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Elements of Quasiconvex Subdifferential Calculus

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A number of rules for the calculus of subdifferentials of generalized convex functions are displayed. The subdifferentials we use are among the most significant for this class of functions, in particular for quasiconvex functions: we treat the Greenberg-Pierskalla's subdifferential and its relatives and the Plastria's lower subdifferential. We also deal with a recently introduced subdifferential constructed with the help of a generalized derivative. We emphasize the case of the sublevel-convolution, an operation analogous to the infimal convolution, which has proved to be of importance in the field of quasiconvex functions. We provide examples delineating the limits of the rules we provide.

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1. Introduction

It is the purpose of the present paper to give some calculus rules for known subdifferentials which are used for generalized convex functions. We feel that without such rules, a number of problems involving generalized convex functions can hardly be dealt with. Some rules already exist (see [10], [5] for instance); but we endeavour to make a systematic study.

In fact we concentrate on the most usual subdifferentials: the subdifferentials of the Greenberg-Pierskalla type and the lower subdifferential of Plastria. We also deal with a notion introduced in [5] which has not received much attention yet, although it seems to be well adapted to the case of quasiconvex functions.

The operations we consider are the most useful ones: composition with a linear map into the source, composition with an increasing function at the range, suprema, performance functions. We also deal with an operation which has attracted some attention during the last few years, the sublevel-convolution $h := f \diamond g$ of the two functions f, g given by

$$h(w) := \inf_{x \in X} f(w - x) \vee g(x).$$

This operation, which is the analogue of the infimal convolution of convex analysis (the

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sum being replaced by the supremum operation $\vee$) is of fundamental importance for quasiconvex analysis inasmuch as the usual sum does not preserve quasiconvexity whereas supremum does (see [1], [11], [12], [13], [14], [L-V]...). Moreover, one can check that the strict sublevel sets of h are given by

$$[h < r] = [f < r] + [g < r],$$

whereas the sublevel sets satisfy

$$[h \leq r] = [f \leq r] + [g \leq r]$$

whenever the infimum is attained in the formula defining h (then one says that the sublevel-convolution is exact). The interest of such an operation for regularization purposes is shown elsewhere (see [11], [9]).

Our study relies on techniques from convex analysis which are described in our paper [8].

Section 2 is devoted to the case of subdifferentials of Greenberg-Pierskalla type. Such subdifferentials are easy to deal with, but they differ drastically from the usual Fenchel subdifferential as they are cones. The lower subdifferential of Plastria which is studied in section 3 also suffers from such a difference, but to a less extent, as it is a shady subset of the dual (i.e. is stable under homotheties of rate greater than 1), hence is unbounded or empty. The last section is devoted to a subdifferential which is a kind of compromise between the Plastria subdifferential and the incident subdifferential. For a Lipschitzian quasiconvex function, it has always nonempty values.

A number of subdifferentials which have been proposed for the study of generalized quasiconvex functions have not been considered here for the sake of brevity (see [5] for a recent survey). However, we hope the present study will enable the reader to get a more precise idea of the possibilities and the limitations of what is called "quasiconvex analysis" in [6].

2. Subdifferentials of normal type

Hereafter, f is a function defined on a Hausdorff locally convex space X with dual X^*, with values in $\mathbb{R} \cup \{\infty\}$ or $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, and we suppose that $f(x_0)$ is finite. As in [8] we denote by $\mathbb{R}_+$ (resp. $\mathbb{P}$) the set of nonnegative (resp. positive) real numbers. We devote this section to some rules for the calculus of subdifferentials of Greenberg-Pierskalla type. These subdifferentials are among the simplest concepts. They are akin to the normal cones to the corresponding sublevel sets $[f \leq f(x_0)]$:

$$\partial^\nu f(x_0) := N([f \leq f(x_0)], x_0),$$

where the normal cone to a subset C of X at $x_0 \in X$ is given by

$$N(C, x_0) := \{x^* \in X^* \mid \langle x^*, x - x_0 \rangle \leq 0 \ \forall x \in C\}.$$

We also consider the case of the *star subdifferential* introduced in [5], or rather a slight modification of it (it differs from the definition in [5] by $\{0\}$ when x_0 is not a minimizer). It is as follows: $x_0^* \in \partial^{\circledast} f(x_0)$ iff $\langle x_0^*, x - x_0 \rangle \leq 0$ for each $x \in [f < f(x_0)]$. In other terms

$$\partial^{\circledast} f(x_0) := N([f < f(x_0)], x_0).$$

Both definitions are variants of the classical *Greenberg-Pierskalla's subdifferential*, given by $x_0^* \in \partial^* f(x_0)$ iff $\langle x_0^*, x - x_0 \rangle < 0$ for each $x \in [f < f(x_0)]$. These subdifferentials are cones and thus they are large and quite different from the usual Fenchel subdifferential. Obviously

$$\partial^* f(x_0) \subset \partial^\circledast f(x_0), \qquad \partial^\nu f(x_0) \subset \partial^\circledast f(x_0).$$

Observe that x_0 is a minimizer of f iff $0 \in \partial^* f(x_0)$ iff $\partial^* f(x_0) = X^*$, iff $\partial^\circledast f(x_0) = X^*$. In contrast, one always has $0 \in \partial^\nu f(x_0)$, $0 \in \partial^\circledast f(x_0)$. Thus, although these subdifferentials are similar, they have distinct features. Let us give a criterion for their coincidence up to zero. Here f is said to be *radially u.s.c.* (upper semicontinuous) at some point x if for any $y \in X$ the function $t \mapsto f(x + ty)$ is u.s.c. at 0.

Lemma 2.1. *If $\partial^* f(x_0)$ is nonempty, then $\partial^\circledast f(x_0) = \mathrm{cl}(\partial^* f(x_0))$. If f is radially u.s.c. at each point of $[f < f(x_0)]$ then $\partial^\circledast f(x_0) = \partial^* f(x_0) \cup \{0\}$. If there is no local minimizer of f in $f^{-1}(f(x_0))$ then $\partial^\circledast f(x_0) = \partial^\nu f(x_0)$.*

Proof. The first assertion is easy. For the second assertion, it suffices to observe that a linear form which is nonnegative on an absorbant subset is 0.

If there is no local minimizer of f in $f^{-1}(f(x_0))$ then the sublevel set $[f \leq f(x_0)]$ is contained in the closure of the strict level set $[f < f(x_0)]$. Thus, any $x_0^* \in \partial^\circledast f(x_0)$ being bounded above by $\langle x_0^*, x_0 \rangle$ on $[f < f(x_0)]$ is also bounded above by $\langle x_0^*, x_0 \rangle$ on $[f \leq f(x_0)]$: $x_0^* \in N([f \leq f(x_0)], x_0)$. □

We observe that each subdifferential $\partial^?$ among the three we consider is *homotone*, i.e. given f, g such that $f \leq g$ and $f(x_0) = g(x_0)$ one has $\partial^? f(x_0) \subset \partial^? g(x_0)$. The first assertion of the next lemma follows from this observation.

Lemma 2.2. *Let $(f_i)_{i \in I}$ be an arbitrary family of functions finite at x_0 and let $f := \inf_{i \in I} f_i$. Suppose $I(x_0) := \{i \in I \mid f_i(x_0) = f(x_0)\}$ is nonempty. Then, for $\partial^? = \partial^*, \partial^\circledast, \partial^\nu$ one has*

$$\partial^? f(x_0) \subset \bigcap_{i \in I(x_0)} \partial^? f_i(x_0).$$

If $I(x_0) = I$, then equality holds for $\partial^*, \partial^\circledast$; if moreover $I(x)$ is nonempty for each $x \in X$ (in particular if I is finite) then equality holds for ∂^ν.

