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Rank-One Connections at Infinity and Quasiconvex Hulls

Kewei Zhang
Department of Mathematics, Macquarie University, Sydney, NSW 2109, Australia.
e-mail: kewei@ics.mq.edu.au

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We define p-rank-one connections at infinity for an unbounded set K in $M^{N\times n}$ and show that the quasiconvex hull $Q_p(K)$ may be bigger than K if K has a p-rank-one connection, where $Q_p(K)$ is the zero set of the quasiconvex relaxation of the p-distance function to K. We examine some examples and compare $Q_p(K)$ with $\mathbf{Q}_p(K)$ - a more restrictive quasiconvex hull of K.

1. Introduction

For a compact set $K \subset M^{N\times n}$, the quasiconvex hull records the locations of all possible microstructures generated by K in the variational approach to martensitic phase transformation [8, 9, 23, 27, 33, 38]. Related notions such as the rank-one convex hull are also useful for the study of microstructures. There are three different but connected definitions of the quasiconvex hull and the rank-one convex hull.

Definition 1.1 ([27]). Let $K \subset M^{N\times n}$ be a non-empty closed set. The **quasiconvex hull** $Q(K)$ of K is defined by

$$Q(K) = \{X \in M^{N\times n},\ f(X) \le \sup_{Y\in K} f(Y),\ f : M^{N\times n} \to \mathbb{R} \text{ quasiconvex}\}.$$

The **rank-one convex hull** $R(K)$ is defined by

$$R(K) = \{X \in M^{N\times n},\ f(X) \le \sup_{Y\in K} f(Y),\ f : M^{N\times n} \to \mathbb{R} \text{ rank-one convex}\}.$$

Definition 1.2 ([38]). For any $1 \le p < \infty$, the p-quasiconvex hull and p-rank one convex hull of K are defined by

$$Q_p(K) = \{X \in M^{N\times n},\ Q\,\mathrm{dist}^p(X, K) = 0\},$$
$$R_p(K) = \{X \in M^{N\times n},\ R\,\mathrm{dist}^p(X, K) = 0\},$$

where $\mathrm{dist}(\cdot, K)$ is the Euclidean distance function to K and $Q\,\mathrm{dist}^p(\cdot, K)$ and $R\,\mathrm{dist}^p(\cdot, K)$ are the quasiconvex relaxation and rank-one convex relaxation of $\mathrm{dist}^p(\cdot, K)$ respectively.

It was established in [38] that for $1 \le p < \infty$, $Q(K) = Q_p(K)$ whenever K is compact. The advantage of Definition 1.2 is that the quasiconvex hull is determined by the zero set of a single quasiconvex function.

When K is bounded, quasiconvex hulls arise naturally from the study of singular solutions of certain elliptic systems [28], the stability problem of the conformal set [32, 33], and the

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study of minimizing problems with linear growth quasiconvex functions [14]. However, for unbounded K the the situation is quite different [32, 33, 35, 39, 24, 11]. In [33], Yan defined the p-quasiconvex hull for an unbounded set as

Definition 1.3 ([33]). Let $K \subset M^{N\times n}$ be non-empty and closed. The p-quasiconvex hull $\mathbf{Q}_p(K)$ of K is defined by

$$\mathbf{Q}_p(K) = \{X \in M^{N\times n},\ f(X) \leq \sup_{Y\in K} f(Y),\\ f : M^{N\times n} \to \mathbb{R} \text{ quasiconvex, and } 0 \leq f(X) \leq C_f(1+|X|^p)\}$$

The p-rank-one convex hull $\mathbf{R}_p(K)$ of K is defined by

$$\mathbf{R}_p(K) = \{X \in M^{N\times n},\ f(X) \leq \sup_{Y\in K} f(Y),\\ f : M^{N\times n} \to \mathbb{R} \text{ rank-one convex, and } 0 \leq f(X) \leq C_f(1+|X|^p)\}$$

where $C_f \geq 0$ is a constant depending on f.

It is well-known that when f is quasiconvex, it is rank-one convex [21, 4, 10], while the converse is not true [29]. Therefore we have

$$R(K) \subset Q(K), \qquad R_p(K) \subset Q_p(K), \qquad \mathbf{R}_p(K) \subset \mathbf{Q}_p(K).$$

From the definitions above, we also see that

$$Q(K) \subset \mathbf{Q}_p(K) \subset Q_p(K), \qquad R(K) \subset \mathbf{R}_p(K) \subset R_p(K),$$

for all $1 \leq p < \infty$.

Since quasiconvex hulls arise naturally from minimizing problems, let us examine briefly how they might otherwise be defined. Consider the minimizing problem $\inf I(u)$ subject to certain boundary conditions, where

$$I(u) = \int_\Omega F(Du(x))dx. \tag{1.1}$$

We are interested in the situation when $\inf I(u) = 0$ with F satisfying $F(P) = 0$, $P \in K$ and $F(P) > 0$, $P \notin K$. The quasiconvex hull of (K, F) could be defined as

$$Q_F(K) = \{P \in M^{N\times n},\ \inf_{u|_{\partial\Omega}=Px} \int_\Omega F(Du)dx = 0\}. \tag{1.2}$$

Therefore, we see that $Q(K)$, given by Definition 1.1, is the smallest quasiconvex hull while $\mathbf{Q}_p(K)$ is the smallest among all $Q_F(K)$ with $F(P)$ has p-th order growth at infinity. However, for general K, it is hard to find enough quasiconvex functions to estimate $Q(K)$ or $\mathbf{Q}_p(K)$, while the p-distance function $\mathrm{dist}^p(\cdot, K)$ characterizes the geometry of K and can always be defined. The study of $Q_p(K)$ depends only on the behaviour of the quasiconvex function $Q\,\mathrm{dist}^p(\cdot, K)$. This function has a very interesting property which is not shared by other type of nonnegative quasiconvex functions vanishing on K, manely:

$$Q\,\mathrm{dist}^p(\cdot, K) = Q\,\mathrm{dist}^p(\cdot, Q_p(K)),$$

for all $1 \leq p < \infty$. A matrix P is in $Q_p(K)$ if there is a sequence (ϕ_j) in C_0^∞, such that

$$\lim_{j\to\infty} \int_\Omega \operatorname{dist}^p(P + D\phi_j(x), K)dx = 0.$$

Hence for every $\epsilon > 0$, $P + D\phi_j(x)$ stays in the ϵ-neighbourhood of K except on a small set H_j whose measure tends to 0 as $j \to 0$. For functions F other than those of distance type, if $F(P) = 0$, $P \in K$ and $F(P) > 0$, $P \notin K$, the behaviour of the minimizing sequences is more difficult to characterize. Therefore, the study of $Q_p(K)$ could also be considered as the first step toward the understanding of $Q(K)$ and $\mathbf{Q}_p(K)$.

In this paper, we define the notion of rank-one connections at infinity, and use it as a tool to study the quasiconvex hull $Q_p(K)$ without knowing too many details of K. It is well known that rank-one connections are the simplest type of algebraic conditions on a closed set K which are sufficient to ensure a nontrivial quasiconvex hull $Q_p(K)$. Rank-one connections at infinity are a natural generalization of the notion of the rank-one connection. We have

Definition 1.4. Suppose $K \subset M^{N\times n}$ is closed and unbounded. K has a p-rank-one connection at infinity if there exists a sequence (A_j) in K with eigenvalues of $A_j^T A_j$

$$\lambda_j^{(1)} \leq \lambda_j^{(2)} \leq \cdots \leq \lambda_j^{(n-1)} \leq \lambda_j^{(n)} =: \Lambda_j,$$

such that

(i) $\Lambda^j \to \infty$ as $j \to \infty$;

(ii) Let $\lambda_j = \lambda_j^{(1)} + \cdots + \lambda_j^{(n-1)}$, then

$$\frac{\lambda_j}{(\Lambda_j)^{1/p}} \to 0, \qquad \text{as } j \to \infty;$$

(iii) There exists a rank-one matrix A_0 such that

$$\frac{A_j}{\sqrt{\Lambda_j}} \to A_0, \qquad \text{as } j \to \infty.$$

When $p = 1$ and K has a 1-rank-one connection at infinity, we simply say that K has a rank-one connection at infinity. For simplicity we say 'p-rank-one connection' rather than 'p-rank-one connection at infinity'.

Intuitively, a rank-one connection at infinity just means that the greatest eigenvalue of $A_j^T A_j$ goes to infinity faster than all other eigenvalues. Hence the limit of $A_j/\sqrt{\Lambda_j}$ is a rank-one matrix.

Before we give examples of sets with rank-one connections at infinity, we state the main theorem of this paper.

Theorem 1.5. *Suppose K is closed, unbounded and has a p-rank-one connection at infinity. Then*

$$\{P + tA_0;\ P \in K,\ t \geq 0\} \subset R_p(K) \subset Q_p(K), \tag{1.3}$$

where A_0 is the rank-one matrix given by Definition 1.4.

This result shows what type of extra points will be in $R_p(K)$ and $Q_p(K)$. If K already contains the set in (1.3), this theorem does not provide any information on $R_p(K)$ or $Q_p(K)$. The following result is a direct consequence of the proof of Theorem 1.5.

Proposition 1.6. *Let S_+, S_- and S be given by*

$$S_+ = \{P \in M^{2\times 2},\ P^T = P,\ \det P = 1,\ \textit{and } P \textit{ is positive definite}\}$$
$$S_- = \{P \in M^{2\times 2},\ P^T = P,\ \det P = 1,\ \textit{and } P \textit{ is negative definite}\},$$
$$S = S_+ \cup S_-.$$

Then

$$R_p(S_+) = Q_p(S_+) = C(S_+), \quad R_p(S_-) = Q_p(S_-) = C(S_-), \quad R_p(S) = Q_p(S) = C(S)$$

for all $p \geq 1$, where $C(K)$ is the closed convex hull of set K.

Remark 1.7. Since the structure of S_+ is simple, we can show that at least for $p \geq 2$, $\mathbf{Q}_p(S_+) = S_+$. Let

$$f(P) = \operatorname{dist}^2(P, C(S_+)) + |\det P - 1|.$$

Then F is quasiconvex (in fact, polyconvex), $F \geq 0$ and $F^{-1}(0) = S_+$.

Theorem 1.5 does not apply to the quasiconformal set $R_+SO(n)$ where the eigenvalues do not satisfy assumption (ii) of Definition 1.4 [32, 33, 22].

