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Minimal Paths between Maximal Chains in Finite Rank Semimodular Lattices*

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Abstract. We study paths between maximal chains, or "flags," in finite rank semimodular lattices. Two flags are adjacent if they differ on at most one rank. A path is a sequence of flags in which consecutive flags are adjacent. We study the union of all flags on at least one minimum length path connecting two flags in the lattice. This is a subposet of the original lattice. If the lattice is modular, the subposet is equal to the sublattice generated by the flags. It is a distributive lattice which is determined by the "Jordan-Hölder permutation" between the flags. The minimal paths correspond to all reduced decompositions of this permutation. In a semimodular lattice, the subposet is not uniquely determined by the Jordan-Hölder permutation for the flags. However, it is a join sublattice of the distributive lattice corresponding to this permutation. It is semimodular, unlike the lattice generated by the two flags, which may not be ranked. The minimal paths correspond to some reduced decompositions of the permutation, though not necessarily all. We classify the possible lattices which can arise in this way, and characterize all possibilities for the set of shortest paths between two flags in a semimodular lattice.

Keywords: semimodular lattice, maximal chain, Jordan-Hölder permutation, reduced decomposition

1. Introduction

In this paper, we study relationships between maximal chains, or *flags*, in a finite rank semimodular lattice. We develop a generalization to semimodular lattices of the sublattice generated by two flags of a modular lattice. We consider the Jordan-Hölder function for two flags X and Y in a semimodular lattice as developed by Stanley in [11] and [12] and by Björner in [4], and show that this gives a permutation in the symmetric group S_n, where n is the rank of the lattice. It is commonly known that in a modular lattice, this permutation determines the lattice structure of the lattice generated by the flags X and Y, and that this lattice is a finite distributive lattice.

For semimodular lattices the situation is more complex. In Section 4, we give an example of a finite rank semimodular lattice in which two flags generate a sublattice which is not ranked; hence, it cannot be semimodular.

Our main object of study is a join sublattice of the original semimodular lattice and of the lattice generated by the two flags. It is a semimodular lattice which is related to the Jordan-Hölder permutation (though not determined by it). The lattice we obtain from the flags X and Y can be embedded as a join sublattice into the distributive lattice corresponding

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to the permutation when the underlying lattice is modular, and we classify the lattices which occur as one of these join sublattices.

We say two flags are i-*adjacent* if they agree except possibly at level i. This notion is related to the Jordan-Hölder permutation; X and Y are distinct i-adjacent flags if and only if the permutation is the adjacent transposition $r_i = (i\ i+1)$. When X and Y are arbitrary flags in the lattice, we define a *path from X to Y* as a sequence of flags beginning with X and ending with Y such that consecutive flags in the sequence are adjacent. We call the path a *minimal path from X to Y* if no path from X to Y has shorter length. We study the collection of minimal paths between two flags.

For modular lattices, it is known that the points on a flag on some minimal paths between X and Y are precisely those points in the sublattice generated by the flags. For semimodular lattices, the points on minimal X-Y paths are still in the sublattice generated by X and Y, but there may be points in the sublattice which are not on a minimal path. As we noted, the sublattice need not be ranked, whereas the subposet of points on reduced paths clearly is ranked, since every point in the subposet is on a flag of the original lattice that is contained in the subposet.

We contend that this subposet, the union of all flags on reduced paths from X to Y, is the natural semimodular analog of the sublattice generated by X and Y in a modular lattice. We show that it is a join sublattice of the distributive lattice which corresponds to the Jordan-Hölder permutation when the underlying lattice is modular. We also give examples to show that the sublattice generated by X and Y in a semimodular lattice is not as appropriate a generalization as one might expect.

The minimal paths can be represented by *reduced decompositions* of the Jordan-Hölder permutation, or expressions of this permutation as a minimal length product of adjacent transpositions. In a modular lattice, it is known that there is a one-to-one correspondence between minimal paths and reduced decompositions. We classify the collection of paths which can occur between two flags in a semimodular lattice by classifying the set of reduced decompositions which correspond to these paths.

Many of these results correspond to results of Abels in [1] and [2], though he approached these problems from a more geometric viewpoint. He considers semimodular lattices from the point of view of chamber systems, since the notion of i-adjacency makes a semimodular lattice into a chamber system. The paper [8] considers axioms which define a building, and uses similar axioms involving chamber systems to define many classes of semimodular lattices.

The remainder of this paper is structured as follows: in Section 2, we present the results for modular lattices. In Section 3, we give some preliminary notions and results for the semimodular case. In Section 4, we show that our poset is a join sublattice of the lattice generated by two flags, and in Section 5, we develop the concept of the label of a point with respect to a flag, and use this to derive an explicit lattice expression for every point in the poset we study. In Section 6, we show that this poset is a join sublattice of the distributive lattice determined by Jordan-Hölder permutation. In Section 7, we classify the sets reduced decompositions which can correspond to paths between two flags in a semimodular lattice, and in Section 8, we use our results to derive the corresponding results in the modular case presented in Section 2.

2. Modular lattices and Jordan-Hölder permutations

In this section, we present some commonly known results concerning the sublattice generated by two maximal chains, or *flags*, in a rank n modular lattice. Although these results are well known, the only references this author has been able to locate for them are Abels' papers [1] and [2]. In this paper we use these results to develop a generalization to semimodular lattices of the lattice generated by two flags in a modular lattice. We also compare our generalization to the lattice generated by two flags in a semimodular lattice.

In this section, X, Y, and Z represent flags in a modular or a semimodular lattice, and x_i, y_j, and z_k represent the rank i, j, and k points on the flags X, Y, and Z respectively. For two flags X and Y, we discuss the lattice generated by X and Y, i.e., the lattice of all meets and joins of points on X and Y. We let $L(X, Y)$ denote this lattice. In a modular lattice, $L(X, Y)$ is determined by a permutation called the *Jordan-Hölder permutation*. We define this function for semimodular lattices and prove that it is a permutation.

Definition If X and Y are two flags in a rank n semimodular lattice, the *Jordan-Hölder function of Y relative to X* from $[n] = \{1, 2, \ldots, n\}$ to itself is denoted by $\pi(X, Y)$ and is given by:

$$\pi(X, Y)(j) = \min\{i : y_j \leq x_i \vee y_{j-1}\} = \min\{i : x_i \vee y_{j-1} = x_i \vee y_j\}.$$

Proposition 2.1 *For all flags X and Y in a rank n semimodular lattice, $\pi(X, Y)$ is a permutation. Its inverse is $\pi(Y, X)$.*

Proof: If $\pi(X, Y)(j) = i$, we have the inequality

$$x_{i-1} \vee y_{j-1} < x_{i-1} \vee y_j \leq x_i \vee y_j = x_i \vee y_{j-1}, \tag{1}$$

since i is the smallest number such that $x_i \vee y_{j-1} = x_i \vee y_j$. Now by semimodularity, $x_i \vee y_{j-1}$ covers $x_{i-1} \vee y_{j-1}$. Therefore, $x_{i-1} \vee y_j = x_i \vee y_j$ in (1), but $x_{i-1} \vee y_{j-1} < x_i \vee y_{j-1}$. In other words, j is the smallest number such that $x_{i-1} \vee y_j = x_i \vee y_j$, so $\pi(Y, X)(i) = j$. □

We now define the lattice $J(\tau)$. This lattice is isomorphic to $L(X, Y)$ when X and Y are flags in a modular lattice and $\tau = \pi(X, Y)$.

Definition Given τ, let $J(\tau)$ be the subsets of $[n]$ with the following property: for all i and j in $[n]$, if $i < j$ and $\tau(i) < \tau(j)$, then every set in $J(\tau)$ which includes $\tau(j)$ also includes $\tau(i)$. If these sets are ordered by inclusion, $J(\tau)$ is a distributive lattice; joins and meets correspond to unions and intersections, respectively. In this lattice, we let X and Y be the flags given by

$$x_i = [i] = \{1, 2, \ldots, i\},$$
$$y_j = \tau([j]) = \{\tau(1), \tau(2), \ldots, \tau(j)\}.$$

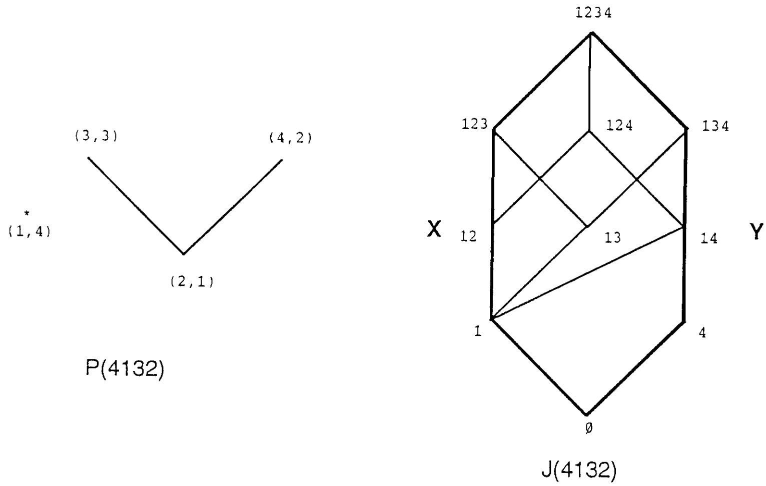

Figure 1. $P(\tau)$ and $J(\tau)$ for $\tau = (4132)$.