Proof. For the second assertion, one observes that if $I(x_0) = I$ and if $x \in [f < f(x_0)]$ (resp. $x \in [f \leq f(x_0)]$ and if $I(x)$ is nonempty), then, for some $i \in I$ one has $x \in [f_i < f_i(x_0)]$ (resp. $x \in [f_i \leq f_i(x_0)]$). □

A similar result holds for suprema. Here we denote by $\mathrm{co}\, A$ (resp. $\overline{\mathrm{co}} A$) the convex hull (resp. the closed convex hull) of A.

Proposition 2.3. *Let $(f_i)_{i \in I}$ be an arbitrary family of functions finite at x_0 and let $f := \sup_{i \in I} f_i$. Suppose $I(x_0) := \{i \in I \mid f_i(x_0) = f(x_0)\}$ is nonempty. Then, for $\partial^? = \partial^*, \partial^\circledast, \partial^\nu$ one has*

$$\partial^? f(x_0) \supset \mathrm{co} \left(\bigcup_{i \in I(x_0)} \partial^? f_i(x_0) \right). \tag{2.1}$$

Equality holds for ∂^ν *if* I *is finite and equal to* $I(x_0)$, *if* $C_i := [f_i \leq f_i(x_0)]$ *is convex for each* $i \in I(x_0)$ *and if either for some* $k \in I(x_0)$ *one has* $C_k \cap (\bigcap_{i \neq k} \operatorname{int} C_i) \neq \emptyset$, *or* X *is a Banach space, each* C_i *is closed and* $X^I = \mathbb{R}_+ \left(\Delta - \prod_{i \in I} C_i\right)$, *where* Δ *is the diagonal of* X^I.

When $I(x_0)$ has two elements j, k only, the qualification condition of the preceding statement can be rewritten in the simpler (and more familiar) form

$$X = \mathbb{R}_+(C_j - C_k).$$

Proof. Again the first assertion is a consequence of homotonicity. When $\partial^? f(x_0)$ is closed, one can replace co by $\overline{\text{co}}$ in (2.1). The second assertion is a consequence of Lemma 2.1 and of the calculus of normal cones ([15] for instance) since $[f \leq f(x_0)] = \bigcap_{i \in I}[f_i \leq f_i(x_0)]$ when $I = I(x_0)$ is finite. $\square$

Proposition 2.4. *Let* $f : X \to \mathbb{R} \cup \{\infty\}$, $g : Y \to \mathbb{R} \cup \{\infty\}$ *and* $M : X \times Y \to \mathbb{R} \cup \{\infty\}$, $M(x,y) = f(x) \vee g(y)$. *Consider* $z_0 := (x_0, y_0) \in X \times Y$ *such that* $r_0 := f(x_0) = g(y_0) \in \mathbb{R}$. *Then for* $\partial^? = \partial^*, \partial^\circledast, \partial^\nu$ *one has*

$$\partial^? M(x_0, y_0) \supset \partial^? f(x_0) \times \partial^? g(y_0),$$

with equality for $\partial^? = \partial^\nu$. *When* x_0 *and* y_0 *are not local minimizers of* f *and* g, *respectively, equality holds for* $\partial^? = \partial^\circledast$, *and*

$$\partial^* M(x_0, y_0) = \left(\partial^* f(x_0) \times \partial^\circledast g(y_0)\right) \cup \left(\partial^\circledast f(x_0) \times \partial^* g(y_0)\right).$$

Proof. The first assertions are easy consequences of the preceding result. Let $z^* := (x^*, y^*) \in \partial^\circledast M(z_0)$. Thus

$$\langle x - x_0, x^* \rangle + \langle y - y_0, y^* \rangle \leq 0 \quad \forall x \in [f < f(x_0)],\ \forall y \in [g < g(y_0)]. \tag{2.2}$$

Since x_0 and y_0 are not local minimizers of f and g, respectively, we get that

$$\langle x - x_0, x^* \rangle \leq 0 \quad \forall x \in [f < f(x_0)], \qquad \langle y - y_0, y^* \rangle \leq 0 \quad \forall y \in [g < g(y_0)],$$

i.e. $x^* \in \partial^\circledast f(x_0)$ and $y^* \in \partial^\circledast g(y_0)$. Now let us suppose that $z^* \in \partial^* M(z_0)$. If $x^* \in \partial^\circledast f(x_0) \backslash \partial^* f(x_0)$ there exists $u \in [f < f(x_0)]$ such that $\langle u - x_0, x^* \rangle = 0$; then, as $\langle u - x_0, x^* \rangle + \langle y - y_0, y^* \rangle < 0$ for each $y \in [g < g(y_0)]$, we get that $y^* \in \partial^* g(y_0)$. $\square$

The following result about performance functions is simple and useful.

Proposition 2.5. *Let* W *and* X *be locally convex spaces and let* $F : W \times X \to \overline{\mathbb{R}}$,

$$p(w) := \inf_{x \in X} F(w, x), \qquad S(w) := \{x \in X \mid F(w,x) = p(w)\}.$$

Then for $\partial^? = \partial^*,\ \partial^\circledast$, *for any* $w_0 \in W$ *such that* $p(w_0)$ *is finite, and for any* $x_0 \in S(w_0)$ *one has* $w_0^* \in \partial^? p(w_0)$ *iff* $(w_0^*, 0) \in \partial^? F(w_0, x_0)$. *When* $S(w)$ *is non-empty for each* $w \in W$, *the same result holds for* ∂^ν.

Recall that if F is (convex) quasiconvex then p is (convex) quasiconvex.

Proof. The result is essentially a consequence of the fact that $w \in [p < p(w_0)]$ iff there exists $x \in X$ such that $(w, x) \in [F < F(w_0, x_0)]$ and of the relation

$$\langle (w_0^*, 0), (w - w_0, x - x_0) \rangle = \langle w_0^*, w - w_0 \rangle.$$

The last assertion is obvious. □

We also observe that when $(w_0^*, 0) \in \partial^* F(w_0, x_0)$ then $x_0 \in S(w_0)$.

In the sequel, given another locally convex space Y, we denote by $L(X, Y)$ the class of continuous linear maps from X into Y and for $A \in L(X, Y)$, A^t stands for the transpose of A.

Proposition 2.6. *Let $f = g \circ A$, where $A \in L(X, Y)$, and $g : Y \to \overline{\mathbb{R}}$ is finite at $y_0 = Ax_0$. Then, if $\partial^?$ is one of the subdifferentials $\partial^*, \partial^\circledast, \partial^\nu$, one has*

$$A^t \left(\partial^? g(y_0) \right) \subset \partial^? f(x_0).$$

If A is onto then

$$(A^t)^{-1} \left(\partial^? f(x_0) \right) = \partial^? g(Ax_0).$$

If A is onto and $R(A^t)$ is w^*-closed (in particular if X and Y are Banach spaces) one has $A^t (\partial^\nu g(y_0)) = \partial^\nu f(x_0)$; if moreover y_0 is not a minimizer of g, then $A^t (\partial^\circledast g(y_0)) = \partial^\circledast f(x_0)$ and $A^t (\partial^* g(y_0)) = \partial^* f(x_0)$.

Proof. Let us consider the case of $\partial^\circledast$. Given $y^* \in \partial^\circledast g(y_0)$, for any $x \in [f < f(x_0)]$ we have $y := A(x) \in [g < g(y_0)]$ hence $\langle y^* \circ A, x - x_0 \rangle = \langle y^*, y - y_0 \rangle \le 0$ and $x^* := y^* \circ A \in \partial^\circledast f(x_0)$.