We also have

Proposition 1.8. *Let $S_+^{(n)}$ be given by*

$$S_+^{(n)} = \{P \in M^{n\times n},\ P^T = P,\ \det P = 1,\ \textit{and } P \textit{ is positive definite}\},$$

Then

$$R_p(S_+^{(n)}) = Q_p(S_+^{(n)}) = C(S_+^{(n)}) = \{P \in M^{n\times n},\ P^T = P,\ \det P \geq 1,\ \textit{and } P \textit{ is positive definite}\}$$

for all $p \geq 1$.

We establish Theorem 1.5 by using an upper bound of the rank-one convex relaxation of the p-distance function from a two point set, $Q\operatorname{dist}^p(\cdot, \{A, B\})$ based on the translation method [13]. The bound recovers the explicit quasiconvex relaxation when $p = 2$, obtained by Kohn [17].

In Section 2, we give some preliminaries and establish an upper bound of the quasiconvex relaxation of $\operatorname{dist}^p(\cdot, \{A, B\})$ which is essential for the proof of Theorem 1.5. We prove Theorem 1.5 and Proposition 1.6 in Section 3. In Section 4, we apply Theorem 1.5 to various situations. We will study quasiconvex hulls for closed, connected and unbounded sets in $M^{2\times 2}$ without rank-one connections. We will also study the quasiconvex hulls related to singular solutions of elliptic systems proposed in [28], and quasimonotone mappings.

2. Preliminaries

Throughout the rest of this paper Ω is a bounded open subset of $\mathbb{R}^n$. We denote by $M^{N\times n}$ the space of real $N\times n$ matrices with the $\mathbb{R}^{Nn}$ metric; hence the norm of $P\in M^{N\times n}$ is defined by $\mid P\mid = (\mathrm{tr}\, P^T P)^{1/2}$, where tr is the trace operator and P^T is the transpose of P. We denote by $\mathrm{diag}(a_1, a_2, \cdots, a_n)$ an $n\times n$ diagonal matrix with diagonal entries $a_1, a_2, \cdots, a_n$. The inner product of two matrices in $M^{N\times n}$ is $P\cdot Q = \mathrm{tr}\, P^T Q$. For an $n\times n$ matrix P, denote by $\mathrm{adj}\, P$ the transpose of the cofactors of P. $O(n)$ is the set of all $n\times n$ orthogonal matrices while $SO(n)\subset O(n)$ is the set of all orthogonal matrices with determinant 1. For a compact subset $K\subset M^{N\times n}$, let $C(K)$ be the convex hull of K. We write $C_0(\Omega)$ for the space of continuous functions $\phi:\Omega\to\mathbb{R}$ having compact support in Ω, and define $C_0^1(\Omega)=C^1(\Omega)\cap C_0(\Omega)$. If $1\le p\le\infty$ we denote by $L^p(\Omega;\mathbb{R}^N)$ the Banach space of mappings $u:\Omega\to R^N$, $u=(u_1,\cdots,u_N)$, such that $u_i\in L^p(\Omega)$ for each i, with norm $\|u\|_{L^p(\Omega;\mathbb{R}^N)}=\sum_{i=1}^N\|u_i\|_{L^p(\Omega)}$. Similarly, we denote by $W^{1,p}(\Omega,\mathbb{R}^N)$ the usual Sobolev space of mappings $u\in L^p(\Omega;\mathbb{R}^N)$ all of whose distributional derivatives $\frac{\partial u_i}{\partial x_j}=D_j u_i$, $1\le i\le N$, $1\le j\le n$, belong to $L^p(\Omega)$. $W^{1,p}(\Omega,\mathbb{R}^N)$ is a Banach space under the norm

$$\|u\|_{W^{1,p}(\Omega,\mathbb{R}^N)}=\|u\|_{L^p(\Omega;\mathbb{R}^N)}+\|Du\|_{L^p(\Omega;M^{N\times n})},$$

where $Du=(D_j u_i)$, and we define, as usual, $W_0^{1,p}(\Omega;\mathbb{R}^N)$ to be the closure of $C_0^\infty(\Omega;\mathbb{R}^N)$ in the topology of $W^{1,p}(\Omega;\mathbb{R}^N)$.

Weak and weak $*$ convergence of sequences are written as $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$, respectively. If $H\subset M^{N\times n}$, $P\in M^{N\times n}$, then we write $H+P$ to denote the set $\{P+Q : Q\in H\}$, $jH=\{jQ,\, Q\in H\}$ for an integer $j>0$. We define the distance function for a set $K\subset M^{N\times n}$ by

$$f(P)=\mathrm{dist}(P,K):=\inf_{Q\in K}\mid P-Q\mid .$$

Definition 2.1 ([21, 4, 10, 1]). A continuous function $f: M^{N\times n}\to\mathbb{R}$ is **quasiconvex** if

$$\int_U f(P+D\phi(x))\,dx\ge f(P)\,\mathrm{meas}(U)$$

for every $P\in M^{N\times n}$, $\phi\in C_0^1(U;\mathbb{R}^N)$, and every open bounded subset $U\subset\mathbb{R}^n$.

For a given function, we can consider its quasiconvexification (quasiconvex relaxation):

Definition 2.2 ([10]). Suppose $f: M^{N\times n}\to\mathbb{R}$ is a continuous function. The **quasiconvexification** of f is defined by

$$\sup\{g\le f;\ g \text{ quasiconvex }\}$$

and will be denoted denoted by Qf.

Proposition 2.3 ([10]). *Suppose $f: M^{N\times n}\to\mathbb{R}$ is continuous, then*

$$Qf(P)=\inf_{\phi\in C_0^\infty(\Omega;\mathbb{R}^N)}\frac{1}{\mathrm{meas}(\Omega)}\int_\Omega f(P+D\phi(x))\,dx, \tag{2.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain. In particular the infimum in (2.1) is independent of the choice of Ω.

We use the following theorem concerning the existence and properties of Young measures from Tartar [30]. For results in a more general context and their proofs, the reader is referred to [5, 31].

Theorem 2.4. *Let $(z^{(j)})$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^s)$. Then there exist a subsequence $(z^{(\nu)})$ of $(z^{(j)})$ and a family $\{\nu_x\}_{x\in\Omega}$ of probability measures on $\mathbb{R}^s$, depending measurably on $x\in\Omega$, such that*

$$f(z^{(\nu)}) \overset{*}{\rightharpoonup} \langle \nu_x, f(\cdot)\rangle \qquad \text{in } L^\infty(\Omega)$$

for every continuous function $f:\mathbb{R}^s\to\mathbb{R}$.

We say that ν_x is a trivial Young measure at $x\in\Omega$ if $\nu_x=\delta_A$ for some $A\in\mathbb{R}^s$, where δ_A is the Dirac mass at A.

Suppose that $\Omega\subset\mathbb{R}^n$. A family of parametrized measures $\{\nu_x\}_{x\in\Omega}$ is called a **Young measure limit of gradients** [7, 18, 19], if it is generated by a sequence of gradients Du_j with (u_j) bounded in $W^{1,p}(\Omega,\mathbb{R}^N)$.

We will need a consequence of the following theorem, but will not introduce the more general notion of normal integrals to which it applies (see [12, page 234]).

Theorem 2.5 (The measurable selection theorem (see [12, page 236])). *Let B be a compact subset of $\mathbb{R}^p$ and g a Carathéodory function of $\Omega\times B$. Then, there exists a measurable mapping $\tilde u:\Omega\to B$ such that for all $x\in\Omega$:*

$$g(x,\tilde u(x)) = \min_{a\in B}\{g(x,a)\}.$$

A direct consequence of Theorem 2.5 is the following:

Proposition 2.6. *Let $B\subset\mathbb{R}^p$ be a compact subset and let $u:\Omega\to\mathbb{R}^p$ be an integrable mapping. Then there exists a measurable mapping $\tilde u:\Omega\to B$ such that for all $x\in\Omega$*

$$|u(x)-\tilde u(x)| = \operatorname{dist}(u(x),B).$$

The following simple result will provide us with a guide for the estimates in the proof of Theorem 1.5.

Lemma 2.7. *Let $a<0<b$ and $0<\lambda<1$ be fixed numbers. Define*

$$f(x)=\min\{|x-a|^p,|x-b|^p\}+\lambda|x|^p.$$

Then

$$Cf(0)\le \lambda\frac{b|a|^p-a|b|^p}{b-a}.$$

Proof. The proof is very simple. Let $\theta=b/(b-a)$, then, $0<\theta<1$ and $\theta a+(1-\theta)b=0$. We then have

$$\begin{aligned}Cf(0)&\le \theta Cf(a)+(1-\theta)Cf(b)\le \theta f(a)+(1-\theta)f(b)\\ &= \theta\lambda|a|^p+(1-\theta)|b|^p=\lambda\frac{b|a|^p-a|b|^p}{b-a}.\end{aligned}$$

□

We conclude this section by examining the structure of closed connected sets in $M^{2\times 2}$ without rank-one connections.

It was established in [26] that if $K \subset M^{2\times 2}$ is closed, connected and has no rank-one connections, then $\det(A-B) > 0$ for all $A, B \in K$, $A \neq B$, or $\det(A-B) < 0$ for all $A, B \in K$, $A \neq B$. Such sets are characterized in [36].

Let $E_\partial = \mathrm{span}\{E_1, E_2\}$, $E_{\bar\partial} = \mathrm{span}\{E_3, E_4\}$ be the pair of two dimensional subspaces of $M^{2\times 2}$ where

$$E_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

Notice that E_∂ and $E_{\bar\partial}$ are orthogonal to each other and $E_1, \ldots, E_4$ is an orthonormal basis of $M^{2\times 2}$. Let P_{E_∂} and $P_{E_{\bar\partial}}$ be the orthogonal projections from $M^{2\times 2}$ to E_∂ and $E_{\bar\partial}$ respectively. It is easy to check that for any $\phi \in C_0^\infty(\Omega, \mathbb{R}^2)$, with $\Omega \subset \mathbb{R}^2$ open,

$$\int_\Omega |P_{E_\partial}(D\phi(x))|^2 dx = \int_\Omega |P_{E_{\bar\partial}}(D\phi(x))|^2 dx.$$

The following was established in [36, Theorem 3.2].

