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Distance-Regular Graphs with Strongly Regular Subconstituents

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Abstract. In [3] Cameron et al. classified strongly regular graphs with strongly regular subconstituents. Here we prove a theorem which implies that distance-regular graphs with strongly regular subconstituents are precisely the Taylor graphs and graphs with $a_1 = 0$ and $a_i \in \{0, 1\}$ for $i = 2, \ldots, d$.

Keywords: distance-regular graph, strongly regular graph, association scheme

1. Introduction

Let Γ be a connected graph without loops and multiple edges, $d = d(\Gamma)$ be the diameter of Γ, $V(\Gamma)$ be the set of vertices and $v = |V(\Gamma)|$. For $i = 1, \ldots, d$ let $\Gamma_i(u)$ be the set of vertices at distance i from u ('subconstituent') and $k_i = |\Gamma_i(u)|$. We use the same notation $\Gamma_i(u)$ for the subgraph of Γ induced by the vertices in $\Gamma_i(u)$. Distance between vertices u and v in Γ will be denoted by $\partial(u, v)$.

Recall that a connected graph is said to be *distance regular* if it is regular and for each $i = 1, \ldots, d$ the numbers $a_i = |\Gamma_i(u) \cap \Gamma_1(v)|$, $b_i = |\Gamma_{i+1}(u) \cap \Gamma_1(v)|$, $c_i = |\Gamma_{i-1}(u) \cap \Gamma_1(v)|$ are independent of the particular choice of u and v with $v \in \Gamma_i(u)$. It is well known that in this case all numbers $p^l_{i,j} = |\Gamma_i(u) \cap \Gamma_j(v)|$ do not depend on the choice of the pair u, v with $v \in \Gamma_l(u)$. The valency of a distance regular graph is $k = k_1 = b_0$.

A regular graph is called *strongly regular* if there exist nonnegative integers λ and μ such that $|\Gamma_1(u) \cap \Gamma_1(v)| = \lambda$ or μ depending on whether $\{u, v\}$ is an edge or a non-edge. Connected strongly regular graph is a distance-regular graph of diameter 2; disconnected strongly regular graph is a disjoint union of equal cliques. For a strongly regular graph we use notation $\mathrm{srg}(v, k, \lambda, \mu)$.

A distance-regular graph in which $x, y \in \Gamma_d(u)$ with $x \neq y$ implies $x \in \Gamma_d(y)$ is called antipodal. Distance-regular graph Γ of diameter 3 such that $|\Gamma_3(u)| = 1$ is called a *Taylor graph*; it is an antipodal 2-cover of a complete graph on $k_1 + 1$ vertices. In a Taylor graph $\Gamma_1(u)$ and $\Gamma_2(u)$ are two copies of a strongly regular graph Δ with parameters $k = 2\mu$ (these are k and μ of Δ). Vice versa, given a strongly regular graph Δ with $k = 2\mu$ one can construct a Taylor graph with subconstituents isomorphic to Δ (on Taylor graphs see [1, 4–6]).

We prove the following theorem.

Theorem 1.1 *Let Γ be a distance-regular graph of diameter $d \geq 3$. Suppose that for some vertex u for every i the subgraph $\Gamma_i(u)$ is either a disjoint union of cliques or else all vertices of $\Gamma_i(u)$ are at distance at most 2 from each other inside Γ (not necessarily so inside $\Gamma_i(u)$).*

If $\Gamma_1(v)$ is a (possibly disconnected) strongly-regular graph for every vertex v then one of the following holds

(i) $a_1 = 0$ *and* $a_i \leq 1$ *for* $i = 2, \ldots, d$ *or*
(ii) Γ *is a Taylor graph.*

In terms of structure constants the first part of the hypothesis says that $p^i_{i,j} = 0$ for all $j \geq 3$ whenever $\Gamma_i(u)$ is not a disjoint union of cliques.

Note that if Γ is a distance-regular graph of diameter at least 3 in which for every vertex u the subconstituents $\Gamma_1(u), \ldots, \Gamma_d(u)$ are strongly regular (possibly disconnected) then Γ obviously satisfies hypothesis of Theorem 1.1. In this sense our theorem is similar to the result of Cameron et al. [3] who classified strongly regular graphs with strongly regular subconstituents.