Alternatively, we could define $P(\tau)$ as the set of points $P(\tau) = \{(i, \tau(i))\}$ ordered by $(i, \tau(i)) \leq (j, \tau(j))$ if $i \leq j$ and $\tau(i) \leq \tau(j)$, and let $J(\tau)$ be the lattice of order ideals of $P(\tau)$. For example, Figure 1 shows $P(\tau)$ and $J(\tau)$ for $\tau = (4132)$ in one line notation, (i.e., $\tau(1) = 4$, $\tau(2) = 1$, $\tau(3) = 3$ and $\tau(4) = 2$), and Figure 2 does the same for all τ in S_3. In our examples, when a lattice consists of a collection of sets, we eliminate the set brackets and commas to label the sets; for example, in Figure 1, we write 124 for the subset $\{1, 2, 4\} \subseteq [4]$.

Theorem 2.2 *Suppose X and Y are flags in a modular lattice with $\tau = \pi(X, Y)$. Then $L(X, Y)$ is isomorphic to $J(\tau)$ via an isomorphism which maps x_i to $[i]$ and y_j to $\tau([j])$ for every i and j.*

Definition Two flags in a finite rank lattice are *i-adjacent* if they are equal except (possibly) at rank i. They are *adjacent* if they are i-adjacent for some i. A *path of length n from X to Y* is a sequence of flags $(X = X_0, X_1, X_2, \ldots, X_n = Y)$ such that consecutive flags are adjacent. Such a path is a *minimal X-Y path* if its length is minimal.

Theorem 2.3 relates minimal X-Y paths to reduced decompositions of the Jordan-Hölder permutation $\pi(X, Y)$. Thus, we define reduced decompositions.

Definition A *simple reflection* in S_n is a permutation of the form $r_i = (i\ i+1)$. We also call these permutations *adjacent transpositions*. A *decomposition* of a permutation τ is an expression of τ as a product of simple reflections, i.e., $\tau = s_1 s_2 \cdots s_k$. The decomposition is *reduced* if every decomposition of τ has at least k simple reflections. In this case, we say k is the *length* of τ, and we write $\ell(\tau) = k$. We generally write r_i for the simple reflection which switches i and $i + 1$, and s_j for an arbitrary simple reflection.

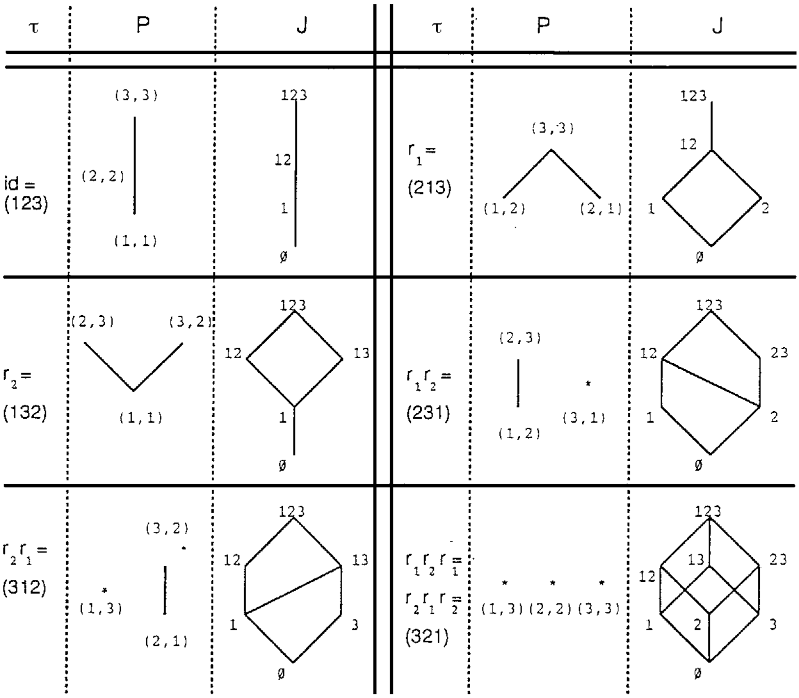

Figure 2. $P(\tau)$ and $J(\tau)$ for all τ in S_3.

Theorem 2.3 *In a finite rank modular lattice, minimal X-Y paths are related to $\pi(X, Y)$ and $L(X, Y)$ as follows.*

(i) *The point z is on a minimal X-Y path (more precisely, z is on a flag which is on a minimal X-Y path) if and only if z is in $L(X, Y)$.*

(ii) *There is a natural one-to-one correspondence between minimal X-Y paths and reduced decompositions of $\pi(X, Y)$. A step between i-adjacent flags in a minimal path corresponds to an occurrence of r_i in the corresponding reduced decomposition.*

For example, consider the lattice $J(4132)$ (see Figure 1). The minimal paths from X to Y are the following.

$$
\begin{array}{rrr}
X = \{\emptyset, 1, 12, 123, 1234\} & X = \{\emptyset, 1, 12, 123, 1234\} & X = \{\emptyset, 1, 12, 123, 1234\} \\
\{\emptyset, 1, 12, 124, 1234\} & \{\emptyset, 1, 12, 124, 1234\} & \{\emptyset, 1, 13, 123, 1234\} \\
\{\emptyset, 1, 14, 124, 1234\} & \{\emptyset, 1, 14, 124, 1234\} & \{\emptyset, 1, 13, 134, 1234\} \\
\{\emptyset, 1, 14, 134, 1234\} & \{\emptyset, 4, 14, 124, 1234\} & \{\emptyset, 1, 14, 134, 1234\} \\
Y = \{\emptyset, 4, 14, 134, 1234\} & Y = \{\emptyset, 4, 14, 134, 1234\} & Y = \{\emptyset, 4, 14, 134, 1234\}
\end{array}
$$

For (i), note that every point of $J(\tau)$ is listed in at least one of these sets. As for (ii), note that in the first path listed, we change the rank 3 point, then the rank 2 point, then the rank

3 point, and finally the rank 1 point. The corresponding reduced decomposition of τ is therefore $\tau = r_3r_2r_3r_1$. The other paths give $\tau = r_3r_2r_1r_3$ and $\tau = r_2r_3r_2r_1$, respectively. These are the only reduced decompositions of τ. For another example, we have listed the reduced decompositions of permutations in S_3 in Figure 2.

3. Reduced decomposition paths in semimodular lattices

To generalize Theorems 2.2 and 2.3 to semimodular lattices, we relate minimal paths between two flags in a semimodular lattice to reduced decompositions of the Jordan-Hölder permutation between the flags. The flags along a minimal path are related to the *weak Bruhat order* on S_n. This partial order on permutations in S_n is also related to the notion of *inversions*. We now define these concepts.

Definition If τ and τ' are permutations in S_n with $\tau = \tau' r_i$, then we define $\tau < \tau'$ if $\ell(\tau) < \ell(\tau')$. The *weak Bruhat order* is the transitive closure of this relation. Equivalently, we have $\rho \leq \sigma$ in the weak Bruhat order if some reduced decomposition of σ begins with a reduced decomposition of ρ. An *inversion in* τ is a pair $(\tau(i), \tau(j))$ such that $i < j$ but $\tau(i) > \tau(j)$.

Proposition 3.1 is a standard result which relates inversions to the weak Bruhat order and to length of a permutation. We use this result to give an alternate characterization of $J(\tau)$.

Proposition 3.1 *In a reduced decomposition of τ, each simple reflection adds one inversion to the permutation. Hence, we have $\rho \leq \tau$ in the weak Bruhat order if and only if every inversion in ρ is also an inversion in τ. Furthermore, $\ell(\tau)$ equals the number of inversions in τ.*

Corollary 3.2 $J(\tau) = \{\rho([k]) : 0 \leq k \leq n$ *and* $\rho \leq \tau$ *in the weak Bruhat order*$\}$.

Proof: Suppose $\rho \leq \tau$ in the weak Bruhat order. Then every inversion in ρ is also in τ. Hence, if $i < j$ and $\tau(i) < \tau(j)$, then $(\tau(j), \tau(i))$ cannot be an inversion in ρ. Therefore, $\tau(i)$ precedes $\tau(j)$ in the one-line expression for ρ. Thus if $\rho([k])$ includes $\tau(j)$, it also includes $\tau(i)$, so every $\rho([k])$ is in $J(\tau)$.

Conversely, let S be a set in $J(\tau)$ with cardinality k. Let ρ be the permutation in which $\rho(1)$ through $\rho(k)$ are the elements of S in increasing order, and $\rho(k+1)$ through $\rho(n)$ are the remaining elements in increasing order. Thus, $S = \rho([k])$. To show that $\rho \leq \tau$ in the weak Bruhat order, suppose $(\tau(j), \tau(i))$ is an inversion in ρ. From the definition of ρ, this can only happen if $\tau(j)$ is in S and $\tau(i)$ is not. But since S is in $J(\tau)$ and $\tau(i) < \tau(j)$, we cannot have $i < j$. Hence, $(\tau(j), \tau(i))$ is also an inversion in τ. □

We now relate these notions to arbitrary semimodular lattices.