Suppose now that A is onto and $y^* \in (A^t)^{-1} (\partial^\circledast f(x_0))$. Let us show that y^* belongs to $\partial^\circledast g(Ax_0)$. Indeed, $x^* := A^t y^* \in \partial^\circledast f(x_0)$ and for each $y \in Y$ such that $g(y) < g(Ax_0)$ there exists $x \in X$ with $y = Ax$. So,

$$\langle y - Ax_0, y^* \rangle = \langle Ax - Ax_0, y^* \rangle = \left\langle x - x_0, A^t y^* \right\rangle \le 0.$$

Therefore $y^* \in \partial^\circledast g(Ax_0)$. The other cases are similar.

Suppose A is onto and $R(A^t)$ is w^*-closed; let $x^* \in \partial^\nu f(x_0)$. Given $x \in A^{-1}(0)$ and $\varepsilon \in \{-1, 1\}$ we have $f(x_0 + \varepsilon x) = f(x_0)$ hence $\langle x^*, \varepsilon x \rangle \le 0$, and so $\langle x^*, x \rangle = 0$. Thus $x^* \in N(A)^\perp = A^t(Y^*)$ and there exists a continuous linear form y^* on Y such that $x^* = y^* \circ A$. It follows that $y^* \in (A^t)^{-1}(\partial^\nu f(x_0)) = \partial^\nu g(y_0)$.

When x_0 is not a minimizer of f and $x^* \in \partial^\circledast f(x_0)$, taking $x \in X$ such that $f(x) < f(x_0)$ one obtains that for any $u \in A^{-1}(0)$ one has $f(x + u) = g(Ax) = f(x) < f(x_0)$ hence $\langle x^*, x + u - x_0 \rangle \le 0$. It follows that $\langle x^*, u \rangle = 0$ and $x^* = y^* \circ A$ for some $y^* \in Y^*$. As above one gets that $y^* \in \partial^\circledast g(y_0)$ (and $y^* \in \partial^* g(y_0)$ when $x^* \in \partial^* f(x_0)$). □

Another important chain rule is the following one.

Proposition 2.7. *Let $g : X \to \mathbb{R} \cup \{\infty\}$ and let $\varphi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a nondecreasing function. Set $\varphi(\infty) = \infty$ and consider $x_0 \in X$ such that $\varphi(t_0) \in \mathbb{R}$, where $t_0 := g(x_0)$. Let $f = \varphi \circ g$. Then*

$$\partial^* g(x_0) \subset \partial^* f(x_0), \quad \partial^\circledast g(x_0) \subset \partial^\circledast f(x_0) \quad \text{and} \quad \partial^\nu g(x_0) \supset \partial^\nu f(x_0).$$

If $g(x_0)$ is not a local minimizer of φ the first two inclusions are equalities while if $g(x_0)$ is not a local maximizer of φ one has $\partial^\nu g(x_0) = \partial^\nu f(x_0)$. If there is no local minimizer of f in $f^{-1}(f(x_0))$ then $\partial^\nu f(x_0) = \partial^\nu g(x_0) = \partial^\circledast g(x_0) = \partial^\circledast f(x_0)$.

Proof. The inclusions are immediate since $[f < f(x_0)] \subset [g < g(x_0)]$ and $[f \leq f(x_0)] \supset [g \leq g(x_0)]$. When $g(x_0)$ is not a local minimizer (resp. maximizer) of φ one has $[f < f(x_0)] = [g < g(x_0)]$ (resp. $[f \leq f(x_0)] = [g \leq g(x_0)]$). The last assertion is a consequence of the other ones, and of the relations $\partial^\nu g(x_0) \subset \partial^\circledast g(x_0)$, $\partial^\nu f(x_0) = \partial^\circledast f(x_0)$ when f satisfies the last assumption of Lemma 2.1. □

The previous results can be used to compute the subdifferential of the sublevel-convolution $h := f \Diamond g$. The following rule, which mimics the classical rule for the infimal convolution, is a simple consequence of the formulas about sublevel sets.

Proposition 2.8. *Given $x_0, y_0 \in X$ with $x_0 + y_0 = z_0$, $f(x_0) = g(y_0) = h(z_0) \in \mathbb{R}$ one has for $h := f \Diamond g$*

$$\begin{aligned} \partial^\circledast f(x_0) \cap \partial^\circledast g(y_0) &\subset \partial^\circledast h(z_0), \\ \big(\partial^\circledast f(x_0) \cap \partial^* g(y_0)\big) \cup \big(\partial^* f(x_0) \cap \partial^\circledast g(y_0)\big) &\subset \partial^* h(z_0), \\ \partial^\nu f(x_0) \cap \partial^\nu g(y_0) &\supset \partial^\nu h(z_0), \end{aligned}$$

with equality in this last relation when for each $z \in X$ there exist $x, y \in X$ such that $x + y = z$, $f(x) \vee g(y) = h(z)$ i.e. the sublevel-convolution is exact. If x_0 and y_0 are not local minimizers of f and g, respectively, equality holds in the first and the second relations.

Proof. Since $[h < h(z_0)] = [f < f(x_0)] + [g < g(y_0)]$, for any $z_0^* \in \partial^\circledast f(x_0) \cap \partial^\circledast g(y_0)$ and any $x \in [f < f(x_0)]$, $y \in [g < g(y_0)]$ we have

$$\langle z_0^*, x + y - z_0 \rangle = \langle z_0^*, x - x_0 \rangle + \langle z_0^*, y - y_0 \rangle \leq 0,$$

hence $z_0^* \in \partial^\circledast h(z_0)$. Similarly, when $z_0^* \in \partial^\circledast f(x_0) \cap \partial^* g(y_0)$ or $z_0^* \in \partial^* f(x_0) \cap \partial^\circledast g(y_0)$ one gets $z_0^* \in \partial^* h(x_0)$.

Conversely, when x_0 and y_0 are not local minimizers of f and g, respectively, given $z_0^* \in \partial^\circledast h(z_0)$, for any $x \in [f < f(x_0)]$, taking a sequence (y_n) in $[g < g(y_0)]$ with limit y_0 we observe that

$$\langle z_0^*, x - x_0 \rangle = \lim_n \langle z_0^*, x + y_n - x_0 - y_0 \rangle \leq 0,$$

hence $z_0^* \in \partial^\circledast f(x_0)$. Similarly, $z_0^* \in \partial^\circledast g(y_0)$. An argument similar to the one in the last part of the proof of Proposition 2.4 yields the case $z_0^* \in \partial^* h(z_0)$.

The assertion about ∂^ν is a consequence of the inclusion $[f \leq f(x_0)] + [g \leq g(y_0)] \subset [h \leq h(z_0)]$ which is an equality when the sublevel convolution is exact. □

3. Subdifferential calculus for lower subdifferentials

We devote the present section to the calculus with the *lower subdifferential* of Plastria [10]. It is defined by $x_0^* \in \partial^< f(x_0)$ iff

$$\langle x_0^*, x - x_0 \rangle \leq f(x) - f(x_0) \quad \forall x \in [f < f(x_0)].$$

We leave to the reader most of the task of writing the necessary modifications for the *infradifferential* of Gutiérrez [2] given by $x_0^* \in \partial^{\leq} f(x_0)$ iff

$$\langle x_0^*, x - x_0 \rangle \leq f(x) - f(x_0) \quad \forall x \in [f \leq f(x_0)].$$

Obviously, one has

$$\partial^{\leq} f(x_0) \subset \partial^{<} f(x_0), \quad \partial^{\leq} f(x_0) \subset \partial^{\nu} f(x_0), \quad \partial^{<} f(x_0) \subset \partial^{*} f(x_0).$$

Again, we start with the observation that these subdifferentials are homotone. Now we consider the performance function p associated to a perturbation $F : W \times X \to \overline{\mathbb{R}}$ by $p(w) = \inf_{x \in X} F(w, x)$.