Note also that all graphs in (i) and (ii) clearly satisfy the assumptions. In (i) we have all bipartite distance-regular graphs (case $a_i = 0$ for all i). Classification of Taylor graphs and bipartite distance-regular graphs are well known open problems. If $a_i = 1$ for some i then $\Gamma_i(u)$ is a matching. Besides odd cycles there are three such graphs in (i) known to the author: the dodecahedron, the Coxeter graph and the Biggs-Smith graph. Their parameters are $v = 20$, $k = 3$ and girth 5 for the dodecahedron, $v = 28$, $k = 3$ and girth 7 for the Coxeter graph and $v = 102$, $k = 3$ and girth 9 for the Biggs-Smith graph. We were not able to show completeness of this list nor to find other examples.

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2. Proof of Theorem 1.1

We prove Theorem 1.1 by way of contradiction. Using Proposition 2.2, Lemma 2.3 and Lemma 2.4 we show first that if a counter-example exists it has to have parameters $a_1 = 0$, $a_i \leq 1$ for $i = 2, \ldots, d-1$ and $a_d \geq 2$. Then we eliminate this possibility with the help of Lemma 2.5. We give complete proofs of all results for the convinience of the reader although some of them are known or are quite easy to prove.

The following lemma is needed in the proof of Proposition 2.2.

Lemma 2.1 *Let Γ be a distance-regular graph of diameter d. If $p^{j+1}_{2,j} = 0$ for some $1 \leq j < d$, then $a_1 = a_j$ and $a_{j+1} = 0$.*

Proof: By Lemma 4.1.7 of [1] $c_2 p^{j+1}_{2,j} = c_{j+1}(a_j + a_{j+1} - a_1)$. If $p^{j+1}_{2,j} = 0$ then $a_j + a_{j+1} = a_1$. If $a_1 = 0$ the assertion is obvious.

On the other hand, if $a_1 > 0$ we can apply Proposition 5.5.1(i) of [1] which says that $2a_i \geq a_1 + 1$ for $i = 1, \ldots, d-1$ to obtain $j = d-1$. Thus we have $p^d_{2,d-1} = 0$ and $a_{d-1} + a_d = a_1$.

Let u be a vertex of Γ and pick $v \in \Gamma_d(u)$. Since $p^d_{2,d-1} = 0$, every vertex of $\Gamma_1(v) \cap \Gamma_{d-1}(u)$ is adjacent to every vertex of $\Gamma_1(v) \cap \Gamma_d(u)$. If $\Gamma_1(v) \cap \Gamma_d(u)$ is non-empty, then complement of $\Gamma_1(v)$ is disconnected. Then Lemma 1.1.7 of [1] implies that Γ is complete multipartite, in particular $a_d = 0$, a contradiction. Hence $\Gamma_d(u) \cap \Gamma_1(v)$ is empty, that is $a_d = 0$. Then $a_1 = a_{d-1}$. (Argument of this paragraph follows the proof of Proposition 5.5.1(ii) of [1].) □

Proposition 2.2 *Let Γ be a distance-regular graph of diameter d. Suppose that for some i $p^i_{i,j} = 0$ for all $j \geq 3$. Then one of the following holds*

(i) $i = 0, 1$ *or* d,

(ii) $i = d-1$ *and* Γ *is an antipodal* 2*-cover, that is* $k_d = 1$, *or*

(iii) $i = 2, d = 3$ *and* Γ *is bipartite.*

Proof: Suppose $2 \leq i \leq d-2$. Let u, v and w be vertices of Γ such that $\partial(u, w) = i+2$, $\partial(u, v) = i$ and $\partial(v, w) = 2$. Then one can find a vertex x such that $\partial(v, x) = i$ and $\partial(w, x) = i-2$. But then $\partial(x, v) = \partial(u, v) = i$ and $\partial(x, u) \geq \partial(u, w) - \partial(x, w) = 4$. This contradicts the hypothesis. Hence $i = 0, 1, d-1$ or d.