Lemma 2.8. *Suppose $K \subset M^{2\times 2}$, and $\det(P-Q) > 0$ for all $P, Q \in K$, $P \neq Q$. Then*

$$K = \{A + f(A),\ A \in P_{E_\partial}(K)\}$$

where E_∂ is the two dimensional subspace defined in Lemma 3.1, *$P_{E_\partial}(K)$ is the image of K under P_{E_∂} and $f : P_{E_\partial}(K) \to E_{\bar\partial}$ satisfies*

$$\mid f(A) - f(B) \mid < \mid A - B \mid \tag{2.2}$$

for all $A, B \in K$, $A \neq B$.

We call the mapping f obtained in Theorem 3.2 the Lipschitz mapping associated to K. Notice that without loss of generality, we may assume that $\det(P-Q) > 0$ for $P, Q \in K$, $P \neq Q$ by multiplying K by J if necessary, where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Lemma 2.9 (see [37, Lemma 2.9] and its proof). *Suppose $f : E_\partial \to E_{\bar\partial}$ is a Lipschitz mapping with Lipschitz constant $k \leq 1$. Let $K = \{P + f(P),\ P \in E_\partial\}$. Then*

$$|P_{E_{\bar\partial}}(A) - f(P_{E_\partial}(A))| \leq \sqrt{2}\,\mathrm{dist}(A, K)$$

for all $A \in M^{2\times 2}$.

3. Proof of the main results

Proof of Theorem 1.5. Since for any $P \in M^{N\times n}$ and $f : M^{N\times n} \to \mathbb{R}+$, we have

$$Rf(P) \leq \inf\{\theta f(P+A) + (1-\theta) f(P+B),\ 0 \leq \theta \leq 1,\ \theta A + (1-\theta)B = 0,\ \mathrm{rank}(A-B) = 1\}.$$

Let $K_j = \{P, A_j\}$. Since an everywhere finite rank-one convex function is continuous, we estimate the value of $R\operatorname{dist}^p(\cdot, K_j)$ at $B_j = P + t(A_j - P)/|A_j - P|$ instead of at $P + tA_0$, because it is easy to see that $B_j \to P + tA_0$ as $j \to \infty$. This follows from the facts that $|A_j| \to \infty$, P is fixed and $A_j/|A_j| \to A_0$ as $j \to \infty$.

We consider two different situations.

Case (1) If there exists a sequence of A_j, such that $\operatorname{rank}(A_j - P) = 1$. then obviously

$$R\operatorname{dist}^p(P + t(A_j - P)/|A_j - P|, K) \leq R\operatorname{dist}^p(P + t(A_j - P)/|A_j - P|, K_j) = 0.$$

Passing to the limit $j \to \infty$, we have $P + t(A_j - P)/|A_j - P| \to P + tA_0$. Hence, $R\operatorname{dist}^p(P + tA_0, K) = 0$.

Case (2) If for sufficiently large j, $\operatorname{rank}(A_j - P) > 1$ we choose

$$U_j = \alpha(A_j - P)v \otimes v, \qquad V_j = \beta(A_j - P)v \otimes v$$

with α, β to be determined, where v is a unit eigenvector corresponding to the greatest eigenvalue μ_j of $(A_j - P)^T(A_j - P)$. Obviously, $\operatorname{rank}(U_j - V_j) = 1$. Let us estimate $\operatorname{dist}^p(B_j + U_j, K_j)$.

$$\begin{aligned}\operatorname{dist}^p(B_j + U_j, K_j) &= \min\{|B_j + U_j - P|^p,\ |B_j + U_j - A_j|^p\} \\ &= \min\left\{\begin{array}{l}\left|\dfrac{t(A_j - P)}{|A_j - P|} + \alpha(A_j - P)v \otimes v\right|^p, \\ \left|\dfrac{t(A_j - P)}{|A_j - P|} + \alpha(A_j - P)v \otimes v - (A_j - P)\right|^p\end{array}\right\}\end{aligned} \tag{3.1}$$

Now, let E_j be the one dimensional subspace of $M^{N\times n}$ spanned by $A_j - P$. We denote by P_{E_j} and $P_{E_j^\perp}$ the orthogonal projection from $M^{n\times N}$ to E_j and its orthogonal complement $E_j^\perp$ respectively. We have, in (3.1) that

$$\begin{aligned}&\min\left\{\begin{array}{l}\left|\dfrac{t(A_j - P)}{|A_j - P|} + \alpha(A_j - P)v \otimes v\right|^p, \\ \left|\dfrac{t(A_j - P)}{|A_j - P|} + \alpha(A_j - P)v \otimes v - (A_j - P)\right|^p\end{array}\right\} \\ &= \min\left\{\begin{array}{l}\left|\dfrac{t(A_j - P)}{|A_j - P|} + P_{E_j}(\alpha(A_j - P)v \otimes v) + P_{E_j^\perp}(\alpha(A_j - P)v \otimes v)\right|^p, \\ \left|\dfrac{t(A_j - P)}{|A_j - P|} + P_{E_j}(\alpha(A_j - P)v \otimes v) + P_{E_j^\perp}(\alpha(A_j - P)v \otimes v) - (A_j - P)\right|^p\end{array}\right\} \\ &\leq 2^{p-1}\left[\min\left\{\begin{array}{l}\left|\dfrac{t(A_j - P)}{|A_j - P|} + P_{E_j}(\alpha(A_j - P)v \otimes v)\right|^p, \\ \left|\dfrac{t(A_j - P)}{|A_j - P|} + P_{E_j}(\alpha(A_j - P)v \otimes v) - (A_j - P)\right|^p\end{array}\right\}\right. \\ &\qquad \left. + \left|P_{E_j^\perp}(\alpha(A_j - P)v \otimes v)\right|^p\right].\end{aligned} \tag{3.2}$$

Let us calculate $P_{E_j}((A_j-P)v\otimes v)$ and $|P_{E_j^\perp}((A_j-P)v\otimes v)|$. We have

$$\begin{aligned} P_{E_j}((A_j-P)v\otimes v) &= \left(\frac{(A_j-P)}{|A_j-P|}\cdot (A_j-P)v\otimes v\right)\frac{(A_j-P)}{|A_j-P|}\\ &= \frac{\mu_j}{|A_j-P|^2}(A_j-P). \end{aligned} \tag{3.3}$$

Since

$$|(A_j-P)v\otimes v|^2 = |P_{E_j}((A_j-P)v\otimes v)|^2 + |P_{E_j^\perp}((A_j-P)v\otimes v)|^2, \tag{3.4}$$

while

$$|(A_j-P)v\otimes v|^2=\mu_j \qquad \text{and} \qquad |P_{E_j}((A_j-P)v\otimes v)|^2 = \frac{\mu_j^2}{|A_j-P|^2}. \tag{3.5}$$

Consequently,

$$|P_{E_j^\perp}((A_j-P)v\otimes v)|^2 = \mu_j - \frac{\mu_j^2}{|A_j-P|^2} = \frac{\mu_j(|A_j-P|^2-\mu_j)}{|A_j-P|^2}.$$

Therefore

$$\frac{|P_{E_j^\perp}((A_j-P)v\otimes v)|^2}{|P_{E_j}((A_j-P)v\otimes v)|^2} = \frac{\frac{\mu_j(|A_j-P|^2-\mu_j)}{|A_j-P|^2}}{\frac{\mu_j^2}{|A_j-P|^2}} = \frac{|A_j-P|^2-\mu_j}{\mu_j}.$$

We then have

$$|P_{E_j^\perp}((A_j-P)v\otimes v)|^p = \left(\frac{|A_j-P|^2-\mu_j}{\mu_j}\right)^{p/2}|P_{E_j}((A_j-P)v\otimes v)|^p. \tag{3.6}$$

Now we substitute (3.6) into (3.2),

$$\begin{aligned} &\operatorname{dist}^p(B_j+U_j,K_j)\\ &\le 2^{p-1}\min\left\{\begin{array}{l} \left|\dfrac{t(A_j-P)}{|A_j-P|}+P_{E_j}(\alpha(A_j-P)v\otimes v)\right|^p,\\ \left|\dfrac{t(A_j-P)}{|A_j-P|}+P_{E_j}(\alpha(A_j-P)v\otimes v)-(A_j-P)\right|^p \end{array}\right\}\\ &\quad + 2^{p-1}\left|P_{E_j^\perp}(\alpha(A_j-P)v\otimes v)\right|^p\\ &= 2^{p-1}\min\left\{\begin{array}{l} \left|\dfrac{t(A_j-P)}{|A_j-P|}+P_{E_j}(\alpha(A_j-P)v\otimes v)\right|^p,\\ \left|\dfrac{t(A_j-P)}{|A_j-P|}+P_{E_j}(\alpha(A_j-P)v\otimes v)-(A_j-P)\right|^p \end{array}\right\}\\ &\quad + 2^{p-1}\left(\frac{|A_j-P|^2-\mu_j}{\mu_j}\right)^{p/2}\left|P_{E_j}(\alpha(A_j-P)v\otimes v)\right|^p \end{aligned}$$

$$
\begin{aligned}
&= 2^{p-1}\min\left\{\begin{array}{l}\left|\dfrac{t(A_j-P)}{|A_j-P|}+\dfrac{\alpha\mu_j}{|A_j-P|^2}(A_j-P)\right|^p,\\ \left|\dfrac{t(A_j-P)}{|A_j-P|}+\dfrac{\alpha\mu_j}{|A_j-P|^2}(A_j-P)-(A_j-P)\right|^p\end{array}\right\}\\
&\quad+2^{p-1}\left(\frac{|A_j-P|^2-\mu_j}{\mu_j}\right)^{p/2}\left|\frac{\alpha\mu_j}{|A_j-P|^2}(A_j-P)\right|^p\\
&= 2^{p-1}|A_j-P|^p\left[\min\left\{\begin{array}{l}\left|\dfrac{\alpha\mu_j}{||A_j-P|^2}-\left(-\dfrac{t}{|A_j-P|}\right)\right|^p,\\ \left|\dfrac{\alpha\mu_j}{|A_j-P|^2}-\left(1-\dfrac{t}{|A_j-P|}\right)\right|^p\end{array}\right\}\right.\\
&\quad\left.+\left(\frac{|A_j-P|^2-\mu_j}{\mu_j}\right)^{p/2}\left|\frac{\alpha\mu_j}{|A_j-P|^2}\right|^p\right].
\end{aligned}
\tag{3.7}
$$

If we replace α by β, we can obtain a similar estimate of $\operatorname{dist}^p(B_j+V_j,K_j)$. Now for large $j>0$,

$$\left(\frac{|A_j-P|^2-\mu_j}{\mu_j}\right)^{p/2}<1.$$

We may use Lemma 2.7 to estimate $R\operatorname{dist}^p(B_j,K_j)$ by careful choice of α, β and θ_j .