Lemma 3.1.
(i) *Suppose that $p(w_0) = F(w_0, x_0) \in \mathbb{R}$. If $w^* \in \partial^{<} p(w_0)$ then $(w^*, 0) \in \partial^{<} F(w_0, x_0)$.*
(ii) *Let $F(w_0, x_0) \in \mathbb{R}$ and $(w^*, 0) \in \partial^{<} F(w_0, x_0)$. Then $p(w_0) = F(w_0, x_0)$ and $w^* \in \partial^{<} p(w_0)$.*

Proof. (see also [5]). (i) Let $F(w, x) < F(w_0, x_0)$; it follows that $p(w) < p(w_0)$, whence

$$\langle w - w_0, w^* \rangle + \langle x - x_0, 0 \rangle \leq p(w) - p(w_0) \leq F(w, x) - F(w_0, x_0),$$

which shows that $(w^*, 0) \in \partial^{<} F(w_0, x_0)$.

(ii) Suppose that $p(w_0) < F(w_0, x_0)$. Then there exists $x \in X$ such that $F(w_0, x) < F(w_0, x_0)$, whence

$$0 = \langle w_0 - w_0, w^* \rangle + \langle x - x_0, 0 \rangle \leq F(w_0, x) - F(w_0, x_0) < 0,$$

a contradiction. Therefore $p(w_0) = F(w_0, x_0)$. Let now $w \in W$ be such that $p(w) < p(w_0)$. Then there exists $(x_n) \subset X$ such that $(F(w, x_n)) \to p(w)$; we may suppose that $F(w, x_n) < F(w_0, x_0) = p(w_0)$ for every n. Hence

$$\langle w - w_0, w^* \rangle + \langle x_n - x_0, 0 \rangle \leq F(w, x_n) - F(w_0, x_0) \quad \forall n \in \mathbb{N}.$$

Taking the limit we get $\langle w - w_0, w^* \rangle \leq p(w) - p(w_0)$. Therefore $w^* \in \partial^{<} p(w_0)$. □

Let us turn to composition with a continuous linear map.

Proposition 3.2. *Let $g : Y \to \overline{\mathbb{R}}$, $A \in L(X, Y)$ and $x_0 \in X$ be such that $g(Ax_0) \in \mathbb{R}$. Then, for $f := g \circ A$*

$$\partial^{<} f(x_0) \supset A^t \left(\partial^{<} g(Ax_0) \right).$$

If A is onto (or even if $A(X) \supset [g < g(Ax_0)]$) then

$$(A^t)^{-1} \left(\partial^{<} f(x_0) \right) = \partial^{<} g(Ax_0). \tag{3.1}$$

Moreover, if $R(A^t)$ is w^*-closed (in particular if X and Y are Banach spaces) and Ax_0 is not a minimizer of g, then

$$\partial^{<} f(x_0) = A^t \left(\partial^{<} g(Ax_0) \right).$$

Proof. The first assertion is contained in [10] Th. 3.5. Let us recall its simple proof. Let $y^* \in \partial^< g(Ax_0)$ and $f(x) < f(x_0)$. Then $g(Ax) < g(Ax_0)$, and so

$$\left\langle x - x_0, A^t y^* \right\rangle = \langle Ax - Ax_0, y^* \rangle \leq g(Ax) - g(Ax_0) = f(x) - f(x_0).$$

Therefore $A^t y^* \in \partial^< f(x_0)$.

Suppose now that A is onto and $y^* \in (A^t)^{-1}\left(\partial^< f(x_0)\right)$. Then $x^* := A^t y^* \in \partial^< f(x_0)$. Let $y \in Y$ be such that $g(y) < g(Ax_0)$. There exists $x \in X$ such that $y = Ax$. So,

$$\langle y - Ax_0, y^* \rangle = \langle Ax - Ax_0, y^* \rangle = \left\langle x - x_0, A^t y^* \right\rangle \leq f(x) - f(x_0) = g(y) - g(Ax_0).$$

Therefore $y^* \in \partial^< g(Ax_0)$.

Suppose now that $R(A^t)$ is w^*-closed and Ax_0 is not a minimizer for g. It follows that $R(A^t) = (\ker A)^\perp$. Let $x^* \in \partial^< f(x_0) \subset \partial^* f(x_0)$. By the proof of Proposition 2.6 $x^* \in R(A^t)$. Thus $x^* = A^t y^*$ for some $y^* \in Y^*$. By (3.1) we have that $y^* \in \partial^< g(Ax_0)$, and so $x^* \in A^t(\partial^< g(Ax_0))$. $\square$

Remark 3.3. The same conclusions hold for $\partial^<$ replaced by $\partial^\leq$ in the preceding statement, even without asking that Ax_0 is not a minimizer of f in the last part, with the same proof.

The following result is a special case of what precedes, taking for A a projection. It can also be proved directly.

Corollary 3.4. *Let $f : X \to \overline{\mathbb{R}}$ and let $\widetilde{f} : X \times Y \to \overline{\mathbb{R}}$ be defined by $\widetilde{f}(x, y) = f(x)$. If $(x_0, y_0) \in X \times Y$ with $f(x_0) \in \mathbb{R}$ and if x_0 is not a minimizer of f, then*

$$\partial^< \widetilde{f}(x_0, y_0) = \partial^< f(x_0) \times \{0\}.$$

Composition with a nondecreasing function is also an important operation for quasiconvex functions. The following result completes [10] Th. 3.3 which corresponds to assertion (i) below.

Proposition 3.5. *Let $g : X \to \mathbb{R} \cup \{\infty\}$ and let $\varphi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a nondecreasing function. Set $\varphi(\infty) = \infty$ and consider $x_0 \in X$ such that $\varphi(t_0) \in \mathbb{R}$, where $t_0 := g(x_0)$.*

(i) *Then*

$$\partial^< \varphi(t_0) \cdot \partial^< g(x_0) \subset \partial^< (\varphi \circ g)(x_0). \tag{3.2}$$

(ii) *Suppose that g is sublinear, l.s.c. at 0, $g(x_0) > \inf g$ and t_0 is not a local minimizer of φ. If either* (a) *$g(x_0) \leq 0$ or* (b) *$g(x_0) > 0$ and $\varphi(t) = 0$ for every $t \in \mathbb{R}_-$ then*

$$\partial^< \varphi(t_0) \cdot \partial^< g(x_0) = \partial^< (\varphi \circ g)(x_0). \tag{3.3}$$

Proof. (i) Let us denote $\varphi \circ g$ by f. Suppose first that t_0 is a minimizer of φ. Then x_0 is a minimizer of f, and so relation (3.2) is obvious.

Suppose now that t_0 is not a minimizer of φ, and let $t^* \in \partial^< \varphi(t_0)$ and $x^* \in \partial^< g(x_0)$. Since t_0 is not a minimizer and φ is nondecreasing, $t^* > 0$. Consider $x \in X$ such that $f(x) < f(x_0)$. Then $g(x) < g(x_0)$, whence

$$t^*(g(x) - g(x_0)) \leq \varphi(g(x)) - \varphi(g(x_0)), \quad \langle x - x_0, x^* \rangle \leq g(x) - g(x_0).$$

Multiplying the second relation by $t^* > 0$, we obtain that $\langle x - x_0, t^*x^* \rangle \leq f(x) - f(x_0)$. Therefore $t^*x^* \in \partial^< f(x_0)$.

