle -1 \rangle}(z) = \varphi_{f_a}^{\langle -1 \rangle}(z). \tag{3.17}$$

But (3.17) follows from (2.18), because $f_a = h_a \star \mu$. □

The argument in 3.6 completes the combinatorial proof of the Theorem 3.1.5 of Voiculescu.

We would like to conclude by signaling some developments related to our work here, which have occurred recently, or/and were brought to our attention by the referees.

(a) An elegant analytical proof for the multiplicativity of the S-transform (Theorem 3.1.5) was given by Haagerup [5], via calculations involving the resolvent functions of certain Toeplitz operators.

(b) In [2], Biane has given another proof for the (main) Theorem 1.6 of the paper, in an equivalent form where it appears as a result in the harmonic analysis of the group $\mathcal{S}_\infty$ (of finitely supported permutations of $\mathbf{N}$). The counterpart for the convolution of multiplicative functions on non-crossing partitions is provided, in Biane's approach, by the restricted convolution of "multiplicative central functions on $\mathcal{S}_\infty$", and the combinatorial Fourier transform is viewed as acting on the latter set of functions.

(c) The considerations involving the block-structure of a non-crossing partition, considered at the same time with its Kreweras complement—as in Eqs. (1.8), (1.9), and (2.1), for instance—are related to the study of the minimal factorizations of a full cycle in a (finite) symmetric group (see e.g., [1], Theorem 1). As it was pointed out to us by one of the referees, this yields a natural connection between the topics of the present paper and earlier work of Jackson [6], Goulden and Jackson [4], where factorization problems in the symmetric group were studied, by using methods from the theory of the symmetric functions. The study of this connection might open an interesting line of research, but its possible implications are not so clear at present, and await further investigation.

(d) The statement made in (3.11), together with the interpretation (see Remark 3.3) of the multiplicative function f_a appearing in (3.11) as a relative of the R-transform R_a, lead to the possibility of studying the multiplication of free elements by using the R-transform (instead of the S-transform). In a subsequent paper [9] we have shown how this possibility can be exploited in a "multivariable" setting, in order to obtain interesting free probabilistic applications (to compressions with free projections, and to a realization of the free analogue for the Poisson process). Oddly enough, the problem of extending to several variables the S-transform itself (or the combinatorial Fourier transform $\mathcal{F}$ considered in this paper) appears to be sensibly harder than the multivariable R-transform approach, and we weren't able to solve it up to present.

Note

1. $\mathcal{M}$ is not closed under addition or multiplication by scalars, it is merely a multiplicative subsemigroup of the large incidence algebra $\mathcal{L}$.

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