Lemma 3.3 *Let X and Y be two flags in a finite rank semimodular lattice with $\tau = \pi(X, Y)$. Then if $(\tau(j), \tau(j+1))$ is not an inversion, we have*

$$y_j = \left(x_{\tau(j)} \vee y_{j-1}\right) \wedge y_{j+1}.$$

Proof: Let $a = \tau(j)$ and $b = \tau(j+1)$. Now $y_j \le x_a \vee y_{j-1}$, since $a = \tau(j)$. Therefore, $(x_a \vee y_{j-1}) \wedge y_{j+1}$ equals either y_j or y_{j+1}, but since $a < b = \tau(j+1)$, we have $y_{j+1} \not\le x_{b-1} \vee y_j$, so $y_{j+1} \not\le x_a \vee y_{j-1}$. Hence, the meet is y_j. □

Proposition 3.4 *Suppose X, Y, and Y' are flags in a finite rank semimodular lattice with $\tau = \pi(X, Y)$ and $\tau' = \pi(X, Y')$, and suppose Y and Y' are i-adjacent. Then either $\tau = \tau'$ or $\tau = \tau' r_i$. Furthermore, if $(\tau(i), \tau(i+1))$ is not an inversion, then $\tau = \tau'$ if and only if $Y = Y'$.*

Proof: Since Y and Y' are i-adjacent, we have $y_k = y'_k$ and $y_{k-1} = y'_{k-1}$ for $k \ne i$ and $k \ne i+1$. Thus, $\tau^{-1}(k) = \tau'^{-1}(k)$ for $k \ne i$ and $k \ne i+1$, so $\tau = \tau'$ or $\tau = \tau' r_i$. From Lemma 3.3, if $\tau = \tau'$ and $(\tau(i), \tau(i+1))$ is not an inversion, then $y_i = (x_{\tau(i)} \vee y_{i-1}) \wedge y_{i+1} = (x_{\tau(i)} \vee y'_{i-1}) \wedge y'_{i+1} = y'_i$, so $Y = Y'$. □

We want some terminology to discuss the relationship of a reduced decomposition to the path to which it corresponds.

Definition We say the decomposition $s_1 s_2 \cdots s_m$ *takes X to Y* along the path $(X = Z_0, Z_1, \ldots, Z_m = Y)$ if $\pi(X, Z_k) = s_1 s_2 \cdots s_k$ for all k. If the decomposition is reduced, we call the path a *reduced decomposition path from X to Y*, or simply a *reduced X-Y path.*

Corollary 3.5 *If X and Y are flags in a semimodular lattice with $\tau = \pi(X, Y)$, then every path from X to Y is at least as long as $\ell(\tau)$. Furthermore, if a path is the same length as $\ell(\tau)$, it is a reduced decomposition path. For every flag Z on a reduced path, if $\rho = \pi(X, Z)$, then $\rho \le \tau$ in the weak Bruhat order.*

Proof: Let $(X = Z_0, Z_1, \ldots, Z_m = Y)$ be a path from X to Y. By Proposition 3.4, either $\pi(X, Z_k) = \pi(X, Z_{k-1})$ or $\pi(X, Z_k) = \pi(X, Z_{k-1}) s_k$ for some simple reflection s_k. Therefore, $\tau = \pi(X, Z_m)$ can be expressed as a product of m or fewer simple reflections, and $\ell(\tau) \le m$. If $\ell(\tau) = m$ then every s_k is included in the product for τ and this product is a reduced decomposition. In this case, $\pi(X, Z_k) = s_1 s_2 \cdots s_k \le s_1 s_2 \cdots s_m = \tau$ in the weak Bruhat order. □

Corollary 3.5 shows that if there is a reduced path from X to Y in a semimodular lattice, then every minimal path is a reduced path. Theorem 3.6 shows that this applies to every pair of flags in every semimodular lattice. In [1] (proof of Theorem 3.3), Abels constructed the path and showed that its length is the same as $\ell(\pi(X, Y))$ without referring to the reduced decomposition. He also showed that the entire path constructed in this manner is contained in the join sublattice $X \vee Y = \{x_i \vee y_j\}$.

Theorem 3.6 *For every pair of flags X and Y in a semimodular lattice, some reduced decomposition of $\tau = \pi(X, Y)$ takes X to Y. Furthermore, if j is the largest number such that $\tau r_j < \tau$, we can choose a decomposition ending with r_j.*

Proof: It suffices to find a flag Y' such that $\tau' = \pi(X, Y') = \tau r_j < \tau$ and $\pi(Y', Y) = r_j$, since by induction on the length of τ, we may assume some reduced decomposition of τ' takes X to Y', and appending r_j to this decomposition gives a reduced decomposition of τ which takes X to Y (through Y').

Since $\tau r_j < \tau$, we have $\tau(j+1) < \tau(j)$, and since j is the largest such number, we find $\tau(j+1) < \tau(j+2) < \cdots < \tau(n)$. Thus, letting $i = \tau(j+1)$ gives $[i-1] \subseteq \tau([j-1]) = l_X(y_{j-1})$, so $x_{i-1} \leq y_{j-1}$. Similarly, we have $x_i \not\leq y_j$, but $x_i \leq y_{j+1}$.

By semimodularity, the point $y'_j = x_i \vee y_{j-1}$ covers $x_{i-1} \vee y_{j-1} = y_{j-1}$, so y'_j has rank j. Since $x_i \leq y_{j+1}$, we have $y'_j < y_{j+1}$. Hence, $Y' = \{\hat{0} = y_0 < y_1 < \cdots < y_{j-1} < y'_j < y_{j+1} < \cdots < y_n = \hat{1}\}$ is a well defined flag. Now Y' is j-adjacent to Y, so either $\tau' = \tau$ or $\tau' = \tau r_j$, where $\tau' = \pi(X, Y')$. Since $y'_j \not\leq x_{i-1} \vee y_{j-1} = y_{j-1}$, but $y'_j = x_i \vee y_{j-1}$, we find that $\pi(X, Y')(j) = i = \tau(j+1)$; thus, $\tau' = \tau r_j$, as desired. □

4. *R*(*X*, *Y*) and *L*(*X*, *Y*)

We now begin generalizing Theorems 2.2 and 2.3 to semimodular lattices. We study the subposet $R(X, Y)$, defined below, and give a detailed comparison of this poset to the sublattice $L(X, Y)$.

Definition If X and Y are two flags in a semimodular lattice, then $R(X, Y)$ is the subposet of all points on at least one reduced X-Y path.

In a modular lattice, $R(X, Y) = L(X, Y)$ by Theorem 2.3. In a semimodular lattice, they need not be equal. We wish to generalize to semimodular lattices the description of $L(X, Y)$ for modular lattices. In a modular lattice, $L(X, Y)$ is finite and distributive, even when the original lattice is not. Thus, constructing $L(X, Y)$ produces a lattice with more restrictive conditions than the original lattice. The examples of this section show that $L(X, Y)$ can lose some properties of the original semimodular lattice, including semimodularity. We do not know whether $L(X, Y)$ must be finite for a semimodular lattice. Therefore, we contend that $R(X, Y)$ is a more appropriate generalization than $L(X, Y)$.

Furthermore, through the notion of i-adjacency, we can view the flags of a finite rank semimodular lattice as elements of a chamber system. Abels did this in [1] and [2], and this idea is also explored in [8]. In this context, $R(X, Y)$ is a natural concept, and $L(X, Y)$ appears meaningless, especially if it is not ranked. Thus, in several ways, $R(X, Y)$ is a more natural generalization to semimodular lattices of $L(X, Y)$ from modular lattices than $L(X, Y)$.

We begin with an example, in which $R(X, Y) \neq L(X, Y)$. This example was also discovered by Abels ([2], Remark 3.13). Let X and Y be the flags $X = \{\emptyset, 1, 12, 123, 1234\}$ and $Y = \{\emptyset, 4, 42, 423, 4231\}$ in the semimodular lattice on the left in Figure 3. In this case, we have $\tau = \pi(X, Y) = (4231)$ in one-line notation. The point 3 is in $L(X, Y)$; it is the meet of 123 and 234. We show that $R(X, Y)$ does not contain the point 3.

Let Z be a flag in this lattice which contains 3, and let $\rho = \pi(X, Z)$. By definition of the Jordan-Hölder permutation, $\rho(1) = 3$ (since $z_0 = \emptyset$), so (3, 2) is an inversion in ρ, but not in τ. Hence, by Proposition 3.1, $\rho \not\leq \tau$ in the weak Bruhat order. Therefore, no reduced

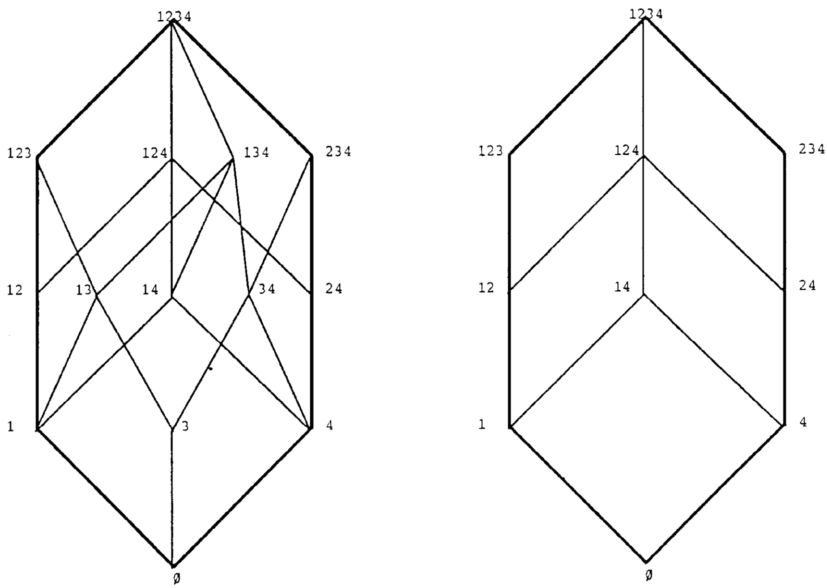

Figure 3. $L(X, Y) \neq R(X, Y)$.

decomposition that takes X to Y goes through Z, and so 3 is not an element of $R(X, Y)$. $R(X, Y)$ is on the right in Figure 3; $L(X, Y)$ is the entire lattice on the left.