We choose α, such that

$$\frac{\alpha\mu_j}{|A_j-P|^2}-\left(-\frac{t}{|A_j-P|}\right)=0,$$

that is,

$$\alpha=-\frac{t|A_j-P|}{\mu_j},$$

so that

$$U_j=-\frac{t|A_j-P|}{\mu_j}(A_j-P)v\otimes v.$$

Similarly, we take β, such that

$$\frac{\beta\mu_j}{|A_j-P|^2}-\left(1-\frac{t}{|A_j-P|}\right)=0,$$

which gives

$$\beta=\left(1-\frac{t}{|A_j-P|}\right)\frac{|A_j-P|^2}{\mu_j}.$$

Similarly, we choose

$$V_j=\left(1-\frac{t}{|A_j-P|}\right)\frac{|A_j-P|^2}{\mu_j}(A_j-P)v\otimes v.$$

We see that 0 is a convex combination of U_j and V_j if

$$1-\frac{t}{|A_j-P|}>0,\qquad \text{or, equivalently}\qquad \frac{t}{|A_j-P|}<1.$$

This is possible for large $j > 0$ because $|A_j| \to \infty$ as $j \to \infty$ and $t > 0$ is fixed. For U_j and V_j given above, let us find θ_j, such that $0 < \theta_j < 1$ and $\theta_j U_j + (1-\theta_j)V_j = 0$. This is equivalent to

$$-\frac{t|A_j - P|}{\mu_j}\theta_j + (1-\theta_j)\left(1 - \frac{t}{|A_j - P|}\right)\frac{|A_j - P|^2}{\mu_j} = 0$$

which implies

$$\theta_j = 1 - \frac{t}{|A_j - P|}, \qquad 1 - \theta_j = \frac{t}{|A_j - P|}.$$

Obviously, $0 < \theta_j < 1$ for large $j > 0$. Now we have

$$\begin{aligned}
&R\,\mathrm{dist}^p(B_j, K) \le \mathrm{dist}^p(B_j, K_j) \le \\
&\theta_j\,\mathrm{dist}^p(B_j + U_j, K_j) + (1-\theta_j)\,\mathrm{dist}^p(B_j + V_j, K_j) \\
&\le 2^{p-1}|A_j - P|^p\left[\theta_j\left(\frac{|A_j - P|^2 - \mu_j}{\mu_j}\right)^{p/2}|\frac{\alpha\mu_j}{|A_j - P|^2}|^p\right. \\
&\left.+(1-\theta_j)\left(\frac{|A_j - P|^2 - \mu_j}{\mu_j}\right)^{p/2}|\frac{\beta\mu_j}{|A_j - P|^2}|^p\right] \\
&= 2^{p-1}|A_j - P|^p\left(\frac{|A_j - P|^2 - \mu_j}{\mu_j}\right)^{p/2}\left(\frac{\mu_j}{|A_j - P|^2}\right)^p \\
&(\theta_j|\alpha|^p + (1-\theta_j)|\beta|^p) \\
&\le 2^{p-1}|A_j - P|^p\left(\frac{|A_j - P|^2 - \mu_j}{\mu_j}\right)^{p/2}\left(\frac{\mu_j}{|A_j - P|^2}\right)^p \\
&\left[\left(\frac{t|A_j - P|}{\mu_j}\right)^p + \frac{t}{|A_j - P|}\right] \\
&= 2^{p-1}t^p\left(\frac{|A_j - P|^2 - \mu_j}{\mu_j}\right)^{p/2} \\
&+ 2^{p-1}t\left(\frac{|A_j - P|^2}{\mu_j}\right)^{(p-1)/2}\left(\frac{|A_j - P|^2 - \mu_j}{\mu_j^{1/p}}\right)^{p/2} \\
&\le 2^{p-1}t^p\left(\frac{|A_j - P|^2 - \mu_j}{\mu_j}\right)^{p/2} \\
&+ 2^{p-1}tn^{(p-1)/2}\left(\frac{|A_j - P|^2 - \mu_j}{\mu_j^{1/p}}\right)^{p/2} =: I_1 + I_2,
\end{aligned} \tag{3.8}$$

where we have used the facts that $1 - t/|A_j - P| < 1$ and $\frac{|A_j - P|^2}{\mu_j} \le n$. The second inequality follows from the fact that μ_j is the greatest eigenvalue of $(A_j - P)^T(A_j - P)$.

From (3.8) we see that the conclusion will follow if we can prove that

$$\lim_{j\to\infty} \frac{|A_j - P|^2 - \mu_j}{\mu_j^{1/p}} = 0. \tag{3.9}$$

Let us consider the polar decomposition of A_j as

$$A_j = R|A_j| = RQ^T \operatorname{diag}(\sqrt{\lambda_j^{(1)}}, \cdots, \sqrt{\lambda_j^{(n-1)}}, \sqrt{\Lambda_j})Q,$$

where R is a $\max\{N,n\} \times n$ matrix with columns orthogonal to one another and with the norm of each column 1, Λ_j is the greatest eigenvalue of $A_j^T A_j$, $|A_j| = \sqrt{A_j^T A_j}$ and Q is an $n \times n$ orthogonal matrix such that

$$|A_j| = Q^T \operatorname{diag}(\sqrt{\lambda_j^{(1)}}, \cdots, \sqrt{\lambda_j^{(n-1)}}, \sqrt{\Lambda_j})Q.$$

Now, $|A_j - P|^2 = |A_j|^2 - 2\operatorname{tr}(A_j^T P) + |P|^2$ and

$$\begin{aligned}\operatorname{tr}(A_j^T P) &= \operatorname{tr}(Q^T \operatorname{diag}(\sqrt{\lambda_j^{(1)}}, \cdots, \sqrt{\lambda_j^{(n-1)}}, \sqrt{\Lambda_j})QR^T P)\\ &= \operatorname{tr}(\operatorname{diag}(\sqrt{\lambda_j^{(1)}}, \cdots, \sqrt{\lambda_j^{(n-1)}}, \sqrt{\Lambda_j})QR^T PQ^T).\end{aligned}$$

Here we have used the simple fact that $\operatorname{tr} Q^T CQ = \operatorname{tr} C$ when Q is orthogonal and C is a square matrix. If we let $QR^T PQ^T = B$ and let b_{ii} be the (i,i) entry of B for $i = 1, 2, \cdots, n$, we have

$$\operatorname{tr}(A_j^T P) = \operatorname{tr}(\operatorname{diag}(\sqrt{\lambda_j^{(1)}}, \cdots, \sqrt{\lambda_j^{(n-1)}}, \sqrt{\Lambda_j})B) = \left(\sum_{i=1}^{n-1} \sqrt{\lambda_j^{(i)}} b_{ii}\right) + \sqrt{\Lambda_j} b_{nn}. \tag{3.10}$$

Now we estimate μ_j from below. Letting $v = Q^T(0, \cdots, 0, 1)^T$, we have

$$\begin{aligned}\mu_j &\geq |(A_j - P)v|^2 = |A_j v|^2 - 2(v^T A_j^T P v) + |Pv|^2\\ &= \Lambda_j - 2(0, \cdots, 0, 1)QQ^T \operatorname{diag}(\sqrt{\lambda_j^{(1)}}, \cdots, \sqrt{\lambda_j^{(n-1)}}, \sqrt{\Lambda_j})QR^T PQ^T(0, \cdots, 0, 1)^T + |Pv|^2\\ &= \Lambda_j - 2(0, \cdots, 0, 1)\operatorname{diag}(\sqrt{\lambda_j^{(1)}}, \cdots, \sqrt{\lambda_j^{(n-1)}}, \sqrt{\Lambda_j})B(0, \cdots, 0, 1)^T + |Pv|^2\\ &= \Lambda_j - 2\sqrt{\Lambda_j} b_{nn} + |Pv|^2 \geq \Lambda_j - 2\sqrt{\Lambda_j} b_{nn},\end{aligned}$$

so that

$$\mu_j \geq \Lambda_j - 2\sqrt{\Lambda_j} b_{nn}, \tag{3.11}$$

and $|b_{nn}| \le |P|$. Combining (3.10) and (3.11), we see that

$$\begin{aligned}
&\frac{|A_j - P|^2 - \mu_j}{\mu_j^{1/p}} \\
&\quad\le \frac{|A_j|^2 - 2\left(\sum_{i=1}^{n-1}\sqrt{\lambda_j^{(i)}}b_{ii}\right) + \sqrt{\Lambda_j}b_{nn} + |P|^2 - (\Lambda_j - 2\sqrt{\Lambda_j}b_{nn}}{(\Lambda_j - 2\sqrt{\Lambda_j}b_{nn})^{1/p}} \\
&\quad\le \frac{|A_j|^2 - \Lambda_j}{\Lambda_j^{1/p}}\left(\frac{\Lambda_j}{\Lambda_j - 2\sqrt{\Lambda_j}|P|}\right)^{1/p} \\
&\quad+ \frac{2(\sum_{i=1}^{n-1}\lambda_j^{(i)})^{1/2}(\sum_{i=1}^{n-1}b_{ii}^2)^{1/2} + |P|^2}{(\Lambda_j - 2\sqrt{\Lambda_j}|P|)^{1/p}} \\
&\quad\le \left(\frac{\lambda_j}{\Lambda_j^{1/p}} + \frac{2\lambda_j^{1/2}|P| + |P|^2}{\Lambda_j^{1/p}}\right)\left(\frac{\Lambda_j}{\Lambda_j - 2\sqrt{\Lambda_j}|P|}\right)^{1/p}.
\end{aligned} \tag{3.12}$$

Since

$$\lim_{j\to\infty}\Lambda_j = +\infty, \qquad \lim_{j\to\infty}\frac{\lambda_j}{\Lambda_j^{1/p}} = 0 \qquad \text{and} \qquad \lim_{j\to\infty}\frac{\Lambda_j}{\Lambda_j - 2\sqrt{\Lambda_j}|P|} = 1,$$

we see that (3.9) follows. Therefore in (3.8), $I_2 \to 0$, as $j \to \infty$, where

$$I_2 = 2^{p-1}tn^{(p-1)/2}\left(\frac{|A_j - P|^2 - \mu_j}{\mu_j^{1/p}}\right)^{p/2}.$$