It remains only to consider the case $i = d-1$. First we claim that $p^{d-1}_{d,j} \neq 0$ implies $j = 1$ or $d = j = 3$, that is for every u all vertices of $\Gamma_d(u)$ are at distance 1 or 3 from the vertices of $\Gamma_{d-1}(u)$ and distance 3 can occur only when $d = 3$.

Suppose there are vertices u, v and w such that $\partial(u, v) = d-1$, $\partial(u, w) = d$ and $\partial(v, w) = j \geq 2$. We need to show that $j = d = 3$. Pick a vertex x such that $\partial(v, x) = d-1$ and $\partial(w, x) = |d-1-j|$.

If $j \leq d-1$ then $\partial(x, u) \geq \partial(u, w) - \partial(x, w) = j+1 \geq 3$, a contradiction. If $j = d$ then $\partial(u, x) \geq d-1$. Hence, we must have $d-1 \leq 2$ (i.e., $d = 3$) to avoid a contradiction. Thus we have proved our claim.

Suppose $d \geq 4$. In this case, by the claim we just proved, if u is a vertex of Γ, then all vertices of $\Gamma_d(u)$ are at distance 1 from the vertices of $\Gamma_{d-1}(u)$. Let v be a vertex of $\Gamma_d(u)$. Then $\Gamma_d(u) \subseteq \{v\} \cup \Gamma_1(v) \cup \Gamma_2(v)$, $\Gamma_{d-1}(u) \subseteq \Gamma_1(v)$, $\Gamma_{d-2}(u) \subseteq \Gamma_2(v), \ldots, \Gamma_1(u) \subseteq \Gamma_{d-1}(v)$. So u is the only vertex of Γ at distance d from v. Hence $k_d = 1$ and Γ is an antipodal 2-cover.

Now suppose $d = 3$. Then $i = 2$. By the claim above $p^3_{2,2} = 0$. Applying Lemma 2.1 we obtain $a_3 = 0$ and $a_1 = a_2$.

Let u be a vertex of Γ and $v \in \Gamma_d(u)$. Since $p^3_{2,2} = 0$, we have $\Gamma_2(v) \subseteq \Gamma_1(u) \cup \Gamma_3(u)$. Suppose first that $\Gamma_2(v) \cap \Gamma_3(u) \neq \emptyset$. Pick a vertex $w \in \Gamma_2(v) \cap \Gamma_3(u)$. As $a_3 = 0$, this w has no neighbours in $\Gamma_2(v) \cap \Gamma_3(u)$, hence no neighbours in $\Gamma_2(v)$ at all. This implies that $a_2 = 0$ and, as $a_1 = a_2$, also $a_1 = 0$. Hence Γ is bipartite in this case.

If $\Gamma_2(v) \cap \Gamma_3(u) = \emptyset$ then $\Gamma_2(v) \subseteq \Gamma_1(u)$. Since every vertex of $\Gamma_3(v)$ lies at distance 1 from some vertex of $\Gamma_2(v)$, we obtain $\Gamma_3(v) \cap \Gamma_3(u) = \emptyset$. Since $\Gamma_1(v) \cap \Gamma_3(u)$ is also empty (as $a_3 = 0$), we have in fact $\Gamma_3(u) = \{v\}$. This means that Γ is an antipodal 2-cover. □

Lemma 2.3 *Let Γ be a distance-regular graph of diameter $d \geq 3$ and suppose that for every v subgraph $\Gamma_1(v)$ is strongly regular.*

If there exists u such that $\Gamma_2(u)$ is a disjoint union of at least two cliques then $\Gamma_1(v)$ is a disjoint union of cliques for every v.

Proof: Suppose first that there is a vertex x such that $\Gamma_1(x)$ is a disjoint union of cliques. Then connectedness of Γ and our assumption that $\Gamma_1(v)$ is strongly regular for every v imply that $\Gamma_1(v)$ is a disjoint union of cliques for every v.

Assume now that $\Gamma_1(v)$ is connected for every v. We are going to show that this assumption leads to a contradiction.

Let u be as in the hypothesis, $v \in \Gamma_1(u)$ and $\Delta = \Gamma_1(v)$. Since Γ has diameter at least 3, it can not be complete multipartite. Therefore Lemma 1.1.7 of [1] implies that the complement of Δ, denote it Φ, is connected. Thus Φ is a connected strongly regular graph and we can apply to it Lemma 1.1.7 of [1] to obtain that either (1) Φ is complete multipartite (in this case Δ is a disjoint union of cliques) or (2) $\Phi_1(u)$ is coconnected (in this case $\Delta_2(u)$ is connected).