We might also look for conditions on the lattice under which $R(X, Y)$ and $L(X, Y)$ must be equal. Since semimodularity places a bound on the join of two points in a lattice, a natural attempt would be requiring every point in the lattice to be the join of rank 1 points, or *atoms*. Such a lattice is called *atomic*, or *geometric*. These lattices can also be viewed as the lattice of flats (or closed sets) of a matroid (see [5] for more details). For our examples of matroids, we limit ourselves to faces of three-dimensional complexes.

However, geometric lattices do not necessarily satisfy $R(X, Y) = L(X, Y)$. Consider the lattice in Figure 4. As a matroid, this lattice can be represented as the faces of the pyramid in Figure 4; if we let a face be denoted by its vertices, the faces in this diagram are all subsets of $\{A, B, C, D, E\}$ which obey the condition that if any three points of $\{A, B, D, E\}$ are in a subset, then all four points are included, since the plane determined by the three points includes the whole square. Let X and Y be the flags $X = \{\emptyset, A, AB, ABC, ABCDE\}$ and $Y = \{\emptyset, E, DE, CDE, ABCDE\}$. Then $L(X, Y)$ is the bold part of the lattice in Figure 4; it is isomorphic to the lattice on the left in Figure 3, and $R(X, Y)$ is isomorphic to $R(X, Y)$ from the example in Figure 3 so $R(X, Y) \neq L(X, Y)$.

In [2] (Proposition 3.11, part (ii)), Abels proved that for every pair of flags in a semimodular lattice, $R(X, Y)$ is a join sublattice of the original lattice. He did this by proving a more general result for every *convex* set of flags.

Definition A set $\mathcal{F}$ of flags in a semimodular lattice is *convex* if whenever X and Y are in $\mathcal{F}$, every flag on a minimal X-Y path is also in $\mathcal{F}$.

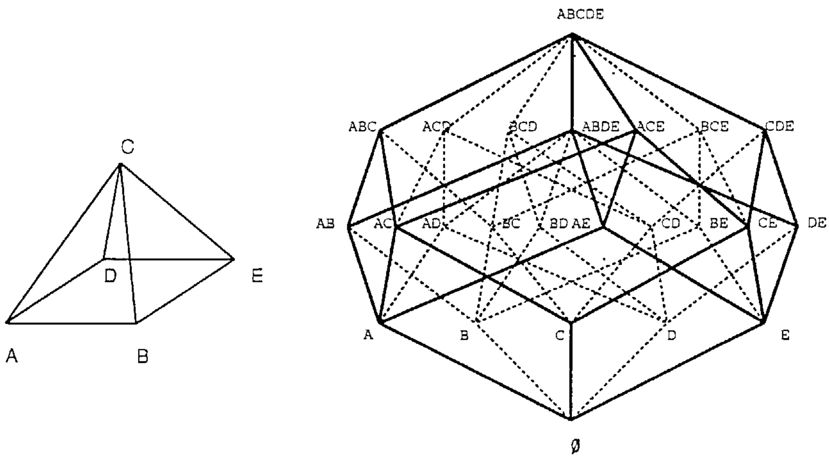

Figure 4. A geometric lattice where $L(X, Y) \neq R(X, Y)$.

Proposition 4.1 (Abels) *Let $\mathcal{F}$ be a convex set of flags in a semimodular lattice. Then the collection of all points on some flag in $\mathcal{F}$ forms a join sublattice of the original lattice. In particular, if we apply this to the convex hull of X and Y (the smallest convex set containing X and Y), we may conclude that $R(X, Y)$ is a join sublattice of the original lattice for every pair of flags X and Y.*

Combining Proposition 4.1 with Lemma 3.3, we obtain the following.

Corollary 4.2 *If X and Y are two flags in a semimodular lattice, then $R(X, Y)$ is a join sublattice of $L(X, Y)$.*

Proof: By Proposition 4.1, it suffices to show that $R(X, Y)$ is a subposet of $L(X, Y)$. For any z in $R(X, Y)$, let $X = Z_0, Z_1, \ldots, Z_m = Y$ be a reduced decomposition path from X to Y in which at least one flag contains z. By Lemma 3.3, every point on Z_{k-1} is in the lattice generated by X and Z_k, and so by descending induction on k, every point in Z_{k-1} is in $L(X, Y)$. In particular, z is in $L(X, Y)$. □

Another corollary of Proposition 4.1 is that $R(X, Y)$ is semimodular.

Corollary 4.3 *$R(X, Y)$ is a semimodular lattice.*

Proof: $R(X, Y)$ is clearly ranked, and the rank of every z in $R(X, Y)$ is the same as its rank in the original lattice, since z is on some flag of the original lattice which is contained in $R(X, Y)$. Since $R(X, Y)$ is a join sublattice of the original lattice, the join of any two points in $R(X, Y)$ is the same as their join in the original lattice. Although the original meet may not be in $R(X, Y)$, this only means that the rank of the meet in $R(X, Y)$ may be lower than in the original lattice, and this does not alter semimodularity. □

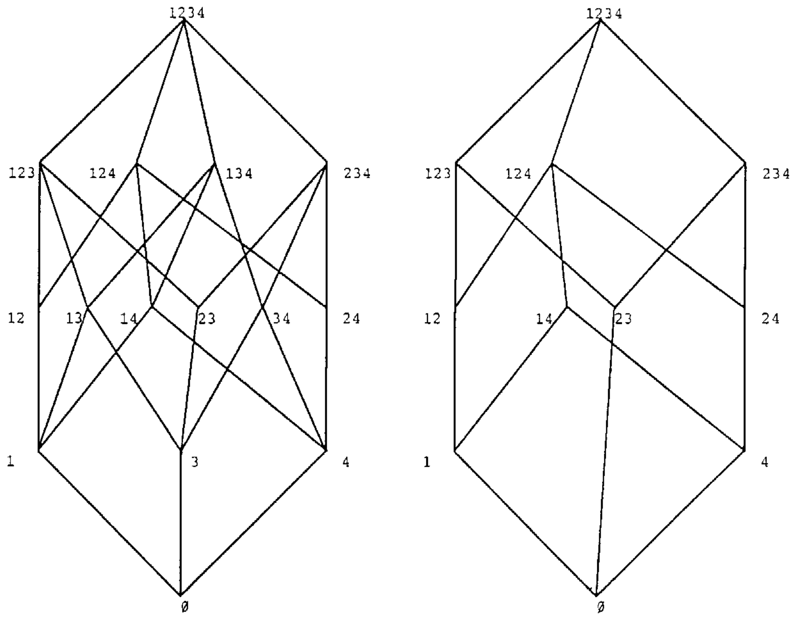

Figure 5. $L(X, Y)$ is unranked.

We could apply the same proof to show that $L(X, Y)$ is a semimodular lattice *if* $L(X, Y)$ *is ranked*. However, this need not be the case. Consider the lattice of all subsets of $\{1, 2, 3, 4\}$ except $\{2\}$, ordered by inclusion. The lattice is drawn on the left in Figure 5. Let $X = \{\emptyset < 1 < 12 < 123 < 1234\}$ and let $Y = \{\emptyset < 4 < 24 < 234 < 1234\}$. The point 23 is the intersection of 123 and 234, so 23 is in $L(X, Y)$. However, the point 3 is not in $L(X, Y)$. This is because every point in either X or Y which contains 3 also contains a 2, so the only way we could get to 3 is by taking a meet of points which contain 23. But the intersection of every such pair of sets also contains 23, and every set with 23 is in the original lattice. $L(X, Y)$ is drawn on the right in Figure 5.

5. Labeling functions for a semimodular lattice

In Section 4, we showed that $R(X, Y)$ is a join sublattice of $L(X, Y)$ whenever X and Y are flags in a semimodular lattice. We now show that $R(X, Y)$ can be embedded as a join sublattice into $J(\pi(X, Y))$. We also obtain an explicit lattice expression for every point in $R(X, Y)$. This expression is independent of the reduced X-Y path that contains the point.

The embedding is given by l_X, the labeling function for a lattice with respect to a flag X. Stanley defined this function in [11] and [12], and Björner developed it further in [4]. We now define this function, and derive some properties which relate it to the Jordan-Hölder permutation.

Definition If X is a flag in a rank n semimodular lattice, we define the *labeling function with respect to* X from points in the lattice to subsets of $[n]$ as follows:

$$l_X(z) = \{i \in [n] : x_i \leq x_{i-1} \vee z\} = \{i \in [n] : x_i \vee z = x_{i-1} \vee z\}.$$

The image $l_X(z)$ is called the *X-label of z*.