From (3.11) we also see that $\mu_j \to \infty$ as $j \to \infty$. This together with (3.9) implies

$$I_1 = 2^{p-1}t^p\left(\frac{|A_j - P|^2 - \mu_j}{\mu_j}\right)^{p/2} \to 0, \qquad \text{as } j \to \infty.$$

therefore, finally we have,

$$R\,\mathrm{dist}^p(P + tA_0, K) \le \limsup_{j\to\infty} R\,\mathrm{dist}^p\left(P + t\frac{A_j - P}{|A_j - P|}, K_j\right) = 0.$$

The conclusion follows. □

Proof of Proposition 1.6. For $A \in S_+$, there is a rotation $R \in M^{2\times 2}$, such that $A = R^T\,\mathrm{diag}(\lambda, 1/\lambda)R$. Let us first examine diagonal matrices in S_+. We have, for every $t > 0$, $A(t) = \mathrm{diag}(t, 1/t) \in S_+$. When $t > 1$, the greatest eigenvalue of $A^T(t)A(t)$ is t^2, the other eigenvalue is $1/t^2$. We see that for every $t > 1$, and $p \ge 1$,

$$\lim_{t\to\infty}\frac{1/t^2}{(t^2)^{1/p}} = 0, \qquad \text{and} \qquad \lim_{t\to\infty}A(t)/\sqrt{t^2} = \mathrm{diag}(1, 0).$$

From Theorem 1.5, we see that

$$\{P + s\,\mathrm{diag}(1,0),\ P \in S_+,\ s>0\} \subset R_p(S_+).$$

Similarly, we see that for every rotation R,

$$\{P + sR^T\,\mathrm{diag}(1,0)R,\ P \in S_+,\ s>0\} \subset R_p(S_+).$$

Therefore $C(S_+) \subset R_p(S_+)$, since we always have $R_p(S_+) \subset Q_p(S_+) \subset C(S_+)$. The first conclusion follows. The proof for S_- is similar.

Now for $R_p(S)$. Since S_+ and S_- are subsets of S, the p-rank-one connections of S_+ and S_- are also p-rank-one connections of S and therefore

$$\{P + sR^T\,\mathrm{diag}(\pm 1,0)R,\ P \in S,\ s>0\} \subset R_p(S),$$

for every orthogonal matrix R. Therefore $C(S) \subset R_p(S)$. The proof is complete. □

Proof of Proposition 1.8. Let

$$K = \{P \in M^{n\times n},\ P^T = P, \det P \geq 1, \text{ and } P \text{ is positive definite}\}.$$

We first prove that $K \subset R_p(S_+^{(n)})$. Then we show that K is convex so that $C(S_+^{(n)}) \subset K$. Since $R_p(S_+^{(n)}) \subset C(S_+^{(n)})$, the conclusion then follows.

In a similar manner to the proof of Proposition 1.6, if we take

$$A(t) = \mathrm{diag}(t, 1/t^{1/(n-1)}, \cdots, 1/t^{1/(n-1)}),$$

we see that $\det A(t) = 1$ for all $t>0$, and if we apply Theorem 1.5 and follow the proof of Proposition 1.6, we see that

$$\{P + sR^T\,\mathrm{diag}(1,0,\cdots,0)R, P \in S_+^{(n)}, s \geq 0, R \in O(n)\} \subset R_p(S_+^{(n)}).$$

Now, for every $A \in K \setminus S_+^{(n)}$, let $\lambda_1 \geq \cdots, \geq \lambda_n$ be its eigenvalues. There is a rotation R_0, such that $A = R_0^T\,\mathrm{diag}(\lambda_1,\cdots,\lambda_n)R_0$. Since $\lambda_1\lambda_2\cdots\lambda_n > 1$, $\lambda_1 > 1/(\lambda_2\cdots\lambda_n)$. Let $\lambda = 1/(\lambda_2\cdots\lambda_n)$ and $s = \lambda_1 - 1/(\lambda_2\cdots\lambda_n) > 0$, we have

$$A = R_0^T\,\mathrm{diag}(\lambda,\lambda_2\cdots,\lambda_n)R_0 + sR_0^T\,\mathrm{diag}(1,0,\cdots,0)R_0,$$

hence

$$A \in \{P + sR^T\,\mathrm{diag}(1,0,\cdots,0)R, P \in S_+^{(n)}, s \geq 0, R \in O(n)\} \subset R_p(S_+^{(n)}).$$

It is well known that K is convex. In fact, let A, $B \in K$, and we let

$$a^n = \det A \geq 1, \qquad b^n = \det B \geq 1$$

and we assume that $1 \leq a \leq b$. Let $0<t<1$, and let $\sigma_1 \geq \cdots \geq \sigma_n$ be the eigenvalues of $C = \sqrt{A^{-1}}B\sqrt{A^{-1}}$, we see that

$$\det C = \sigma_1\cdots\sigma_n b^n/a^n \geq 1,$$

and

$$\det(tA + (1-t)B) = a^n \det(tI + (1-t)C) \geq \det(tI + (1-t)C).$$

We see that $tA + (1-t)B \in K$ if $\det(tI + (1-t)C) \geq 1$. This follows essentially from the arithmetic-geometric inequality. We have

$$\begin{aligned} &\det(tI + (1-t)C) = \det[\operatorname{diag}(t + (1-t)\sigma_1, \cdots, t + (1-t)\sigma_n)] \\ &= [(t + (1-t)\sigma_1] \cdots [(t + (1-t)\sigma_n] \\ &= \sum_{k=0}^{n} \left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} \sigma_{i_1} \cdots \sigma_{i_k} \right) t^{n-k}(1-t)^k. \end{aligned} \tag{3.13}$$

From the arithmetic-geometric means inequality,

$$\begin{aligned} &\sum_{1 \leq i_1 < \cdots < i_k \leq n} \sigma_{i_1} \cdots \sigma_{i_k} \geq \binom{n}{k} \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n} \sigma_{i_1} \cdots \sigma_{i_k} \right)^{\frac{1}{\binom{n}{k}}} \\ &= \binom{n}{k} (\sigma_1 \cdots \sigma_n)^{\frac{k}{\binom{n}{k}}} \geq \binom{n}{k} \end{aligned} \tag{3.14}$$

which follows from the fact that $\sigma_1 \cdots \sigma_n = \det C \geq 1$. Combining (3.13) and (3.14), and applying the binomial theorem, we have

$$\det(tI + (1-t)C) \geq \sum_{k=0}^{n} \binom{n}{k} t^{n-k}(1-t)^k = (t + (1-t))^n = 1.$$

Hence $tA + (1-t)B \in K$, K is convex. The proof is complete. □

Remark 3.1. For $p \geq n$, we know that $\mathbf{Q}_n(S_+^{(n)}) = S_+^{(n)}$ because the following quasiconvex function $f : M^{n\times n} \to R_+$ vanishes only on $S_+^{(n)}$:

$$f(P) = \operatorname{dist}^n(P, C(S_+^{(n)})) + |\det(P) - 1|.$$

4. Connected sets in $M^{2\times 2}$ without rank-one connections

In this section, we study quasiconvex hulls for connected, closed, and unbounded subsets of $M^{2\times 2}$ which do not have rank-one connections. We give conditions such that the corresponding quasiconvex hulls are larger than the sets themselves. We consider this type of sets because they have very simple representations [36] so we can translate the conditions for rank-one connections at infinity into conditions on the assymptotic behaviours of certain Lipschitz mappings at infinity.

Theorem 4.1. *Let $K \subset M^{2\times 2}$ with $0 \in K$ be an unbounded closed set without rank-one connections and $f : P_{E_\partial}(K) \to E_{\bar\partial}$ be its associated Lipschitz mapping (the case $f : P_{E_{\bar\partial}} \to E_\partial$ is similar) satisfying*

$$|f(A) - f(B)| < |A - B|, \qquad A, B \in P_{E_\partial}(K),\ A \neq B.$$

Then Khas a p-rank-one connection at infinity if and only if

$$\limsup_{|A|\to\infty, A \in P_{E_\partial}(K)} \frac{|P| - |f(A)|}{|P|^{1/p}} = 0. \tag{4.1}$$

When (4.1) holds for some $1 \leq p < \infty$, $Q_p(K) \neq K$.

Remark 4.2. The condition $0 \in K$ is a normalization assumption for the function f which does not affect the generality of the result because $Q_p(K+A) = Q_p(K) + A$ for all $1 \le p < \infty$. However, the condition $0 \in K$ implies $|f(A)| < |A|$.

In [37], the Lipschitz condition $|f(A) - f(B)| \le c(p)|A - B|$ for a certain constant $0 < c(p) < 1$, was assumed to ensure that $Q_p(K) = K$ [37]. This Lipschitz condition also implies

$$\limsup_{|A|\to\infty,\, A\in P_{E_\partial}(K)} \frac{|f(A)|}{|A|} \le c(p) < 1$$

so that

$$\limsup_{|A|\to\infty,\, A\in P_{E_\partial}(K)} \frac{|A| - |f(A)|}{|A|^{1/p}} = +\infty.$$

However, since $c(p)$ is obtained as a bound of a certain singular integral, we do not know the value of $c(p)$ explicitly (except for $p = 2$ where $c(2)$ can be chosen as any positive number strictly less than 1). I do not know whether $Q_2(K) = K$ if there exists $\alpha > 0$ such that

$$\limsup_{|A|\to\infty,\, A\in P_{E_\partial}(K)} \frac{|P| - |f(A)|}{|P|^{1/2}} \ge \alpha \tag{4.2}$$

Notice that when $p = 1$, (4.1) is equivalent to

$$\limsup_{|A|\to\infty,\, A\in P_{E_\partial}(K)} \frac{|f(A)|}{|A|} = 1.$$

We would like to know whether

$$\limsup_{|A|\to\infty,\, A\in P_{E_\partial}(K)} \frac{|f(A)|}{|A|} < 1,$$

would imply $Q_1(K) = K$. In [39], $Q_1(K) = K$ was established under the condition that $f \equiv 0$, or, equivalently, $K \subset E_\partial$. However, we have the following result for $p = 2$.

Theorem 4.3. *Let $K \subset M^{2\times 2}$ with $0 \in K$, be an unbounded, closed and connected set without rank-one connections. Let $f : P_{E_\partial}(K) \to E_{\bar\partial}$ be its associated Lipschitz mapping (the case $f : P_{E_{\bar\partial}} \to E_\partial$ is similar) satisfying*

$$|f(A) - f(B)| < |A - B|.$$

Then

$$\limsup_{|A|\to\infty,\, A\in P_{E_\partial}(K)} \frac{|f(A)|}{|A|} = \alpha < 1 \tag{4.3}$$

implies $Q_2(K) = K$.