Since we know that Δ is connected, (2) is the only possibility. Let $S_1, \ldots, S_n$ be the complete set of cliques of $\Gamma_2(u)$ and suppose that $\Gamma_1(v) \cap S_i \neq \emptyset$. Since $\Delta_2(u) = \Gamma_1(v) \cap \Gamma_2(u)$ is connected, we obtain that $\Gamma_1(v) \cap \Gamma_2(u) \subseteq S_i$. This shows that we can partition $\Gamma_1(u)$ into disjoint subsets $C_1, \ldots, C_n$ such that $x \in C_i$ if and only if $\Gamma_1(x) \cap \Gamma_2(u) \subseteq S_i$. Note that every C_i is non-empty.

We claim that $C_1, \ldots, C_n$ disconnect $\Gamma_1(u)$ contrary to our assumption. Indeed, suppose there are $x \in C_i$ and $y \in C_j$, $i \neq j$, such that $\{x, y\}$ is an edge and let $\Sigma = \Gamma_1(x)$. As Σ is a connected strongly regular graph, $\Sigma_2(u) \cap \Sigma_1(y) = \Gamma_2(u) \cap \Gamma_1(y) \cap \Gamma_1(x)$ is non-empty. This contradicts our choice of x and y. Hence $C_1, \ldots, C_n$ disconnect $\Gamma_1(u)$. This is a contradiction, since we assumed that $\Gamma_1(v)$ is connected for every v. □

Lemma 2.4 *Let Γ be a distance-regular graph with $a_1 > 0$ and suppose that for every v subgraph $\Gamma_1(v)$ is a disjoint union of cliques. Let r be the index such that $(1 = c_1, a_1, b_1) = \cdots = (c_r, a_r, b_r) \neq (c_{r+1}, a_{r+1}, b_{r+1})$. Then $\Gamma_1(v), \ldots, \Gamma_r(v)$ are disjoint unions of cliques and $\Gamma_{r+1}(v)$ is not a disjoint union of cliques for every v.*

Proof: Let v be any vertex of Γ. We use induction to show that $\Gamma_1(v), \ldots, \Gamma_r(v)$ are disjoint unions of cliques. The statement is obvious if $i = 1$. So assume that $\Gamma_1(v), \ldots, \Gamma_n(v)$ are disjoint unions of cliques and pick $x \in \Gamma_{n+1}(v)$. Let $y \in \Gamma_1(x) \cap \Gamma_n(v)$, $z \in \Gamma_1(y) \cap \Gamma_{n-1}(v)$ and $S = \Gamma_1(y) \cap \Gamma_n(v)$. Since $a_1 > 0$, subgraph $\Gamma_1(y)$ consists of cliques of cardinality at least 2. Therefore z and S must form a clique of $\Gamma_1(y)$. All other cliques of $\Gamma_1(y)$ have to lie entirely in $\Gamma_{n+1}(v)$. In particular x lies in such a clique T. Since $|T| = a_1 + 1$ and $a_{n+1} = a_1$, we obtain that T is in fact a connected component of $\Gamma_{n+1}(v)$. Since x was arbitrary vertex of $\Gamma_{n+1}(v)$, we have proved that $\Gamma_{n+1}(v)$ is a disjoint union of cliques.

We need to show now that $\Gamma_{r+1}(v)$ is not a disjoint union of cliques. Assume the contrary and let $x \in \Gamma_{r+1}(v)$ and $y \in \Gamma_1(x) \cap \Gamma_r(v)$. Same argument as above shows that x lies inside a complete subgraph $T \subseteq \Gamma_1(y) \cap \Gamma_{r+1}(v)$, $|T| = a_1 + 1$. Since $a_1 > 0$, we can find $x' \in T$ such that $x' \neq x$. Then $\Gamma_1(x) \cap \Gamma_1(x') \supseteq \{y\} \cup (T - \{x, x'\})$. Since

$|\Gamma_1(x) \cap \Gamma_1(x')| = a_1$ and $|\{y\} \cup (T - \{x, x'\})| = a_1$, we obtain that T is in fact a maximal clique of $\Gamma_{r+1}(v)$. This implies $a_{r+1} = a_1$.