Proposition 5.1 *Suppose X and Y are flags in a semimodular lattice with $\tau = \pi(X, Y)$, and let $y_j < y_k$ be points on Y. Then the labeling function l_X has the following properties.*

(i) *The element i is in $l_X(y_j)$ if and only if $i = \tau(m)$ for some $m \leq j$, i.e., $l_X(y_j) = \{\tau(1), \ldots, \tau(j)\} = \tau([j])$. Thus, the cardinality of $l_X(z)$ equals the rank of z for all z.*

(ii) *We have $l_X(y_j) \subset l_X(y_k)$, so l_X is strictly monotonic. Thus, the labels on every flag form a strictly ascending chain of subsets of $[n]$.*

(iii) *The containment $[i] \subseteq l_X(y_j)$ holds if and only if $x_i \leq y_j$.*

Proof: We have i in $l_X(y_j)$ if and only if $x_i \vee y_j = x_{i-1} \vee y_j$, or equivalently, $\tau^{-1}(i) = \pi(Y, X)(i) \leq j$. Letting $m = \tau^{-1}(i)$ proves (i). For (ii), we have $l_X(y_j) = \tau([j]) \subset \tau([k]) = l_X(y_k)$. To show (iii), note that $[i] \subseteq l_X(y_j)$ if and only if $x_i \vee y_j = x_{i-1} \vee y_j = \cdots = x_0 \vee y_j = y_j$, or equivalently, $x_i \leq y_j$. □

We now give a criterion to determine whether a flag is on a reduced X-Y path from the X- and Y-labels of its points.

Proposition 5.2 *Let $\tau = \pi(X, Y)$. The flag Z is on a reduced X-Y path if and only if $l_X(z_k) = \tau(l_Y(z_k))$, and $l_X(z_k)$ is in $J(\tau)$ for each z_k in Z.*

Proof: Let $\rho = \pi(X, Z)$, $\sigma = \pi(Z, Y)$, and let $\tau = s_1 s_2 \cdots s_m$ be a reduced decomposition taking X to Y through Z, i.e., $\rho = s_1 \cdots s_l$ for some l, and $\sigma = s_{l+1} \cdots s_m$. Therefore, $\tau = \rho\sigma$, and $l_X(z_k) = \rho([k]) = (\tau\sigma^{-1})([k]) = \tau(\pi(Y, Z)([k])) = \tau(l_Y(z_k))$. Since $\rho \leq \tau$ in the weak Bruhat order, $\rho([k])$ is in $J(\tau)$ by Corollary 3.2.

Conversely, let $\rho = \pi(X, Z) = s_1 \cdots s_l$, and $\sigma = \pi(Z, Y) = t_1 \cdots t_m$ be reduced decompositions taking X to Z and Z to Y, respectively. Then in $J(\tau)$, these decompositions also take X to Z and Z to Y, since the labels give the same permutation in $J(\tau)$ as in the semimodular lattice. Thus, $\tau = s_1 \cdots s_l t_1 \cdots t_m$ is reduced, since it takes X to Z to Y in $J(\tau)$ along a minimal path with respect to paths through Z, and Z is on a reduced path in $J(\tau)$. □

Proposition 5.2 only applies when the labels of all points on a flag obey its conditions. Having $l_X(z_k) = \tau(l_Y(z_k))$ and $l_X(z_k)$ in $J(\tau)$ does not insure that z_k is in $R(X, Y)$. For a counterexample, we refer back to the lattice on the left in Figure 5. The point $z_2 = 23$ has $l_X(z_2) = \{2, 3\} = \tau(l_Y(z_2))$, and this label is in $J(\tau)$. However, a flag through 23 must also contain $z_1 = 3$, and $l_X(z_1) = \{3\}$. This label is not in $J(\tau)$ since $(3, 2)$ is not an inversion in τ. Alternatively, we could note that 3 cannot be in $R(X, Y)$ because it is not in $L(X, Y)$. Since no reduced X-Y path includes 3, none can include 23; thus, 23 is not in $R(X, Y)$.

Proposition 5.2 shows that l_X takes points in $R(X, Y)$ to $J(\pi(X, Y))$. To show that l_X embeds $R(X, Y)$ as a join sublattice of $J(\pi(X, Y))$, we need to show that it is an injective function, and that $l_X(w_k \vee z_m) = l_X(w_k) \cup l_X(z_m)$ whenever w_k and z_m are in $R(X, Y)$. We first show that the join of any point in $R(X, Y)$ with a point in either X or Y has the proper label.

Proposition 5.3 *If z_k is $R(X, Y)$, then $l_X(x_i \vee z_k) = [i] \cup l_X(z_k)$, and $l_X(y_j \vee z_k) = \tau([j]) \cup l_X(z_k)$.*

Proof: Let Z be a flag that contains z_k, and is on a reduced X-Y path. Also, let $\rho = \pi(X, Z)$. From Theorem 3.6, we can choose a reduced X-Z path whose reduced decomposition ends with r_j, where j is the largest number such that $\rho r_j < \rho$. If $j \geq k$, we may replace Z by the flag before the r_j. Applying this inductively, we may assume ρ has the property that $\rho r_j > \rho$ for all $j > k$. Thus, $\rho(j) < \rho(j+1)$ for $j > k$, and so labels are added in increasing order above rank k. We claim that this flag is

$$\{\emptyset = z_0 < z_1 < \cdots < z_k \leq x_1 \vee z_k \leq \cdots \leq x_n \vee z_k = [n]\}.$$

There are two cases. If $[i] \subseteq l_X(z_k)$ then $x_i \vee z_k = z_k$ which is on Z by definition. Otherwise, let z' be the lowest point on Z such that every label added above z' is greater than i. Now $l_X(z') = [i] \cup l_X(z_k)$. Therefore, $x_i \leq z'$, and $z_k \leq z'$ since z' is on Z. But the label of every upper bound of x_i and z_k contains $[i] \cup l_X(z_k)$ so z' is an upper bound of minimal rank. Hence, $z' = x_i \vee z_k$.

As for $y_j \vee z_k$, we have $l_Y(y_j \vee z_k) = [j] \cup l_Y(z_k)$, and so by Proposition 5.2, we find

$$\begin{aligned} l_X(y_j \vee z_k) &= \tau(l_Y(y_j \vee z_k)) = \tau([j] \cup l_Y(z_k)) \\ &= \tau([j]) \cup \tau(l_Y(z_k)) = \tau([j]) \cup l_X(z_k). \end{aligned}$$ □

Corollary 5.4 *For every z in $R(X, Y)$, $z \leq x_i \vee y_j$ if and only if $l_X(z) \subseteq l_X(x_i \vee y_j) = [i] \cup \tau([j])$.*

Proof: If $l_X(z) \subseteq [i] \cup \tau([j])$, then $l_X(z \vee x_i \vee y_j) = l_X(z) \cup [i] \cup \tau([j]) = [i] \cup \tau([j])$, by Proposition 5.3. Since $x_i \vee y_j \leq z \vee x_i \vee y_j$ and both points have the same label, and hence the same rank, we have $x_i \vee y_j = z \vee x_i \vee y_j$, or $z \leq x_i \vee y_j$. □

Proposition 5.5 gives an explicit lattice expression for an arbitrary point in $R(X, Y)$. Since the order relations in $R(X, Y)$ and $L(X, Y)$ are inherited from the original lattice, this shows that $R(X, Y) \subseteq L(X, Y)$. Corollaries 5.6 and 5.7 show that l_X is an injective function on $R(X, Y)$.

Proposition 5.5 *If X and Y are flags in a finite rank semimodular lattice with $\tau = \pi(X, Y)$, then for all z in $R(X, Y)$, we can write z as the following meet*:

$$z = \bigwedge_{l_X(z) \subseteq [i] \cup \tau([j])} x_i \vee y_j = \bigwedge_{z \leq x_i \vee y_j} x_i \vee y_j. \tag{2}$$

Proof: Let z' be the meet in (2). Clearly, $z \leq z'$, since z' is the meet of points above z. Now suppose $i = \tau(j)$ is in $l_X(z')$. Then $l_X(z) \not\subseteq [i-1] \cup \tau([j-1])$, so $x_{i-1} \vee y_{j-1}$ is not in the meet for z'. Thus, there is some $\tau(k)$ in $l_X(z)$ with $k \geq j$ and $\tau(k) \geq i = \tau(j)$. But $l_X(z)$ is in $J(\tau)$; therefore, if $\tau(k)$ is in $l_X(z)$, then $i = \tau(j)$ is in $l_X(z)$ as well. Hence, $l_X(z') \subseteq l_X(z)$, and $z = z'$. □

Corollary 5.6 *If z and z' are in $R(X, Y)$, then $l_X(z) \subseteq l_X(z')$ if and only if $z \leq z'$.*

Corollary 5.7 *Labels in $R(X, Y)$ are unique—that is, for all points z and z' in $R(X, Y)$, $l_X(z) = l_X(z')$ if and only if $z = z'$. Hence, l_X is an injection of $R(X, Y)$ into $J(\tau)$.*

Proof of Corollaries 5.6 and 5.7: If $l_X(z) \subseteq l_X(z')$, then $z' \leq x_i \vee y_j$ implies $z \leq x_i \vee y_j$. Hence, every point in the meet for z' is also in the meet for z, so $z \leq z'$ in Corollary 5.6, and $z \leq z' \leq z$ in Corollary 5.7. □

By contrast, labels need not be unique in $L(X, Y)$, even when the underlying lattice is geometric. For an example, consider Figure 6. In this lattice, let X and Y be the flags $X = \{\emptyset, A, AB, ABC, ABCDE\}$ and $Y = \{\emptyset, E, DE, CDE, ABCDE\}$. Then the points $CE = (x_3 \wedge y_3) \vee y_1$ and $DE = y_2$ are both in $L(X, Y)$, but $l_X(CE) = l_X(DE) = \{3, 4\}$.