Remark 4.4. Theorem 4.3 improves upon Theorem 4.1 in [37] where a much stronger assumption

$$|f(A) - f(B)| \le k|A - B|$$

with $0 < k < 1$ is used.

Proof of Theorem 4.1. Let us find the eigenvalues Λ, λ (with $\lambda \le \Lambda$) for a general element $(A+f(A))^T(A+f(A))$ and noticing that A and $f(A)$ are orthogonal to each other. We have

$$\Lambda(A)+\lambda(A) = \mathrm{tr}(A+f(A))^T(A+f(A)) = |A|^2+|f(A)|^2,$$
$$\Lambda(A)\lambda(A) = \det(A+f(A))^T(A+f(A)) = (\det(A+f(A)))^2 = \frac{1}{4}(|A|^2-|f(A)|^2)^2.$$

Therefore,

$$\Lambda(A) = \frac{1}{2}\left(|A|^2+|f(A)|^2+\sqrt{(|A|^2+|f(A)|^2)^2-(|A|^2-|f(A)|^2)^2}\right)$$
$$= \frac{1}{2}\left(|A|+|f(A)|\right)^2.$$

Similarly,

$$\lambda(A) = \frac{1}{2}\left(|A|-|f(A)|\right)^2,$$

so that

$$\begin{aligned}\frac{\lambda(A)}{(\Lambda(A))^{1/p}} &= \left(\frac{|A|-|f(A)|}{(|A|+|f(A)|)^{1/p}}\right)^2\\ &= \left(\frac{|A|-|f(A)|}{|A|^{1/p}}\right)^2\frac{1}{\left(1+\frac{|f(A)|}{|A|}\right)^{2/p}}.\end{aligned} \tag{4.4}$$

From Definition 1.4, we see that K has a p-rank-one conncetion at infinity if

$$\limsup_{|A|\to\infty,\, A\in P_{E_\partial}(K)} \frac{\lambda(A)}{(\Lambda(A))^{1/p}} = 0. \tag{4.5}$$

Since $1\le 1+|f(A)|/|A|\le 2$, we have, combining (4.3) and (4.5) that

$$\limsup_{|A|\to\infty,\, A\in P_{E_\partial}(K)} \frac{|A|-|f(A)|}{|A|^{1/p}} = 0.$$

Hence we have a sequence $(A_j+f(A_j))$, such that

$$\lim_{j\to\infty}\frac{|A_j|-|f(A_j)|}{|A_j|^{1/p}} = 0,$$
$$\lim_{j\to\infty}\frac{A_j+f(A_j)}{\sqrt{\Lambda(A_j)}} = A_0+B_0,$$

where $A_0\in E_\partial$, $B_0\in E_{\bar\partial}$ respectively, $\mathrm{rank}(A_0+B_0)=1$ and $|A_0+B_0|=1$. From Theorem 1.5, we have

$$\{A+f(A)+t(A_0+B_0),\ A+f(A)\in K,\ t\ge 0\}\subset Q_p(K).$$

Therefore $Q_p(K)$ has rank-one connections, which implies that $Q_p(K)\ne K$. □

Proof of Theorem 4.3. We extend f to be defined on E_∂ such that $|f(A) - f(B)| \leq |A - B|$, while on $P_{E_\partial}(K)$ we have the strict inequality $|f(A) - f(B)| < |A - B|$. Let $P \in Q_2(K)$. There exists a sequence $\phi_j \in C_0^\infty(U, R^2)$ such that

$$\lim_{j\to\infty} \int_U \operatorname{dist}^2(P + D\phi_j, K)dx = 0,$$

where $U \subset R^2$ is the unit square.

As in the proof of [37, Th.4.1], we first consider

$$I_j = \int_U [|P_{E_{\bar\partial}}(D\phi_j)|^2 - |f(P_{E_\partial}(P + D\phi_j)) - f(P_{E_\partial}(P))|^2]dx, \tag{4.6}$$

to show that $D\phi_j$ is bounded in $L^2(U)$. Since $|f(A)|/|A| \to \alpha < 1$ as $|A| \to \infty$, there exists $M > 0$, whenever $|A| \geq M$, $|f(A)|/|A| \leq (1 + \alpha)/2$. Now, on the one hand, since

$$\int_U |P_{E_{\bar\partial}}(D\phi_j)|^2 dx = \int_U |P_{E_\partial}(D\phi_j)|^2 dx, \tag{4.7}$$

we have

$$\begin{aligned}
I_j &= \int_U [|P_{E_\partial}(D\phi_j)|^2 - |f(P_{E_\partial}(P + D\phi_j)) - f(P_{E_\partial}(P))|^2]dx \\
&\geq \int_{\{x\in U, |P_{E_\partial}(D\phi_j)(x)|\geq M+|P|\}} [|P_{E_\partial}(D\phi_j)|^2 - |f(P_{E_\partial}(P + D\phi_j)) - f(P_{E_\partial}(P))|^2]dx \\
&\geq \int_{\{x\in U, |P_{E_\partial}(D\phi_j)(x)|\geq M+|P|\}} |P_{E_\partial}(D\phi_j)|^2 dx \\
&- \int_{\{x\in U, |P_{E_\partial}(D\phi_j)(x)|\geq M+|P|\}} \left(\frac{(1+\alpha)^2(1+\epsilon)}{4}|P_{E_\partial}(D\phi_j)|^2\right) dx \\
&- C(\epsilon)\int_{\{x\in U, |P_{E_\partial}(D\phi_j)(x)|\geq M+|P|\}} (|P_{E_\partial}(P)| + |f(P_{E_\partial}(P))|)^2 dx \\
&= \left(1 - \frac{(1+\alpha)^2(1+\epsilon)}{4}\right) \int_{\{x\in U, |P_{E_\partial}(D\phi_j)(x)|\geq M+|P|\}} |P_{E_\partial}(D\phi_j)|^2 dx \\
&- C(\epsilon)(|P_{E_\partial}(P)| + |f(P_{E_\partial}(P))|)^2 \\
&= \mu \int_{\{x\in U, |P_{E_\partial}(D\phi_j)(x)|\geq M+|P|\}} |P_{E_\partial}(D\phi_j)|^2 dx - C(\epsilon, P),
\end{aligned} \tag{4.8}$$

where $\epsilon > 0$ is a positive integer such that $\mu = 1 - (1 + \alpha)^2(1 + \epsilon)/4 > 0$; $C(\epsilon) > 0$, $C(\epsilon, P) > 0$ are positive constants.

On the other hand, we may extend f to be defined on E_∂ such that $|f(A) - f(B)| \leq |A - B|$.

From Lemma 2.9 we have

$$|P_{E_{\bar\partial}}(A) - f(P_{E_\partial}(A))| \leq \sqrt{2}\operatorname{dist}(A, K) \tag{4.9}$$

for every $A \in M^{2\times 2}$, so that

$$\begin{aligned}
I_j &= \int_U |P_{E_{\bar\partial}}(D\phi_j)|^2 - |f(P_{E_\partial}(P+D\phi_j)) - f(P_{E_\partial}(P))|^2]dx \\
&= \int_U [|P_{E_{\bar\partial}}(D\phi_j)|^2 dx \\
&- \int_U \left|[f(P_{E_\partial}(P+D\phi_j)) - P_{E_{\bar\partial}}(P+D\phi_j)] + [P_{E_{\bar\partial}}(D\phi_j)] + [P_{E_{\bar\partial}}(P) - f(P_{E_\partial}(P))]\right|^2 dx \\
&\le -\int_U [P_{E_{\bar\partial}}(D\phi_j)]\cdot \big([f(P_{E_\partial}(P+D\phi_j)) - P_{E_{\bar\partial}}(P+D\phi_j)] + [P_{E_{\bar\partial}}(P) - f(P_{E_\partial}(P))]\big)\,dx
\end{aligned}$$

$$\begin{aligned}
&\le 2\sqrt{2}\int_U |P_{E_{\bar\partial}}(D\phi_j)|[\mathrm{dist}(P+D\phi_j,K) + \mathrm{dist}(P,K)]dx \\
&\le \frac{\mu}{2}\int_U |P_{E_{\bar\partial}}(D\phi_j)|^2 dx + C(\mu)\int_U [\mathrm{dist}^2(P+D\phi_j,K) + \mathrm{dist}^2(P,K)]dx \qquad (4.10)\\
&= \frac{\mu}{2}\int_U |P_{E_\partial}(D\phi_j)|^2 dx + C(\mu)\int_U [\mathrm{dist}^2(P+D\phi_j,K) + \mathrm{dist}^2(P,K)]dx
\end{aligned}$$

$$\begin{aligned}
&\le \frac{\mu}{2}\int_{\{x\in U,\,|P_{E_\partial}(D\phi_j)(x)|\ge M+|P|\}} |P_{E_\partial}(D\phi_j)|^2 dx \\
&+ \frac{\mu}{2}\int_{\{x\in U,\,|P_{E_\partial}(D\phi_j)(x)|< M+|P|\}} (M+|P|)^2 dx \\
&+ C(\mu)\left[\mathrm{dist}^2(P,K) + \int_U \mathrm{dist}^2(P+D\phi_j,K)\right]dx,
\end{aligned}$$

where we have used (4.9) and the inequality $ab \le \epsilon a^2 + C(\epsilon)b^2$ for real numbers a, b and $\epsilon > 0$ and $C(\epsilon)$ is a constant. Combining (4.8) and (4.10), we obtain

$$\int_{\{x\in U,\,|P_{E_\partial}(D\phi_j)(x)|\ge M+|P|\}} |P_{E_\partial}(D\phi_j)(x)|^2 dx \le C$$

for some constant $C > 0$ independent of j, because $\int_U \mathrm{dist}^2(P+D\phi_j,K)dx \to 0$ as $j\to\infty$. Therefore $P_{E_\partial}(D\phi_j)$ is bounded in $L^2(U, M^{2\times 2})$. Then (4.7) implies that $D\phi_j$ is bounded in $L^2(U, M^{2\times 2})$.

We may assume, up to a subsequence, that $\phi_j \rightharpoonup 0$ in $W^{1,2}(U,R^2)$ because we can extend ϕ_j periodically outside U and then define $\psi_j = \frac{1}{j}\phi(jx)$. It is then easy to see that $\psi_j \rightharpoonup 0$ in $W^{1,2}(U,R^2)$ and

$$\lim_{j\to\infty}\int_U \mathrm{dist}^2(P+D\psi_j,K)dx = 0,$$

so that (ψ_j) satisfies our requirements.