Let now y' be any vertex of $\Gamma_1(x) \cap \Gamma_r(v)$. Then again same argument as in the first paragraph shows that y' is adjacent to all of T. If y' were different from y then the number of triangles on the edge $\{x, x'\}$ would be at least $|\{y\} \cup \{y'\} \cup (T - \{x, x'\})| = a_1 + 1$ which is impossible. Hence $y' = y$. Thus $c_{r+1} = 1 = c_1$. Then $b_{r+1} = k_1 - c_{r+1} - a_{r+1} = k_1 - c_1 - a_1 = b_1$, i.e., we obtain that $(c_{r+1}, a_{r+1}, b_{r+1}) = (c_1, a_1, b_1)$ contrary to the choice of r. Hence $\Gamma_{r+1}(v)$ is not a disjoint union of cliques. □

Lemma 2.5 *Let Γ be a distance-regular graph and suppose that there is an index i such that $a_i = 0$ but $a_{i+1} \neq 0$. Then $c_{i+1} \leq a_{i+1}$.*

Proof: Pick vertices u, v and v' such that $v, v' \in \Gamma_{i+1}(u)$ and $v' \in \Gamma_1(v)$. Let u' be a vertex in $\Gamma_1(u) \cap \Gamma_i(v')$. The distance between u' and v is not greater than $\partial(u', v') + \partial(v', v) = i + 1$ but not less than i, as $u' \in \Gamma_1(u)$ and $v \in \Gamma_{i+1}(u)$. If $\partial(u', v)$ were i then both v and v' would lie in $\Gamma_i(u')$ in contradiction with $a_i = 0$. Hence $\partial(u', v) = i + 1$.

Thus we have $\Gamma_1(u) \cap \Gamma_i(v') \subseteq \Gamma_{i+1}(v)$. This is equivalent to $\Gamma_1(u) \cap \Gamma_i(v') \subseteq \Gamma_1(u) \cap \Gamma_{i+1}(v)$. Since $\partial(v', u) = \partial(v, u) = i + 1$ the cardinality of the set on the left is c_{i+1} and cardinality of the set on the right is a_{i+1}. Hence $c_{i+1} \leq a_{i+1}$. □

Proof of Theorem 1.1: Let Γ be a counter-example. Throughout the proof vertex u is as in the hypothesis.

First observe that $\Gamma_2(u)$ is a disjoint union of cliques. Indeed, if $\Gamma_2(u)$ were not a disjoint union of more than one clique, then Proposition 2.2(ii) or (iii) would hold for Γ and in either case Γ would not be a counter-example. Lemma 2.3 implies now that for every v subgraph $\Gamma_1(v)$ is a disjoint union of cliques.

Next we are going to show that $a_1 = 0$. Assume the contrary, that is assume $a_1 > 0$. Then, as $\Gamma_1(v)$ is a disjoint union of cliques for every v, we can apply Lemma 2.4 to obtain that $c_1 = c_2 = \cdots = c_r = 1$, subgraphs $\Gamma_1(v), \ldots, \Gamma_r(v)$ are disjoint unions of cliques of size $a_1 + 1$ and $\Gamma_{r+1}(v)$ is not a disjoint union of cliques (here r is as in Lemma 2.4). Note that this holds for every v and that by the above paragraph $r \geq 2$.

Pick three vertices x, y and z in $\Gamma_{r+1}(u)$ such that $z \in \Gamma_2(x)$ and $y \in \Gamma_1(x) \cap \Gamma_1(z)$. Such a triple exists, as $\Gamma_{r+1}(u)$ is not a disjoint union of cliques. By hypothesis of the theorem $\Gamma_{r+1}(u) \subseteq \{x\} \cup \Gamma_1(x) \cup \Gamma_2(x)$. Recall that $\Gamma_1(x)$ and $\Gamma_2(x)$ are both disjoint unions of cliques and, moreover, all neighbours of z in $\Gamma_1(x) \cup \Gamma_2(x)$ lie in the clique through z and y (c.f. proof of Lemma 2.4). This implies that $\Gamma_1(z) \cap \Gamma_{r+1}(u) - \{y\} \subseteq \Gamma_1(y) \cap \Gamma_{r+1}(u) - \{z\}$. Since x is adjacent to y but not to z the inclusion is proper, contradiction with regularity of $\Gamma_{r+1}(u)$. Hence $a_1 = 0$.