Besides showing that Corollaries 5.6 and 5.7 cannot apply to $L(X, Y)$, this example also shows that points in $L(X, Y)$ need not obey Corollary 5.4 or the label criterion of Proposition 5.5. The labels of CE and DE are identical. Since their join is above both of them, the label of the join contains more than the union of the labels. Furthermore, $CE \not\leq DE = y_2$, even though its X-label is contained in $l_X(DE)$. However, we note that $l_Y(DE) = \{1, 2\}$ since $y_2 = DE$, but $l_Y(CE) = \{1, 3\}$. This suggests the following analog of Corollaries 5.6 and 5.7, which is an open question.

Question 1 *If z and z' are in $L(X, Y)$, and if $l_X(z) \subseteq l_X(z')$ and $l_Y(z) \subseteq l_Y(z')$, must we have $z \leq z'$? If $l_X(z) = l_X(z')$ and $l_Y(z) = l_Y(z')$, must $z = z'$?*

We proved Corollaries 5.6 and 5.7 via Proposition 5.5; similarly, question 1 would follow as a corollary to the following question.

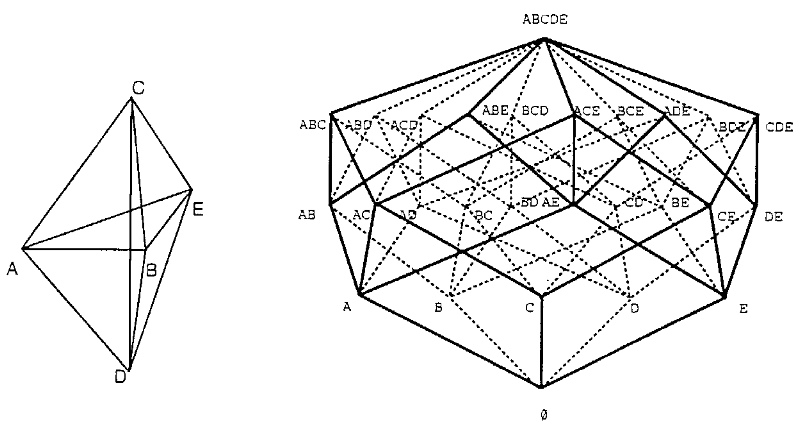

Figure 6. A geometric lattice with duplicated X-labels in $L(X, Y)$.

Question 2 *If z is a point in $L(X, Y)$, can we write always write z as the meet*

$$z = \bigwedge_{z \le x_i \vee y_j} x_i \vee y_j?$$

Finally, we note that $R(X, Y)$ is finite, since it is a sublattice of $J(\tau)$ which is finite. We do not know whether the same holds for $L(X, Y)$. Thus, we are led to ask:

Question 3 *Is it possible to construct a finite rank semimodular lattice such that $L(X, Y)$ is arbitrarily large for some flags X and Y?*

Obviously, an affirmative answer to question 3 would imply a negative answer to questions 1 and 2.

6. *R*(*X*, *Y*) and *J*(*τ*)

Theorem 6.1 *We have $l_X(w_k \vee z_m) = l_X(w_k) \cup l_X(z_m)$. Hence, the labeling function l_X embeds $R(X, Y)$ into $J(\tau)$ as a join sublattice.*

Proof: We have $l_X(w_k) \cup l_X(z_m) \subseteq l_X(w_k \vee z_m)$ by Proposition 5.1(ii). Now assume by induction on k that $l_X(w_{k-1} \vee z_m) = l_X(w_{k-1}) \cup l_X(z_m)$ for all m. By semimodularity, either $w_k \vee z_m$ covers $w_{k-1} \vee z_m$ or the points are equal. If they are equal, then $l_X(w_k \vee z_m) = l_X(w_{k-1} \vee z_m) = l_X(w_{k-1}) \cup l_X(z_m) \subseteq l_X(w_k) \cup l_X(z_m)$. Otherwise, we have $l_X(w_k) \not\subseteq l_X(w_{k-1} \vee z_m)$, by Corollary 5.6, since $w_k \not\leq (w_{k-1} \vee z_m)$, and both points are in $R(X, Y)$. Hence, $l_X(w_k) \cup l_X(z_m)$ has at least one more element than $l_X(w_{k-1} \vee z_m)$. But $l_X(w_k \vee z_m)$ has exactly one more element than $l_X(w_{k-1} \vee z_m)$, since the first point covers the second. Since

$$l_X(w_{k-1} \vee z_m) = l_X(w_{k-1}) \cup l_X(z_m) \subset l_X(w_k) \cup l_X(z_m) \subseteq l_X(w_k \vee z_m)$$

and the last set has exactly one more element than the first, we must have $l_X(w_k) \cup l_X(z_m) = l_X(w_k \vee z_m)$. □

We can also derive a converse to Theorem 6.1.

Proposition 6.2 *Every ranked join sublattice of $J(\tau)$ which contains its distinguished flags X and Y occurs as $R(X, Y)$ for some semimodular lattice.*

Proof: Let M be a join sublattice of $J(\tau)$. The proof of Corollary 4.3 shows that if a ranked join sublattice of a semimodular lattice has the same rank as the original lattice, the join sublattice is semimodular. Hence, M is semimodular, and it suffices to show that $R(X, Y) = M$ in M. Since the labeling functions l_X and l_Y are determined by joins, and since M is a join sublattice of $J(\tau)$, the X- and Y-labels of a point in M is the same as the corresponding labels in $J(\tau)$. As every flag is on a reduced path in $J(\tau)$, we also have $l_X(z) = \tau(l_Y(z))$ for every z in M, and so, by Proposition 5.2, every flag in M is on a reduced X-Y path. Thus, $R(X, Y) = M$. □

7. Classification of reduced paths in semimodular lattices

In this section, we classify the sets of reduced decompositions of a permutation τ which can correspond to a set of reduced X-Y paths in some semimodular lattice. We begin by defining the monoid of all decompositions of permutations in S_n.

Definition Let M be the free monoid on generators $\{r_1, r_2, \ldots, r_{n-1}\}$, where we multiply elements of M by concatenating them, and the identity is $\emptyset$, the empty product. Then for f in M, we define $\bar{f}$ to be the image of f in S_n. In particular, f is a decomposition of $\bar{f}$; $\overline{fg} = \bar{f}\bar{g}$; and $\bar{\emptyset}$ is the identity in S_n.

Proposition 7.1 *Let S be the set of reduced decompositions of τ which take X to Y in a semimodular lattice. Then S is nonempty and has the properties*:

R1. *If fr_ir_jh is in S and r_i and r_j commute, then fr_jr_ih is in S.*

R2. *If $fr_ir_{i+1}r_ih$ is in S then $fr_{i+1}r_ir_{i+1}h$ is in S.*

R3. *If fh and $f'h'$ are in S and $\bar{f} \leq \bar{f}'$ in weak Bruhat order, then some decomposition in S has the form fgh'.*

Proof: S is nonempty by Theorem 3.6. If $\tau = r_ir_j$ and the reflections commute, we have

$$Y = \{\hat{0} < x_1 < \cdots < x_{i-1} < y_i < x_{i+1} < \cdots < x_{j-1} < y_j < x_{j+1} < \cdots < x_n = \hat{1}\},$$

and we can go from x_i to y_i either before or after going from x_j to y_j, so S contains both decompositions. If $\tau = r_ir_{i+1}r_i$, then by Theorem 3.6, $r_{i+1}r_ir_{i+1}$ takes X to Y, so this decomposition must be in S. For longer permutations, if fr_ir_jh (or $fr_ir_{i+1}r_ih$, respectively) is a decomposition in S, applying the above comments letting X' be the flag reached after traversing f and Y' be the flag reached after fr_ir_j (or $fr_ir_{i+1}r_i$, respectively) proves R1 and R2.

As for R3, let X' be the flag in $R(X, Y)$ in position $\bar{f}$, and Y' be the flag in position $\bar{f}'$ from the path ending with h'. By uniqueness of labels in $R(X, Y)$, these flags are well defined. Since $R(X, Y)$ is semimodular, Theorem 3.6 gives a reduced decomposition g which takes X' to Y' in $R(X, Y)$. The decomposition fgh' takes X to X' to Y' to Y in $J(\tau)$. Furthermore, this is a reduced X-Y path in $J(\tau)$, since $\bar{f} \leq \bar{f}' \leq \tau$ in the weak Bruhat order. But we can follow the same steps in $R(X, Y)$ as we do in $J(\tau)$, and this gives a reduced X-Y path in the original lattice as well. □

For the remainder of this section, we prove the conditions in Proposition 7.1 are sufficient as well, i.e., for any nonempty set S of reduced decompositions which obey these conditions, we construct a semimodular lattice $R(S)$ with two distinguished flags X and Y such that the decompositions which take X to Y in $R(S)$ are precisely those in S. For notational convenience, we make the following definition.