Now we show that $D\phi_j \to 0$ almost everywhere in U. Similar to the proof of the boundedness of $D\phi_j$ in L^2, we introduce

$$I_{j,k} = \int_U [|P_{E_{\bar\partial}}(D\phi_j - D\phi_k)|^2 - |f(P_{E_\partial}(P+D\phi_j)) - f(P_{E_\partial}(P+D\phi_k))|^2]dx,$$

motivated from [26]. We use the so-called 'biting' Young-measures [19] to study the limit of $I_{j,k}$. Since $D\phi_j$ is bounded in L^2, there exists a sequence of measurable sets $U_m \subset U$, $U_{m+1} \subset U_m$ and $\text{meas}(U_m) \to 0$, a subsequence of $D\phi_j$ (we still denote it by using subscript j) and a family of Young measures $\{\nu_x\}_{x\in U}$, such that for every Carathéodory function $f : U \times M^{2\times 2}$ satisfying $|f(x,A)| \leq a(x) + C|A|^2$,

$$\lim_{j\to\infty} \int_{U\setminus U_m} f(x, D\phi_j(x))dx = \int_{U\setminus U_m} \left(\int_{M^{2\times 2}} f(x,\lambda)d\nu_x \right) dx.$$

for each fixed $m > 0$.

For $I_{j,k}$, we have, on the one hand,

$$\begin{aligned} I_{j,k} &= \int_U [|P_{E_\partial}(D\phi_j - D\phi_k)|^2 - |f(P_{E_\partial}(P + D\phi_j)) - f(P_{E_\partial}(P + D\phi_k))|^2]dx \\ &\geq \int_{U\setminus U_m} |P_{E_\partial}(D\phi_j - D\phi_k)|^2 - |f(P_{E_\partial}(P + D\phi_j)) - f(P_{E_\partial}(P + D\phi_k))|^2]dx, \end{aligned} \tag{4.11}$$

because f is a 1-Lipschitz function. If we pass to the limits, $j \to \infty$, then $k \to \infty$, we have

$$\begin{aligned} &\liminf_{k\to\infty} \liminf_{j\to\infty} I_{j,k} \\ &\geq \lim_{k\to\infty} \lim_{j\to\infty} \int_{U\setminus U_m} [|P_{E_\partial}(D\phi_j - D\phi_k)|^2 \\ &- |f(P_{E_\partial}(P + D\phi_j)) - f(P_{E_\partial}(P + D\phi_k))|^2]dx \\ &= \int_{U\setminus U_m} \int_{M^{2\times 2}} \int_{M^{2\times 2}} [|P_{E_\partial}(P+\lambda) - P_{E_\partial}(P+\tau)|^2 \\ &- |f(P_{E_\partial}(P+\lambda)) - f(P_{E_\partial}(P+\tau))|^2] d\nu_x(\lambda) d\nu_x(\tau) dx. \end{aligned} \tag{4.12}$$

On the other hand (c.f. the calculations in (4.10)), we can estimate $I_{j,k}$ from above by

$$\begin{aligned} I_{j,k} &\leq 2\sqrt{2} \int_U |P_{E_{\bar\partial}}(D\phi_j - D\phi_k)| [\text{dist}(P + D\phi_j, K) + \text{dist}(P + D\phi_k, K)]dx \\ &\leq 2\sqrt{2} \left(\int_U |P_{E_{\bar\partial}}(D\phi_j - D\phi_k)|^2 dx \right)^{1/2} \\ &\left(\int_U \text{dist}^2(P + D\phi_j, K)dx \right)^{1/2} \left(\int_U \text{dist}^2(P + D\phi_k, K)dx \right)^{1/2}. \end{aligned} \tag{4.13}$$

Since $\int_U |P_{E_{\bar\partial}}(D\phi_j - D\phi_k)|^2 dx$ is bounded and

$$\lim_{j\to\infty} \int_U \text{dist}^2(P + D\phi_j, K)dx = 0, \quad \lim_{k\to\infty} \int_U \text{dist}^2(P + D\phi_k, K)dx = 0,$$

we let $j \to \infty$, then $k \to \infty$,

$$\limsup_{k\to\infty} \limsup_{j\to\infty} I_{j,k} \leq 0.$$

Combining (4.12) and (4.13) we see that

$$\int_U \int_{M^{2\times 2}} \int_{M^{2\times 2}} [|P_{E_\partial}(P+\lambda) - P_{E_\partial}(P+\tau)|^2 \\ - |f(P_{E_\partial}(P+\lambda)) - f(P_{E_\partial}(P+\tau))|^2] d\nu_x(\lambda) d\nu_x(\tau) dx = 0 \tag{4.14}$$

Since the integrand in (4.14) equals 0 if and only if $P_{E_\partial}(\lambda) = P_{E_\partial}(\tau)$, we see that $P_{E_\partial}(D\phi_j) \to 0$ almost everywhere. Now, since

$$0 \le |P_{E_{\bar\partial}}(P + D\phi_j) - f(P_{E_\partial}(P + D\phi_j))| \le \sqrt{2}\,\mathrm{dist}(P + D\phi_j, K) \to 0,$$

almost everywhere, we see that $P_{E_{\bar\partial}}(D\phi_j)$ converges almost everywhere in U. The limit is 0 almost everywhere because $P_{E_{\bar\partial}}(D\phi_j) \rightharpoonup 0$ in L^2.

The conclusion follows from the fact that

$$0 = \lim_{j\to\infty} \int_U \mathrm{dist}^2(P + D\phi_j, K) dx \\ \ge \lim_{j\to\infty} \int_{U\setminus U_m} \mathrm{dist}^2(P + D\phi_j, K) dx = \int_{U\setminus U_m} \mathrm{dist}^2(P, K) dx,$$

for each fixed m, hence $\mathrm{dist}^2(P, K) = 0$, $P \in K$. □

Remark 4.5. The second part of the proof of Theorem 4.3 shows that if we do not assume condition (4.3), and instead we require that the sequence $D\phi_j$ is bounded, we can still show that up to a subsequence $D\phi_j$ converges almost everywhere. In the terminology of Yan [33], the set is $W^{1,2}$-weakly stable. However, when we study quasiconvex hulls, the sequence comes from a minimization problem. Therefore we have to establish boundedness of the sequence separately.

5. Quasiconvex hulls of some graphs in $M^{2n\times 2}$

In this section, we examine the quasiconvex hulls for a class of sets in $M^{2n\times 2}$ proposed in [28] for the study of singular solutions for elliptic systems in two-dimensional spaces. We examine two cases. The first one is when the system is of sublinear growth, so that the quasiconvex hull contains extra matrices. The second is when the system is of linear growth and the corresponding constitutive functions are strongly quasimonotone. In this case we can prove that $Q_2(K) = K$.

We examine the quasiconvex hulls of the unbounded set designed by Šverák [28] in searching of singular solutions of the Euler-Lagrange equations of variational problems with quasiconvex integrands. Let $F : M^{n\times 2} \to \mathbb{R}$ be smooth and quasiconvex, then the weak solution of $\mathrm{div}\, DF(Du) = 0$ is equivalent to a system of first order equations $Dv = DF(Du)J$ where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The method suggested in [28] is to find nontrivial Young measures for the $n\times 2$ dimensional graph $K \subset M^{2n\times 2}$ defined by

$$G_F = \left\{ \begin{pmatrix} X \\ DF(X)J \end{pmatrix},\ X \in M^{n\times 2} \right\}. \tag{5.1}$$

Let us examine a simple example first, which connects the Lipschitz graphs in $M^{2\times 2}$ without rank-one connections studied earlier in this section and monotone gradient mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$.

A mapping $f:\mathbb{R}^2\to\mathbb{R}^2$ is strictly monotone increasing (decreasing) if for all $x, y\in\mathbb{R}^2$, $x\neq y$, $(f(x)-f(y))\cdot(x-y)>0$ ($(f(x)-f(y))\cdot(x-y)<0$ respectively). We have

Proposition 5.1. *Suppose $F:\mathbb{R}^2\to\mathbb{R}$ is of class C^1. Then $DF:\mathbb{R}^2\to\mathbb{R}^2$ is a strictly monotone mapping if and only if $G_F\subset M^{2\times 2}$ defined in (5.1) with $n=1$ does not have rank-one connections.*

The following result is a simple application of Theorem 1.5.

Theorem 5.2. *Suppose $F:M^{n\times 2}\to\mathbb{R}$ is of class C^1 and $|DF(X)|\leq C(1+|X|^\sigma)$ for some $0\leq\sigma<1$. Then*

$$\left\{\begin{pmatrix} Y \\ DF(X)J\end{pmatrix},\ X,\, Y\in M^{n\times 2}\right\}\subset Q_1(G_F).$$

If we further assume that $0\leq\sigma<1/2$, then

$$\left\{\begin{pmatrix} Y \\ DF(X)J\end{pmatrix},\ X,\, Y\in M^{n\times 2}\right\}\subset Q_2(G_F).$$

In the next result, we use the notion of strong quasimonotonicity of a gradient mapping to study quasiconvex hulls. We give conditions such that $Q_2(G_F)=G_F$. An important class of mappings in the study of existence problem for elliptic equations and systems are the so-called pseudo-monotone mappings [25]. Let $A:W_0^{1,2}\to(W_0^{1,2})^*$ be a continuous map. If $u_j\rightharpoonup u$ in $W_0^{1,2}$ and

$$\lim\inf_{j\to\infty}\langle A(u_j),u_j-u\rangle\leq 0$$

imply that $u_j\to u$ strongly in $W_0^{1,2}$, the operator A is called pseudo-monotone. For elliptic systems, a class of mappings $A:M^{N\times n}\to M^{N\times n}$ called quasimonotone mappings was introduced to solve elliptic systems under a weaker monotonicity condition [34] (also see [15]) for the regularity problem). Recently, R. Landes [20] showed that for elliptic systems, quasimonotonicity is also a necessary condition for pseudo-monotonicity. For a gradient mapping $DF(P)$ with $F:M^{N\times n}\to\mathbb{R}$, quasimonotonicity of DF implies quasiconvexity of F while the converse is not true [16].