Thus we have shown that $a_1 = 0$. In this case $a_i \leq 0$ whenever $\Gamma_i(u)$ is a disjoint union of cliques. So, if all $\Gamma_i(u)$, $i = 1, \ldots, d$, were disjoint unions of cliques, then part (i) of the theorem would hold for Γ. Therefore, as Γ is a counter-example, there must be some i such that $\Gamma_i(u)$ is not a disjoint union of cliques. By Proposition 2.2 $i = 1$, d or $d - 1$. We proved at the very beginning that $i \neq 1, 2$. If $i = d - 1$, $i > 2$, then, by Proposition 2.2, Γ is an antipodal 2-cover and $a_{d-1} = a_1$. This is a contradiction, since $a_1 = 0$ forces $\Gamma_{d-1}(u)$ to be a disjoint union of cliques. Hence $i = d$.

Thus the only possibility for a counter-example is $a_1 = 0$, $a_i \leq 1$ for $i = 2, \ldots, d-1$ and $a_d \geq 2$. We are going to show now that in fact $a_d = 2$.

Let v and w be vertices such that $\partial(u, v) = \partial(u, w) = d$ and $\partial(v, w) = 2$. Since $p^d_{d,3} = 0$, we obtain $a_d = |\Gamma_1(w) \cap \Gamma_d(u)| = |\Gamma_1(w) \cap \Gamma_d(u) \cap \Gamma_1(v)| + |\Gamma_1(w) \cap \Gamma_d(u) \cap \Gamma_2(v)| \leq c_2 + a_2$. Thus $a_d \leq c_2 + a_2$.

We have two possibilities, $a_2 = 0$ or $a_2 = 1$. If $a_2 = 0$, then $a_d \leq c_2$ which implies $c_2 \geq 2$. Let r be the first index such that $a_r \neq 0$. Then by Lemma 2.5 we have $2 \leq c_2 \leq c_r \leq a_r$ (the sequence $c_1, \ldots, c_d$ is non-decreasing for every distance-regular graph). Hence $r = d$. Combining $c_2 \leq c_d \leq a_d$ and $a_d \leq c_2$ we obtain $c_d = c_2$. This is impossible, as c_d must be strictly greater than c_2 once $c_2 > 1$ (see Theorem 5.4.1 of [1]).

Thus $a_2 = 1$. An application of Lemma 2.5 gives $c_2 = a_2 = 1$. Then $a_d = 2$.

We have shown that $c_2 = a_2 = 1$ and $a_d = 2$. In this case $\Gamma_d(u)$ is a disjoint union of circuits. Pick $x \in \Gamma_d(u)$ and let C be the unique circuit of $\Gamma_d(u)$ through x. Since $p^d_{d,j} = 0$ whenever $j \geq 3$, we have $\Gamma_d(u) \subseteq \{x\} \cup \Gamma_1(x) \cup \Gamma_2(x)$. Therefore $c_2 = a_2 = 1$ implies that C is a pentagon. Moreover, if $C' \neq C$ were another circuit in $\Gamma_d(u)$, it would have to lie entirely in $\Gamma_2(x)$ in contradiction with $a_2 = 1$. Hence $\Gamma_d(u) = C$ is a pentagon.

Counting in two ways all triples of vertices with distances $\{d, d, 1\}$ and $\{d, d, 2\}$ we obtain $k_1 p^1_{d,d} = k_d p^d_{1,d} = 10$ and $k_2 p^2_{d,d} = k_d p^d_{2,d} = 10$. This means that both k_1 and k_2 divide 10. Since $c_2 = 1$ and $a_1 = 0$, we obtain $k_2 = k_1(k_1 - 1)$. This leaves the only possibility, $k_1 = k_2 = 2$. In this case Γ is a circuit and cannot have $a_d = 2$. □

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