(ii) Let now $u^* \in \partial^< f(x_0)$. As t_0 is not a local minimizer of φ, we have that $[f < f(x_0)] = [g < g(x_0)]$. Thus

$$\langle x - x_0, u^* \rangle \leq f(x) - f(x_0) < 0 \quad \forall x \in [\, g < g(x_0)[. \tag{3.4}$$

(a) Let $t_0 \leq 0$. Taking $x \in [g < 0]$ and $s > 0$, we have $2x_0 + sx \in [g < g(x_0)] = [\, f < f(x_0]$ hence $\langle x_0 + sx, u^* \rangle \leq 0$. Therefore $\langle x_0, u^* \rangle = \lim_{s \to 0} \langle x_0 + sx, u^* \rangle \leq 0$. Let $t < t_0$. From (3.4) we obtain that

$$\langle x, u^* \rangle \leq \langle x_0, u^* \rangle + \varphi(t) - \varphi(t_0) \quad \forall\, x \in [g \leq t].$$

As $\inf g < t < 0$, applying [8] Prop. 5.2 with its notation, we get

$$\sup\{\langle w, u^* \rangle \mid w \in [g \leq t]\} = t\beta_{\partial g(0)}(u^*) \leq \langle x_0, u^* \rangle + \varphi(t) - \varphi(t_0) < 0. \tag{3.5}$$

Thus $\lambda := \beta_{\partial g(0)}(u^*) > 0$. Since $0 \notin \partial g(0)$ and $\partial g(0)$ is w^*-closed, λ is finite. Using again [8] Prop. 5.2 we get $w^* := \lambda^{-1}u^* \in \partial g(0)$. By (3.5) we obtain that

$$t\lambda \leq \lambda \langle x_0, w^* \rangle + \varphi(t) - \varphi(t_0) < \lambda \langle x_0, w^* \rangle \quad \forall\, t < t_0.$$

Dividing by λ, taking the limit as $t \to t_0$ and using the inclusion $w^* \in \partial g(0)$, we get $t_0 = \langle x_0, w^* \rangle = g(x_0) = t_0$, whence $w^* \in \partial g(x_0) \subset \partial^< g(x_0)$ and $\lambda \in \partial^< \varphi(t_0)$. The conclusion follows in this case.

(b) Suppose now that $t_0 > 0$. Thus, for each $t \in]0, t_0[$, using again [8] Prop. 5.2, we have

$$0 \leq \sup\{\langle w, u^* \rangle \mid w \in [g \leq t]\} = t\alpha_{\partial g(0)}(u^*) \leq \langle x_0, u^* \rangle + \varphi(t) - \varphi(t_0) < \infty.$$

Therefore, $u^* \in \mathbb{R}_+ \partial g(0)$. If $u^* \in 0 \cdot \partial g(0) = 0^+ \partial g(0) = (\operatorname{dom} g)^-$, then $\langle x_0, u^* \rangle \leq 0$ and so $0 \leq \varphi(\frac{1}{2}t_0) - \varphi(t_0) < 0$, a contradiction. Therefore $\lambda := \alpha_{\partial g(0)}(u^*) > 0$ and $w^* := \lambda^{-1}u^* \in \partial g(0)$. Thus

$$t\lambda \leq \lambda \langle x_0, w^* \rangle + \varphi(t) - \varphi(t_0) < \lambda \langle x_0, w^* \rangle \;\; \forall\, t \in]0, t_0[.$$

As above, it follows that $\langle x_0, w^* \rangle = g(x_0) = t_0$, whence $w^* \in \partial g(x_0) \subset \partial^< g(x_0)$ and $\lambda(t - t_0) \leq \varphi(t) - \varphi(t_0)$, for each $t \in]0, t_0[$. Taking $x = 0$ in (3.4), we see that the preceding inequality is also valid for $t = 0$, while for $t < 0$ the inequality is obvious. Thus $\lambda \in \partial^< \varphi(t_0)$ again. □

Example 3.6. Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi(t) := t^3$, $g(x) = \max(x, -1)$. Then one has $\partial^< \varphi(0) = \emptyset$ but $\partial^< (\varphi \circ g)(0) = [1, \infty[$. This example shows the necessity of restrictive assumptions such as sublinearity of g.

Let us observe that for any function g and any $x_0 \in X$ such that $t_0 := g(x_0) = \inf g \in \mathbb{R}$, relation (3.3) holds if and only if $\partial^< \varphi(t_0) \neq \emptyset$. When $\dim X = 1$, relation (3.3) holds whenever $t_0 \neq 0$ and g is sublinear. One may ask whether the assumptions of the statement (ii) are crucial.

Example 3.7. Let $\varphi, \overline{\varphi}, \widetilde{\varphi}, \widetilde{g} : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by: $\varphi(t) = \max(t, 0)$, $\overline{\varphi}(t) := \min(1, 2\varphi(t))$, $g(r,s) = r$ and

$$\widetilde{\varphi}(t) = \begin{cases} 2t & \text{if } t \leq 0, \\ t & \text{if } t > 0, \end{cases} \qquad \widetilde{g}(x) = \begin{cases} \frac{1}{2}x & \text{if } x \leq 0, \\ x & \text{if } x > 0. \end{cases}$$

For $x_0 = (0,1)$, $t_0 := g(x_0) = 0$ is a minimizer of φ so that $\partial^{<}\varphi(t_0) = \mathbb{R}$, $\partial^{<}g(x_0) = \{(\lambda, 0) \mid \lambda \geq 1\}$ by Corollary 3.4, and so $\partial^{<}\varphi(t_0) \cdot \partial^{<}g(x_0) = \mathbb{R} \times \{0\} \neq \partial^{<}(\varphi \circ g)(x_0) = \mathbb{R}^2$.

The condition that t_0 is not a local minimizer of φ is also essential in case (ii)(b), as shown by the example of $\overline{\varphi} \circ \|\cdot\|$ and $\|x_0\| = 1$, $\dim X > 1$.

The necessity of the condition $\varphi(t) = 0$ for $t \leq 0$ in case (b) is shown by the function $\widetilde{\varphi} \circ \widetilde{g}$ and $x_0 = 1$: $\partial^{<}\widetilde{g}(x_0) = [1, \infty[$, $\partial^{<}\widetilde{\varphi}(t_0) = [2, \infty[$ and $\partial^{<}(\widetilde{\varphi} \circ \widetilde{g})(x_0) = [1, \infty[$.

An important particular case of the preceding result is when X is a normed vector space and $g(x) = \|x\|$ for every $x \in X$.

Corollary 3.8. *Let X be a normed space and $\varphi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be increasing on $\mathbb{R}_+$ with $\varphi(t) = 0$ for every $t \in \mathbb{R}_-$. Consider $f : X \to \overline{\mathbb{R}}$, $f(x) = \varphi(\|x\|)$ and $x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. Then*

$$\begin{aligned} \partial^{<} f(x_0) &= \partial^{<}\varphi(\|x_0\|) \cdot \partial^{<} \|\cdot\| (x_0) \\ &= \{x^* \in X^* \mid \langle x_0, x^* \rangle = \|x_0\| \, \|x^*\|, \ \|x^*\| \in \partial^{<}\varphi(\|x_0\|)\}. \end{aligned}$$

Proof. The conclusion is obvious for $x_0 = 0$, both sides of the relation being X^*, while for $x_0 \neq 0$ the conclusion follows from assertion (ii)(b) of the preceding proposition. $\square$

One may ask if there are other situations when (3.3) holds. Such a case occurs when φ and g are convex functions. To be more specific, we have:

Proposition 3.9. *Let $\varphi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be nondecreasing and convex, $g : X \to \mathbb{R} \cup \{\infty\}$ be convex, $g(X) \cap \operatorname{int}(\operatorname{dom}\varphi) \neq \emptyset$ and $x_0 \in \operatorname{dom} f$ which is not a minimizer of $f := \varphi \circ g$. Then (3.3) holds.*

Proof. Let $t_0 = g(x_0)$. Using [15, Th. 2.7.5], we have that $\partial f(x_0) = \cup\{\partial(\lambda g)(x_0) \mid \lambda \in \partial\varphi(t_0)\}$. Since x_0 is not a minimizer of f, t_0 is not a minimizer of φ and x_0 is not a minimizer of g. Therefore $\partial\varphi(t_0) \subset]0, \infty[$. It follows that $\partial f(x_0) = \partial\varphi(t_0) \cdot \partial g(x_0)$. Using now [5] Prop. 10, which says that for a convex function $k : X \to \mathbb{R} \cup \{\infty\}$ and $x_0 \in \operatorname{dom} k$ which is not a minimizer of k, we have $\partial^{<}k(x_0) = [1, \infty[\cdot \partial k(x_0)$, we obtain that (3.3) holds. $\square$

Note that with the preceding data, when $x_0 \in \operatorname{dom} f$ is a minimizer of f, then either x_0 is a minimizer of g (and (3.3) holds if $g(x_0) \in \operatorname{int}(\operatorname{dom}\varphi)$), or $g(x_0)$ is a minimizer of φ; in this case it may happen that (3.3) is not satisfied.