Theorem 5.3. *Suppose $F:M^{n\times 2}\to R$ is of class C^1 such that*

$$|DF(X)-DF(Y)|\leq C|X-Y|$$

for some $c_0>0$ and DF is strongly quasimonotonic in the sense that

$$\int_\Omega DF(P+D\phi(x))\cdot D\phi(x)dx\geq c_0\int_\Omega|D\phi(x)|^2dx,$$

for some constant $c>0$ and for every open set $\Omega\subset\mathbb{R}^2$, for every $P\in M^{n\times 2}$ and every $\phi\in C_0^1(\Omega,\mathbb{R}^n)$. Then

$$Q_2(G_F)=G_F$$

and G_F can only support trivial gradient Young measures.

Proof of Proposition 5.1. The statement that G_F is closed, unbounded and does not have rank-one connections is equivalent to the statement that G_F is a Lipschitz graph from $P_{E_\partial}(G_F)$ to $E_{\bar\partial}$, or from $P_{E_{\bar\partial}}(G_F)$ to E_∂. Without loss of generality, we may assume that the first situation occurs. The statement is also equivalent to

$$\begin{aligned}&\left|P_{E_{\bar\partial}}\left(\begin{pmatrix} X \\ DF(X)J\end{pmatrix}\right) - P_{E_{\bar\partial}}\left(\begin{pmatrix} X \\ DF(X)J\end{pmatrix}\right)\right| \\ &< \left|P_{E_\partial}\left(\begin{pmatrix} X \\ DF(X)J\end{pmatrix}\right) - P_{E_\partial}\left(\begin{pmatrix} X \\ DF(X)J\end{pmatrix}\right)\right|.\end{aligned}$$

This is equivalent to

$$\begin{aligned}&\left[(x_1-y_1) - \left(\frac{\partial F(X)}{\partial x_1} - \frac{\partial F(Y)}{\partial x_1}\right)\right]^2 + \left[(x_2-y_2) - \left(\frac{\partial F(X)}{\partial x_2} - \frac{\partial F(Y)}{\partial x_2}\right)\right]^2 \\ &< \left[(x_1-y_1) + \left(\frac{\partial F(X)}{\partial x_1} - \frac{\partial F(Y)}{\partial x_1}\right)\right]^2 + \left[(x_2-y_2) + \left(\frac{\partial F(X)}{\partial x_2} - \frac{\partial F(Y)}{\partial x_2}\right)\right]^2,\end{aligned}$$

which is equivalent to

$$(DF(X) - DF(Y)) \cdot (X - Y) > 0,$$

hence DF is monotone increasing. □

Proof of Theorem 5.2. Let us calculate the eigenvalues $\Lambda(X) \geq \lambda(X)$ of

$$\begin{pmatrix} X \\ DF(X)J\end{pmatrix}^T \begin{pmatrix} X \\ DF(X)J\end{pmatrix}.$$

We obtain

$$\begin{aligned}&\Lambda(X) \\ &= \frac{1}{2}\left(|X|^2 + |DF(X)|^2 + \sqrt{(|X|^2 + |DF(X)|^2) - 4\det(X^TX + J^TDF(X)^TDF(X)J)}\right), \\ &\lambda(X) \\ &= \frac{1}{2}\left(|X|^2 + |DF(X)|^2 - \sqrt{(|X|^2 + |DF(X)|^2) - 4\det(X^TX + J^TDF(X)^TDF(X)J)}\right).\end{aligned}$$

We have

$$\frac{\lambda(X)}{\Lambda(X)} = 4\frac{\det\left(\frac{(X^TX + J^TDF(X)^TDF(X)J)}{|X|^2 + |DF(X)|^2}\right)}{1 + \sqrt{1 - 4\det\left(\frac{(X^TX + J^TDF(X)^TDF(X)J)}{|X|^2 + |DF(X)|^2}\right)}}.$$

Noticing that $|DF(X)| \leq C(1 + |X|^\sigma)$ and $0 < \sigma < 1$, we have, for any rank-one matrix X_0 with unit norm $|X_0| = 1$,

$$\lim_{t\to+\infty} \frac{\lambda(tX_0)}{\Lambda(tX_0)} = 4\frac{\det(X_0^TX_0)}{1 + \sqrt{1 - 4\det(X_0^TX_0)}} = 0.$$

From Theorem 1.5 we see that

$$\{\begin{pmatrix} X \\ DF(X)J \end{pmatrix} + s\begin{pmatrix} X_0 \\ 0 \end{pmatrix},\ X \in M^{n\times 2},\ s \geq 0\} \subset R_1(G_F).$$

Since for any closed set K such that $G_F \subset K \subset R_p(G_F)$, we have $R_p(K) = R_p(G_F)$, we see that for any $X_1, \cdots, X_m$ with X_i a rank-one matrix and $|X_i| = 1$, $i = 1, \cdots, m$, we have

$$\{\begin{pmatrix} X \\ DF(X)J \end{pmatrix} + \sum_{i=1}^{m} s_i\begin{pmatrix} X_i \\ 0 \end{pmatrix},\ X \in M^{n\times 2},\ s_i \geq 0\} \subset R_1(G_F).$$

Notice that for any X, $Y \in M^{n\times 2}$, Y can be represented by

$$Y = X + \sum_{i=1}^{m} s_i X_i$$

for some rank-one matrices X_i with norm 1 and $s_i \geq 0$, $0 \leq m \leq 2n$. Therefore

$$\left\{\begin{pmatrix} Y \\ DF(X)J \end{pmatrix},\ X, Y \in M^{n\times 2}\right\} \subset R_1(G_F).$$

The proof is complete.

If $\sigma < 1/2$, we have

$$\frac{\lambda(X)}{\Lambda(X)^{1/2}} = 4\frac{\det\left(\frac{(X^TX + J^TDF(X)^TDF(X)J)}{(|X|^2+|DF(X)|^2)^{3/4}}\right)}{\left[1+\sqrt{1-4\det\left(\frac{(X^TX+J^TDF(X)^TDF(X)J)}{|X|^2+|DF(X)|^2}\right)}\right]^{3/2}}.$$

As in the above proof, let X_0 be a rank-one matrix with norm 1, we see that

$$\lim_{t\to+\infty}\frac{\lambda(tX_0)}{(\Lambda(tX_0))^{1/2}} = 0$$

if and only if

$$\lim_{t\to\infty}\det\left(\frac{(tX_0^T tX_0 + J^TDF(tX_0)^TDF(tX_0)J)}{(|tX_0|^2+|DF(tX_0)|^2)^{3/4}}\right) = 0.$$

This follows from the growth condition $|DF(X)| \leq C(1+|X|^\sigma)$ with $0 \leq \sigma < 1/2$. □

Proof of Theorem 5.3. Let $\binom{A}{B} \in Q_2(G_F)$ and let $\binom{\phi_j}{\psi_j}$ be a minimizing sequence in C_0^∞ such that

$$\lim_{j\to\infty}\int_D \operatorname{dist}^2\left(\begin{pmatrix} A + D\phi_j \\ B + D\psi_j \end{pmatrix}, G_F\right)dx = 0.$$

Since for each fixed j, $D\phi_j$ and $D\psi_j$ are bounded in L^∞, we can apply the measurable selection lemma to find an measurable mapping $X_j : D \to M^{2\times n}$ such that

$$\operatorname{dist}^2\left(\begin{pmatrix} A + D\phi_j(x) \\ B + D\psi_j(x) \end{pmatrix}, G_F\right) = |A + D\phi_j(x) - X_j(x)|^2 + |B + D\psi_j(x) - DF(X_j(x))J|^2$$

and

$$\lim_{j\to\infty}\int_D |A + D\phi_j(x) - X_j(x)|^2 dx = 0, \qquad \lim_{j\to\infty}\int_D |B + D\psi_j(x) - DF(X_j(x))J|^2 dx = 0.$$

Let $A + D\phi_j(x) - X_j(x) = n_j(x)$, $B + D\psi_j(x) - DF(X_j(x))J = m_j(x)$, then $n_j \to 0$, $m_j \to 0$ strongly in L^2.

Now we have $X_j(x) = A + D\phi_j(x) - n_j(x)$ so that

$$B + D\psi_j(x) = DF(A + D\phi_j(x) - n_j(x))J + m_j(x).$$

Applying the curl operator on both sides of the above equality in the sense of distributions we have

$$-\operatorname{div} DF(A + D\phi_j(x) - n_j(x)) + \operatorname{curl} m_j(x) = 0.$$

Since n_j, $m_j \in L^2$, we have

$$\int_D [DF(A + D\phi_j(x) - n_j(x)) \cdot D\phi_j(x) - m_j(x) \cdot \tilde{D}\phi_j(x)]dx = 0, \tag{5.2}$$

where

$$\tilde{D}\phi_j = \left(m_{12}\frac{\partial\phi_{j1}}{\partial x_1} - m_{11}\frac{\partial\phi_{j1}}{\partial x_2}\right) + \left(m_{22}\frac{\partial\phi_{j2}}{\partial x_1} - m_{21}\frac{\partial\phi_{j2}}{\partial x_2}\right).$$

Since DF is strongly quasimonotone and satisfies Lipschitz condition, we have in (5.2)

$$\begin{aligned}
&\int_D DF(A + D\phi_j(x)) \cdot D\phi_j(x) \\
&= \int_D [DF(A + D\phi_j(x)) - DF(A + D\phi_j(x) - n_j(x))] \cdot D\phi_j(x)dx \\
&+ \int_D m_j(x) \cdot \tilde{D}\phi_j(x)]dx,
\end{aligned}$$

so that

$$c_0 \int_D |D\phi_j|^2 dx \le C\int_D |n_j||D\phi_j(x)|dx + \int_D |m_j||D\phi_j|dx.$$

Hence there exists a constant $C_1 > 0$ depending only on c_0 and C, such that

$$\int_D |D\phi_j|^2 dx \le C_1 \int_D (|n_j|^2 + |m_j|^2)dx \to 0$$

as $j \to \infty$. Therefore $D\phi_j \to 0$ strongly in L^2. Since

$$B + D\psi_j(x) = DF(A + D\phi_j(x) - n_j(x))J + m_j(x),$$

we may pass to the limit $j \to \infty$ in this equality, so that

$$D\psi_j \to DF(A)J - B.$$

Since ψ_j has compact support in D, we see that $B = DF(A)J$. Thus

$$\begin{pmatrix} A \\ B \end{pmatrix} \in G_F.$$

The proof is complete. □

Remark 5.4. Instead of requiring that DF is a Lipschitz mapping, we could require that $DF(X)$ is coercive in the sense that $DF(X) \cdot X \geq C|X|^2 - C_1$. In that situation one could apply the method used in [34] to establish a similar result.

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