Definition Let S be a set of reduced decompositions of some τ in S_n. Then S is an *R-set of τ*, or simply an *R-set*, if S is nonempty and obeys conditions R1, R2, and R3 in Proposition 7.1.

Theorem 7.2 *The R-sets are in one to one correspondence with the isomorphism classes of $R(X, Y)$'s.*

We construct $R(S)$ as follows: for each decomposition $f = s_1 s_2 \dots s_m$ in S, let $f_j = s_1 \dots s_j$, and let f_{jk} be the subset $\bar{f}_j([k]) \subseteq [n]$. The idea is that if f is to represent a path from X to Y, we need a flag at each step of the path. If Z_j is the flag at the jth step, then $\pi(X, Z_j) = \bar{f}_j$, and $l_X(z_k)$ would be f_{jk}. Thus, we define $R(S) = \{f_{jk} : f \in S\}$ ordered by inclusion. Then $R(S)$ is contained in $J(\tau)$ since every f_{jk} is $\rho([k])$ for some $\rho \leq \tau$ in the weak Bruhat order, and X and Y are in $R(S)$ since $X = \{f_{0k} = \bar{f}_0([k]) = \bar{\emptyset}([k]) = [k]\}$ and $Y = \{f_{mk} = \bar{f}_m([k]) = \tau([k])\}$ for any f in S. Also, by construction, every decomposition of S takes X to Y in $R(S)$. We call a reduced path from X to Y in $R(S)$ an *S-path* if the corresponding decomposition is in S, and we say any flag on an S-path is an *S-flag*.

We want to show that every reduced X-Y path is an S-path, and every flag in $R(S)$ is an S-flag, so $R(X, Y) = R(S)$. To do this, we show $R(S)$ has several properties analogous to those of $R(X, Y)$ that we proved in previous sections. We use these properties to show $R(S)$ is a join sublattice of $J(\tau)$ and the only reduced decompositions which take X to Y are those in S. For example, Lemma 7.3 corresponds to Theorem 3.6.

Lemma 7.3 *Let S be an R-set with fh in S. Let i, j, k, and m be the largest numbers such that $\overline{r_i f} < \bar{f}$, $\overline{f r_j} < \bar{f}$, $\overline{r_k h} < \bar{h}$, and $\overline{h r_m} < \bar{h}$. Then there are decompositions in S of the form $r_i f' h$, $f'' r_j h$, $f r_k h'$, and $f h'' r_m$. Conditions* R1 *and* R2 *are sufficient to guarantee the conclusions.*

Proof: We first suppose $h = \emptyset$, the empty decomposition, and use induction on the length of f to prove the statement for i. Let $f = s_1 s_2 \cdots s_m$, and let S' be the set of reduced decompositions f^* of $\overline{s_1 f} < \bar{f}$ such that $s_1 f^*$ is in S. Then S' is nonempty (since f is in S) and obeys conditions R1 and R2. Now $s_1 = r_p$ for some $p \leq i$ by definition of i. If $p < i - 1$, we may assume by induction on the length of $\bar{f}$ that $s_2 = r_i$, since i is the largest number such that $\overline{r_i s_1 f} < \overline{s_1 f}$, and we may exchange s_1 and r_i by condition R1.

If $p = i - 1$, then $\overline{r_{i-1} f} < \bar{f}$ and $\overline{r_i f} < \bar{f}$, so $\bar{f}(i-1) > \bar{f}(i) > \bar{f}(i+1)$. Now by induction, we assume $s_2 = r_i$ and $s_3 = r_{i-1}$. Using R2, we replace the initial $r_{i-1} r_i r_{i-1}$ with $r_i r_{i-1} r_i$ to get a decomposition beginning with r_i. Similarly, we can find a decomposition $f'' r_j$ in S.

If $h \neq \emptyset$, we let S^* be the set of reduced decompositions f^* of $\bar{f}$ such that $f^* h$ is in S. As before, S^* is nonempty, and conditions R1 and R2 hold for S^*, so by the above argument, we can find an appropriate decompositions in S^*, and appending h gives the decomposition in S. Similarly, we can find h' and h''. □

Lemma 7.4 corresponds to Proposition 5.3.

Lemma 7.4 *Suppose $Z = \{\emptyset \subset z_1 \subset \cdots \subset z_n\}$ is an S-flag. Then for all k, we have S-flags of the form*

$$Z'_k = \{\emptyset \subset z_1 \subset \cdots \subset z_k \subseteq z_k \cup [1] \subseteq \cdots \subseteq z_k \cup [n]\}$$

and

$$Z_k'' = \{\emptyset \subset z_1 \subset \cdots \subset z_k \subseteq z_k \cup \tau([1]) \subseteq \cdots \subseteq z_k \cup \tau([n]) = [n]\}.$$

Furthermore, we have $\pi(X, Z_k') \leq \pi(X, Z) \leq \pi(X, Z_k'')$ *in the weak Bruhat order.*

Proof: Choose fh in S such that $f = s_1 \cdots s_p$ takes X to Z and h takes Z to Y. By Lemma 7.3, we may assume s_p is the largest r_i such that $\overline{fr_i} < \bar{f}$. If $i > k$, we suppose by induction on the length of f that the lemma holds for Z^*, the flag immediately before the s_p. But Z^* is identical to Z at rank k and below, so $Z_k' = (Z^*)_k'$, which is an S-flag by induction, and $Z_k' = (Z^*)_k' \leq Z^* < Z$. If $i \leq k$, then the elements $\bar{f}([m+1]) \setminus \bar{f}([m])$ are in increasing order for $m > k$. Therefore, for all i we have $z_k \cup [i] = z_m$ for some m, and $Z_k' = Z$. To find Z_k'', use the same argument and induction on the length of h. □

In $R(X, Y)$, if w_k and z_k both cover $w_k \wedge z_k$, then $w_k \vee z_k$ covers w_k and z_k, by semimodularity. We would like to show the same property holds for $R(S)$. Lemma 7.5 is a partial result in that direction. It applies when there are S-flags through w_k and z_k such that the two flags agree at all levels below k.

Lemma 7.5 *Let W and Z be S-flags, and let k be the smallest rank such that $w_k \neq z_k$. Then there is an S-path through k-adjacent flags W' and Z' with the property that $w_j' = w_j$ and $z_j' = z_j$ for $j \leq k$. In particular, $w_k \cup z_k$ is in $R(S)$ and it covers w_k and z_k.*

Proof: We first show that we can choose W and Z so that $\pi(X, W) < \pi(X, Z)$ in the weak Bruhat order (or vice versa, in which case we reverse the roles of W and Z). Then we let f, h, f', and h' be reduced paths from X to W, W to Y, X to Z, and Z to Y, so fh and $f'h'$ are decompositions in S taking X to Y in $R(S)$ through W and Z, respectively. Since $\bar{f} < \bar{f}'$, condition R3 gives a reduced decomposition fgh' in S which takes X to W to Z to Y. Finally, we show that we can choose a g with exactly one r_k in it. (Note that g cannot have an r_j with $j < k$ since W and Z are equal below level k.) The flags immediately before and after the r_k in the corresponding S-path are W' and Z', respectively.

We write $\rho = \pi(X, W)$, $\sigma = \pi(X, Z)$, and $\tau = \pi(X, Y)$. To choose W and Z such that ρ and σ are related in the Bruhat order, let a and b be the lone elements of $(w_k \setminus w_{k-1})$ and $(z_k \setminus z_{k-1})$, respectively. Without loss of generality, suppose $a < b$ (otherwise, we switch W and Z). By Lemma 7.4, we may assume $W = \{\emptyset \subset w_1 \subset \cdots \subset w_k \subseteq w_k \cup [1] \subseteq \cdots \subseteq w_k \cup [n]\}$ and $Z = \{\emptyset \subset w_1 \subset \cdots \subset w_{k-1} \subset z_k \subseteq z_k \cup \tau[1] \subseteq \cdots \subseteq z_k \cup \tau[n]\}$. We show that every inversion in ρ is also in σ, so $\rho < \sigma$ in weak Bruhat order by Proposition 3.1. Clearly, every inversion of ρ which involves $\rho(j)$ for $j < k$ is also in σ, since $\rho([j]) = \sigma([j])$. The only other inversions in ρ involve $\rho(k) = a$. But if (a, a') is an inversion in ρ, it is also an inversion in τ by Proposition 3.1. Now $a' \neq b$, since $a' < a < b = \sigma(k)$, but the labels above rank k in Z are added in the same order that they appear in τ. Hence, if (a, a') is an inversion in τ, it is also an inversion in σ. Therefore, $\rho < \sigma$, and condition R3 applies.