The fact that in (3.2) one does not have equality in general (even if one of the functions is convex) is shown by the following example.

Example 3.10. Let $\varphi, \psi : \mathbb{R} \to [0, \infty[$, $\varphi(t) = t^p$, $\psi(t) = t^q$ for $t \geq 0$, $\varphi(t) = \psi(t) = 0$ for $t < 0$, where $0 < q < 1 \leq p < \infty$. Then for every $t > 0$ we have that $\partial^< \varphi(t) = [pt^{p-1}, \infty[$, $\partial^< \psi(t) = [t^{q-1}, \infty[$. Taking $q = 1/p$ with $p > 1$ we have that

$$\begin{aligned} \partial^<(\varphi \circ \psi)(t) &= [1, \infty[= \partial^<(\psi \circ \varphi)(t), \\ \partial^< \varphi(\psi(t)) \cdot \partial^< \psi(t) &= [p, \infty[= \partial^< \psi(\varphi(t)) \cdot \partial^< \varphi(t). \end{aligned}$$

In the sequel, for $f : X \to \mathbb{R} \cup \{\infty\}$ and $x_0 \in \operatorname{dom} f$, we consider that

$$0 \cdot \partial^< f(x_0) := N([f < f(x_0)], x_0) = \partial^{\circledast} f(x_0), \qquad 0 \cdot \partial f(x_0) = N(\operatorname{dom} f, x_0), \tag{3.6}$$

even if $\partial^< f(x_0)$ (resp. $\partial f(x_0)$) is empty. These conventions are motivated by the fact that $0^+ \partial^< f(x_0) = N([f < f(x_0)], x_0)$ (see [5] Prop. 7, with a change of notation) and $0^+ \partial f(x_0) = N(\operatorname{dom} f, x_0)$ if $\partial^< f(x_0)$ and $\partial f(x_0)$ are non-empty, respectively.

Proposition 3.11. *Let $f, g : X \to \mathbb{R} \cup \{\infty\}$, $h = f \vee g$ and $x_0 \in X$ be such that $r_0 := h(x_0) \in \mathbb{R}$.*

(i) *If $f(x_0) = g(x_0)$ then*

$$\bigcup_{\lambda \in [0,1]} (\lambda \partial^< f(x_0) + (1 - \lambda) \partial^< g(x_0)) \subset \partial^< h(x_0). \tag{3.7}$$

(ii) *If $f(x_0) < g(x_0)$ then $N([f < r_0], x_0) + \partial^< g(x_0) \subset \partial^< h(x_0)$.*

Proof. (i) Let $x^* \in \partial^< f(x_0)$, $y^* \in \partial^< g(x_0)$, $\lambda \in]0, 1[$ and $z^* = \lambda x^* + (1 - \lambda) y^*$. Consider $x \in X$ such that $h(x) < h(x_0)$. It follows that $f(x) < f(x_0)$ and $g(x) < g(x_0)$. Hence

$$\begin{aligned} \langle x - x_0, x^* \rangle &\leq f(x) - f(x_0) \leq h(x) - h(x_0), \\ \langle x - x_0, y^* \rangle &\leq g(x) - g(x_0) \leq h(x) - h(x_0). \end{aligned}$$

Multiplying the first relation by λ and the second one by $1 - \lambda$, then adding them, we get $\langle x - x_0, z^* \rangle \leq h(x) - h(x_0)$. Therefore $z^* \in \partial^< h(x_0)$. The cases $\lambda = 0, 1$ follow as in the proof of (ii).

(ii) Let $x^* \in N([f < r_0], x_0)$ and $y^* \in \partial^< g(x_0)$. Consider $x \in X$ such that $h(x) < h(x_0)$. It follows that $x \in [f < r_0]$ and $g(x) < g(x_0)$. Hence

$$\langle x - x_0, x^* \rangle \leq 0, \quad \langle x - x_0, y^* \rangle \leq g(x) - g(x_0) \leq h(x) - h(x_0).$$

Adding both relations we get that $x^* + y^* \in \partial^< h(x_0)$. □

Note that when $\partial^< f(x_0)$ and $\partial^< g(y_0)$ are nonempty, the left hand side of relation (3.7) is the convex hull of $\partial^< f(x_0) \cup \partial^< g(y_0)$.

The inclusions above may be strict; it may even happen that the first set is empty and the second one is nonempty.

Example 3.12. $f(x) = x^3$, $g(x) = -x^3$, $x_0 = 0$ for (i), $x_0 = 1$ for (ii).

Example 3.13. Let $\gamma \in]0,1[$, $\eta \in [1,\infty[$ and $f,g:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases} -\eta & \text{if} \quad x\in]-\infty,-\eta], \\ x & \text{if} \quad x\in]-\eta,0], \\ x^2 & \text{if} \quad x\in]0,\infty[, \end{cases} \qquad g(x)=\begin{cases} -\eta & \text{if} \quad x\in]-\infty,0], \\ \frac{\eta+\gamma^2}{\gamma}x-\eta & \text{if} \quad x\in]0,\gamma], \\ (1+\gamma)x-\gamma & \text{if} \quad x\in]\gamma,\infty[. \end{cases}$$

Then

$$(f\vee g)(x)=\begin{cases} -\eta & \text{if} \quad x\in]-\infty,-\eta], \\ x & \text{if} \quad x\in]-\eta,0], \\ x^2 & \text{if} \quad x\in]0,\gamma], \\ (1+\gamma)x-\gamma & \text{if} \quad x\in]\gamma,1], \\ x^2 & \text{if} \quad x\in]1,\infty]. \end{cases}$$

We have $f(1)=g(1)=1$, $\partial^< f(1)=[2,\infty[$, $\partial^< g(1)=[1+\eta,\infty[$ and $\partial^<(f\vee g)(1)=[1+\gamma,\infty[$. Hence $\overline{\text{co}}\,(\partial^< f(1)\cup\partial^< g(1))=[2,\infty[\neq\partial^<(f\vee g)(1)$. Moreover, $\inf f=\inf g\in\mathbb{R}$, f,g are strictly quasiconvex and continuous (even Lipschitzian on sublevel sets; taking $f(x)=x$ for $x\geq 1$, f,g are Lipschitzian on $\mathbb{R}$).

Taking now the same functions and $x_0=(1+\gamma)/2$, $f(x_0)<g(x_0)$, $N([f<f(x_0)],x_0)=\{0\}$, $\partial^< g(x_0)=[\frac{1+\gamma^2+2\eta}{1+\gamma},\infty[$ and $\partial^<(f\vee g)(x_0)=[1+\gamma,\infty[$.

Proposition 3.14. *Let $f:X\to\mathbb{R}\cup\{\infty\}$, $g:Y\to\mathbb{R}\cup\{\infty\}$ and $M:X\times Y\to\mathbb{R}\cup\{\infty\}$, $M(x,y)=f(x)\vee g(y)$. Consider $(x_0,y_0)\in X\times Y$ such that $r_0:=M(x_0,y_0)\in\mathbb{R}$.*

(i) *If $f(x_0)=g(y_0)$, then*

$$\bigcup_{\lambda\in[0,1]}(\lambda\partial^< f(x_0)\times(1-\lambda)\partial^< g(y_0))\subset\partial^< M(x_0,y_0). \tag{3.8}$$

(ii) *If $f(x_0)<g(y_0)$, then*

$$N([f<r_0],x_0)\times\partial^< g(y_0)\subset\partial^< M(x_0,y_0)\subset X^*\times\partial^< g(y_0).$$

Moreover, if y_0 is not a local minimizer of g then

$$\partial^< M(x_0,y_0)=N([f<r_0],x_0)\times\partial^< g(y_0).$$

Proof. The conclusions of (i) and the first part of (ii) are obvious if (x_0,y_0) is a minimizer of M. In the contrary case, let $\widetilde{f},\widetilde{g}:X\times Y\to\overline{\mathbb{R}}$ be given by $\widetilde{f}(x,y)=f(x)$ and $\widetilde{g}(x,y)=g(y)$. Of course, $M=\widetilde{f}\vee\widetilde{g}$. By Proposition 3.2 and Corollary 3.4 we have that $\partial^<\widetilde{f}(x_0,y_0)=\partial^< f(x_0)\times\{0\}$ in case (i) and $\partial^<\widetilde{g}(x_0,y_0)=\{0\}\times\partial^< g(y_0)$ in cases (i) and (ii). As $[\widetilde{f}<r_0]=[f<r_0]\times Y$, the conclusion (i) and the first inclusion of the first part of (ii) follow from Proposition 3.11; the second inclusion in (ii) is immediate: taking $y\in[g<g(y_0)]$ and $x=x_0$ in the definition of $(x^*,y^*)\in\partial^< M(x_0,y_0)$ we get that $y^*\in\partial^< g(y_0)$.

Suppose now that we are in case (ii) and y_0 is not a local minimizer of g. There exists a net $(y_i)_{i\in I}\subset[g<g(y_0)]$ such that $(y_i)\to y_0$. Then for any $(x^*,y^*)\in\partial^< M(x_0,y_0)$ we have $y^*\in\partial^< g(y_0)$ and

$$\langle x-x_0,x^*\rangle+\langle y_i-y_0,y^*\rangle<0 \quad \forall\, x\in[f<r_0],\ i\in I.$$

Passing to the limit for $i \in I$, we obtain that $\langle x - x_0, x^* \rangle \leq 0$ for every $x \in [f < r_0]$, whence $x^* \in N([f < r_0], x_0)$. $\square$

Remark 3.15. By Proposition 3.14 (ii), if

$$\partial^< M(x_0, y_0) \cap \bigcup_{\lambda \in \,]0,1[} (\lambda \partial^< f(x_0) \times (1-\lambda)\partial^< g(y_0)) \neq \emptyset,$$

f and g are upper semi-continuous at x_0 and y_0, and x_0, y_0 are not local minimizers of f, g, respectively, then $f(x_0) = g(y_0)$. Indeed, if $f(x_0) < g(y_0)$ then $N([f < r_0], x_0) = \{0\}$, whence $0 \in \partial^< f(x_0)$ and x_0 is a minimizer, a contradiction.

It is known that for proper convex functions $f_i : X_i \to \overline{\mathbb{R}}$, $1 \leq i \leq n$, and $\psi : \prod_{i=1}^n X_i \to \overline{\mathbb{R}}$, given by $\psi(x_1, \ldots, x_n) = \max_{1 \leq i \leq n} f_i(x_i)$, if $x = (x_1, \ldots, x_n) \in \operatorname{dom} \psi$ and $I(x) = \{i \mid f_i(x_i) = \psi(x)\}$, then

$$\partial \psi(x) = \bigcup \left\{ \prod\nolimits_{i=1}^n \lambda_i \partial f_i(x_i) \mid \lambda_i \geq 0,\ \sum\nolimits_{i=1}^n \lambda_i = 1,\ \lambda_i = 0 \ \text{ if } i \notin I(x) \right\}.$$

From this relation and from [10], [5] Prop. 10, one obtains easily that (3.8) holds with equality, i.e.

$$\partial^< M(x_0, y_0) = \bigcup_{\lambda \in [0,1]} (\lambda \partial^< f(x_0) \times (1-\lambda)\partial^< g(y_0)), \tag{3.9}$$

if f and g are convex functions, $f(x_0) = g(y_0) \in \mathbb{R}$ and (x_0, y_0) is not a minimizer of M (equivalently, x_0 is not a minimizer of f and y_0 is not a minimizer of g). One may ask whether (3.9) holds for arbitrary functions. The following two examples show that this is not the case.

Example 3.16. Let $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, $g(x) = |x|$ and $M(x, y) = f(x) \vee g(y)$. As $M(x, y) = (f(x) \vee 0) \vee g(y)$, and $f \vee 0$, g are convex, M is convex. It follows that $\partial^< M(1,1) = [1, \infty[\cdot \partial M(1,1) = [1, \infty[\cdot \{(3\lambda, 1-\lambda) \mid \lambda \in [0,1]\}$. Moreover, $M(1,1) = f(1) = g(1)$, $\partial^< f(1) = \emptyset$, $\partial^< g(1) = [1, \infty[$, and so

$$\partial^< M(1,1) \neq \bigcup_{\lambda \in [0,1]} (\lambda \partial^< f(1) \times (1-\lambda)\partial^< g(1)) = [0, \infty[\times [1, \infty[.$$

One can say that the above example is not very conclusive because $\partial^< f(x_0) = \emptyset$ or $\inf f \neq \inf g$. In fact one observes that taking $m := \max\{\inf f, \inf g\}$ (supposed to be a real number), we have that $M(x, y) = f_m(x) \vee g_m(y)$, where $f_m(x) = m \vee f(x)$ and $g_m(y) = m \vee g(y)$. In the next example $\inf f = \inf g$ and $\partial^< f(x_0) \neq \emptyset$.

Example 3.17. Let f, $g : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{|x|}$, $g(x) = x^2$ and $M(x, y) = f(x) \vee g(y)$. We have that f is (strictly) quasiconvex and g is convex. Consider $x_0 = y_0 = 1$. We have that $f(x_0) = g(y_0) = M(x_0, y_0)$. Moreover, $\partial^< f(1) = [1, \infty[$ and $\partial^< g(1) = [2, \infty[$, whence

$$\bigcup_{\lambda \in [0,1]} (\lambda \partial^< f(1) \times (1-\lambda)\partial^< g(1)) = \{(a, b) \in \mathbb{R}^2 \mid a, b \geq 0,\ 2a + b \geq 2\}.$$

Some computations show that we have

$$(a, b) \in \partial^< M(1,1) \Leftrightarrow a, b \geq 0 \text{ and } \left[a(y^3 + y^2 + y + 1) + b \geq y + 1 \ \forall y \in [0, 1[\ \right].$$

To obtain the final form of $\partial^< M(1,1)$, for $a,b \geq 0$ we consider the function ζ given by $\zeta(y) := a(y^3+y^2+y+1)+b-y-1$ for $y \in \mathbb{R}$ and we get that $\partial^< M(1,1)$ is the epigraph of the function $\xi : [0,\infty[\to [0,\infty[$,

$$\xi(a) = \begin{cases} 2-4a & \text{if } \ a \in [0,\frac{1}{6}], \\ \frac{2}{3} - \frac{20}{27}a + \frac{6-4a}{27}\sqrt{\frac{3-2a}{a}} & \text{if } \ a \in]\frac{1}{6},1], \\ 0 & \text{if } \ a \in]1,\infty[. \end{cases}$$

Replacing f by $f_\gamma : \mathbb{R} \to \mathbb{R}$, $f_\gamma(x) = \gamma^2 |x|$ for $|x| \leq \gamma^4$, $f_\gamma(x) = \sqrt{|x|}$ for $|x| > \gamma^4$, where $\gamma \in]0,1[$, we find that (3.9) does not hold even if f is quasiconvex and Lipschitzian and g is convex. One easily sees that $\partial^< f_\gamma(1) = \partial^< f(1) = [1,\infty[$, but

$$\begin{aligned} &(\tfrac{1}{6}, \max\{\tfrac{4}{3}, 2\gamma - \tfrac{1}{3}\gamma^4, \tfrac{1}{6}(1+\gamma)(5-\gamma^2)\}) \\ &\qquad\qquad \in \partial^<(f_\gamma \vee g)(1) \setminus \bigcup_{\lambda \in [0,1]} (\lambda \partial^< f_\gamma(1) \times (1-\lambda)\partial^< g(1)) \end{aligned}$$

for every $\gamma \in]0,1]$.