To choose g with only one r_k, take the given g, and look at the first occurrence of a string of the form $r_k r_{k+1} \cdots r_m$. If this string is at the end of g we have only one r_k and there is nothing to prove. Otherwise, let r_p be the simple reflection following this string. We cannot have $p = m$ since the decomposition is reduced. If $p = m+1$, we replace m by $m+1$ and use the longer string. If $p > m+1$, we apply condition R1 repeatedly to replace $r_k r_{k+1} \cdots r_m r_p$ by $r_p r_k r_{k+1} \cdots r_m$. If $p < m$, we replace $r_k r_{k+1} \cdots r_m r_p$ by $r_k r_{k+1} \cdots r_{p-1} r_p r_{p+1} r_p r_{p+2} \cdots r_m$ using condition R1, then replace this by $r_k r_{k+1} \cdots r_{p-1} r_{p+1} r_p r_{p+1} r_{p+2} \cdots r_m$ using condition R2, and replace this with $r_{p+1} r_k r_{k+1} \cdots r_m$ by condition R1 again. In each of these cases, we move the string closer to the end of g. When it gets there, we have a decomposition fgh' in S with only one r_k in g, so we can choose W' and Z' as explained above. □

The requirement in Lemma 7.5 that the S-flags be equal up to the points in question can be rather cumbersome. Lemma 7.6 allows us to sidestep the difficulties, and to complete the proof of Theorem 7.2.

Lemma 7.6 *Suppose W and Z are S-flags such that $w_i = z_i$ for $i < k$ and $w_{k+1} = z_{k+1}$. Then $\{\emptyset \subset w_1 \subset w_2 \subset \cdots \subset w_{k-1} \subset z_k \subset w_{k+1} \subset \cdots \subset w_n\}$ is an S-flag.*

Proof: Let a and b be the lone elements of $(w_k \setminus w_{k-1})$ and $(z_k \setminus w_{k-1})$, respectively. If $a < b$, we assume $Z = \{\emptyset \subset z_1 \subset \cdots \subset z_{k+1} \subseteq z_{k+1} \cup \tau[1] \subseteq \cdots \subseteq z_{k+1} \cup \tau[n]\}$, since this is an S-flag by Lemma 7.4, and the points of Z above z_{k+1} are irrelevant for this lemma. Now every inversion in $\pi(X, W)$ is also in $\pi(X, Z)$, so $\pi(X, W) \leq \pi(X, Z)$ in weak Bruhat order. Thus, condition R3 implies the existence of a decomposition fgh in S which takes X to W to Z to Y. The decomposition g from W to Z has exactly one r_k and no other r_i's for $i \leq k+1$, since W and Z agree up to level $k+1$ except at level k. By condition R1, we may assume the r_k is the first reflection in g. The S-flag in position $\overline{fr_k}$ is the S-flag asserted in the lemma. If $a > b$, we assume $Z = \{\emptyset \subset z_1 \subset \cdots z_{k+1} \subseteq z_{k+1} \cup [1] \subseteq \cdots \subseteq z_{k+1} \cup [n]\}$ and note that $\pi(X, Z) \leq \pi(X, W)$ in weak Bruhat order. We now assume the lone r_k in g (which is now the decomposition from Z to W) is at the end of g, so the flag in position $\overline{fgr_k}$ is the asserted flag. □

Proposition 7.7 shows that $R(S)$ is a join sublattice of $J(\tau)$, and also allows us to prove that every flag of $R(S)$ is an S-flag.

Proposition 7.7 *Suppose W and Z are S-flags. Then for every i, there is an S-flag*

$$U_i = \{\emptyset \subset w_1 \subset \cdots \subset w_i \subseteq w_i \cup z_1 \subseteq \cdots \subseteq w_i \cup z_n\}.$$

Hence, $R(S)$ is a join sublattice of $J(\tau)$.

Proof: Since $U_0 = Z$, we assume by induction that U_{i-1} is an S-flag. We first use Lemma 7.5 and induction on j to extend

$$U_{i-1,j} = \{\emptyset \subset w_1 \subset \cdots \subset w_{i-1} \subseteq w_{i-1} \cup z_1 \subseteq \cdots \subseteq w_{i-1} \cup z_j \subseteq w_i \cup z_j\}$$

to an S-flag, and then we use descending induction on k and Lemma 7.6 to show there is an S-flag

$$V_{ik} = \{\emptyset \subset w_1 \subset \cdots \subset w_{i-1} \subseteq w_{i-1} \cup z_1 \subseteq \cdots \subseteq w_{i-1} \cup z_k \subseteq w_i \cup z_k \subseteq w_i \cup z_{k+1} \subseteq \cdots \subseteq w_i \cup z_n\}.$$

For the induction on j, we note that $U_{i-1,0}$ can be extended to W. Applying Lemma 7.5 to U_{i-1} and the S-flag which results from extending $U_{i-1,j-1}$ gives an S-flag containing $U_{i-1,j}$, completing the induction. We also find an S-flag containing $U_{i-1,j-1}$ and the point $w_i \cup z_j$. We will use this flag in the descending induction on k. We first note that $V_{in} = U_{i-1,n}$, so V_{in} is an S-flag. For $k \leq n$, we apply Lemma 7.6 to $V_{i,k}$ and the S-flag from the induction on j which contains $U_{i-1,k-1}$ and $w_i \cup z_k$. We find that $V_{i,k-1}$ is an S-flag. Since $U_i = V_{i,0}$, this completes the original induction on i. □

Corollary 7.8 *Every flag in $R(S)$ is an S-flag.*

Proof: Let W be a flag in $R(S)$ and suppose by induction that $\{\emptyset \subset w_1 \subset \cdots \subset w_k\}$ can be extended to an S-flag W_k^*. Now w_{k+1} is on some S-flag Z, since it is in $R(S)$, so applying Proposition 7.7 to W_k^* and Z gives the flag W_{k+1}^*, completing the induction. □

Proof of Theorem 7.2: Proposition 7.1 shows that the set of decompositions taking X to Y in a semimodular lattice form an R-set. Conversely, Proposition 7.7 shows that for every R-set S, $R(S)$ is a join sublattice of $J(\tau)$, i.e., a poset isomorphic to some $R(X, Y)$ for two flags X and Y in a semimodular lattice. To complete the proof, we must show that every reduced decomposition which takes X to Y in $R(S)$ is in S, i.e., if $s_1 \cdots s_m$ is a reduced decomposition which takes X to Y in $R(S)$, then this decomposition is in S. Thus, suppose by induction that some decomposition in S begins with $s_1 \cdots s_k$. By Corollary 7.8, there must be an S-flag Z in $R(S)$ such that $\pi(X, Z) = \overline{s_1 \cdots s_{k+1}}$, so there is another decomposition $t_1 \cdots t_m$ in S such that $\overline{t_1 \cdots t_{k+1}} = \overline{s_1 \cdots s_{k+1}}$. Now applying R3 to $f = s_1 \cdots s_k$ and $h' = t_{k+2} \cdots t_m$, we see that $s_1 \cdots s_{k+1} t_{k+2} \cdots t_m$ is in S, and by induction, $s_1 \cdots s_m$ is in S. □

8. Proofs of Theorems 2.2 and 2.3

We note that Theorems 2.2 and 2.3 follow from our results on semimodular lattices. We prove Theorem 2.3 first.

Proof of Theorem 2.3: From Proposition 4.2, we know that in an upper semimodular lattice, $R(X, Y)$ is a join sublattice of $L(X, Y)$. By duality, in a lower semimodular lattice, $R(X, Y)$ is a meet sublattice of $L(X, Y)$. Hence, if the original lattice is modular, $R(X, Y)$ must be a sublattice of $L(X, Y)$. But $L(X, Y)$ is the smallest sublattice of the original lattice that contains X and Y, so $R(X, Y) = L(X, Y)$. This proves Theorem 2.3(i).

As for Theorem 2.3(ii), Proposition 7.1 applies to lower semimodular lattices, except that we must replace condition R2 by condition R2′.

R2′. If $f r_{i+1} r_i r_{i+1} h$ is in S then $f r_i r_{i+1} r_i h$ is in S.

For a modular lattice, this condition becomes

R2″. If either $fr_i r_{i+1} r_i h$ or $fr_{i+1} r_i r_{i+1} h$ is in S then both decompositions are in S.

But by a standard result (see, for example [9], Theorem 2.11) we can transform any reduced decomposition of τ into any other reduced decomposition of τ by a sequence of replacements allowed by conditions R1 and R2″. Since S is nonempty, S must therefore consist of *every* reduced decomposition of $\pi(X, Y)$. This proves Theorem 2.3(ii). □

Proof of Theorem 2.2: We show that in a modular lattice, the labeling function l_X is an isomorphism between $R(X, Y)$ and $J(\tau)$ when $\tau = \pi(X, Y)$. Since $R(X, Y) = L(X, Y)$ by Theorem 2.3(i), and since $l_X(x_i) = [i]$ and $l_X(y_j) = \tau([j])$, this is sufficient.

By Theorem 6.1, $R(X, Y)$ can be embedded as a join sublattice of $J(\tau)$. Conversely, choose $\rho \leq \tau$ in the weak Bruhat order. By definition of the weak Bruhat order, some reduced decomposition of τ begins with a reduced decomposition of ρ. By Theorem 2.3(ii), this decomposition takes X to Y. Thus, some flag Z along this path has $\rho = \pi(X, Z)$. The X-label of rank k point of this flag is $l_X(z_k) = \rho([k])$. Therefore, every label of the form $\rho([k])$ with $\rho \leq \tau$ in the weak Bruhat order occurs. By Corollary 3.2, l_X is a surjection, and so by Corollary 5.7, it is an isomorphism. □

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