A natural question is whether there are situations in which (3.9) holds besides the case where f and g are convex. The next result gives such a case; another one will be provided later on. It uses the lower quasi-inverse g^e (see [7]) of a nondecreasing function $g : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by

$$g^e(t) := \inf\{s \in \mathbb{R} \mid t \leq g(s)\}.$$

Lemma 3.18. *Let $f : X \to \mathbb{R}\cup\{\infty\}$, let $g : \mathbb{R} \to \mathbb{R}\cup\{\infty\}$ be nondecreasing and let $M : X \times \mathbb{R} \to \mathbb{R}\cup\{\infty\}$ be given by $M(x,y) = f(x) \vee g(y)$. Consider $x_0 \in X$, $y_0 \in \mathbb{R}$ with $t_0 = f(x_0) = g(y_0) \in \mathbb{R}$ and $(x^*,y^*) \in \partial^< M(x_0,y_0)$. Suppose $g(\mathbb{R}) - \mathbb{R}_+ \supset f([f < t_0])$. If x_0 and y_0 are not local minimizers of f and g respectively, then for each $s^* \in \partial^< g^e(t_0)$, either $s^*y^* \geq 1$ and $x^* \in N([f < f(x_0)], x_0)$, or $s^*y^* \in [0,1[$ and $(1-s^*y^*)^{-1}x^* \in \partial^< f(x_0)$. So, in the case $g(y) = y$ for each $y \in \mathbb{R}$, relation* (3.9) *holds.*

Proof. Taking $(x_n)_{n\geq 1} \subset [f < f(x_0)]$ with $(x_n) \to x_0$ and $y < y_0$, we get that $y^* \geq 0$, while taking $x \in [f < f(x_0)]$ and $(y_n) \nearrow y_0$, we get that $x^* \in N([f < f(x_0)], x_0)$. Suppose that $s^*y^* \in [0,1[$ and take $x \in [f < f(x_0)]$. By assumption, there exists $y \in]-\infty, y_0[$ such that $g(y) \geq t$ with $t := f(x)$. Then we have $g^e(t) \leq y$. Moreover, as $g(y') < g(y_0)$ for $y' < y_0$, we have $g^e(t_0) = y_0$. Thus, as $g^e(t) < g^e(t_0)$, we obtain

$$s^*(f(x) - f(x_0)) = s^*(t - t_0) \leq g^e(t) - g^e(t_0) \leq y - y_0.$$

Hence

$$\langle x - x_0, x^* \rangle + y^*s^*(f(x) - f(x_0)) \leq \langle x - x_0, x^* \rangle + y^*(y - y_0) \leq f(x) - f(x_0)$$

and so $u^* := (1 - s^*y^*)^{-1}x^* \in \partial^< f(x_0)$, x being arbitrary in $[f < f(x_0)]$. When g is the identity mapping, g^e is also the identity mapping and $\partial^< g^e(t_0) = [1,\infty[= \partial^< g(y_0)$. Taking $s^* = 1$ we have that $(x^*,y^*) = ((1-y^*)u^*, y^*1)$ belongs to the right-hand side of relation (3.9) when $y^* \in [0,1[$. The same conclusion holds in the case $y^* \geq 1$, in view of our convention $0 \cdot \partial^< f(x_0) = N([f < f(x_0)], x_0)$. Taking (3.8) into account, relation (3.9) holds. $\square$

Example 3.16 shows that (3.9) does not hold in general for $g(t) = |t|^p$ with $p \neq 1$.

We take now g as a composition of an increasing function with the norm.

Proposition 3.19. *Let $\varphi : [0, \infty[\to [0, \infty[$ be an increasing function with $\varphi(0) = 0$, $f : X \to \overline{\mathbb{R}}$ and Y a normed vector space. Consider $x_0 \in X$, $y_0 \in Y$, $x^* \in X^*$ and $y^* \in Y^*$ such that $\langle y_0, y^* \rangle = \|y_0\| \cdot \|y^*\|$ and $(x^*, \|y^*\|) \in \partial^<(f \vee \varphi)(x_0, \|y_0\|)$. Then $(x^*, y^*) \in \partial^<(f \vee \varphi \circ \|\cdot\|)(x_0, y_0)$.*

Proof. Let $(x, y) \in X \times Y$ be such that $f(x) \vee \varphi(\|y\|) < f(x_0) \vee \varphi(\|y_0\|)$. Then

$$\begin{aligned} \langle x - x_0, x^* \rangle + \langle y - y_0, y^* \rangle &\leq \langle x - x_0, x^* \rangle + \|y^*\| \left(\|y\| - \|y_0\| \right) \\ &\leq f(x) \vee \varphi(\|y\|) - f(x_0) \vee \varphi(\|y_0\|), \end{aligned}$$

i.e. $(x^*, y^*) \in \partial^<(f \vee \varphi \circ \|\cdot\|)(x_0, y_0)$. □

The converse is also true under some additional hypotheses.

Proposition 3.20. *Let $\varphi : [0, \infty[\to [0, \infty[$ be an increasing function with $\varphi(0) = 0$, $f : X \to \overline{\mathbb{R}}$ be such that $\inf f = 0$ and Y a normed space. Consider also $x_0 \in X$, $y_0 \in Y$, $x^* \in X^*$ and $y^* \in Y^* \setminus \{0\}$ such that x_0 is not a local minimizer for f and φ is right-continuous at $\|y_0\|$. If $(x^*, y^*) \in \partial^<(f \vee \varphi \circ \|\cdot\|)(x_0, y_0)$ and $\gamma_0 := f(x_0) \vee \varphi(\|y_0\|)$, then*

$$\begin{aligned} f(x_0) &\leq \varphi(\|y_0\|), \quad (x^*, \|y^*\|) \in \partial^<(f \vee \varphi)(x_0, \|y_0\|), \\ \langle y_0, y^* \rangle &= \|y_0\| \cdot \|y^*\|, \quad x^* \in N([f < \gamma_0], x_0). \end{aligned}$$

Moreover, if $f(x_0) < \varphi(\|y_0\|)$ then $\|y^*\| \in \partial^< \varphi(\|y_0\|)$ and so $y^* \in \partial^<(\varphi \circ \| \cdot \|)(y_0)$.

Proof. Since φ is right-continuous at $t_0 := \|y_0\|$, $g := \varphi \circ \|\cdot\|$ is upper semi-continuous at y_0. Supposing that $g(y_0) < f(x_0)$, by Proposition 3.14, we get the contradiction $y^* = 0$ because $N([g < f(x_0)], y_0) = \{0\}$. Therefore $f(x_0) \leq \varphi(\|y_0\|)$. In particular $y_0 \neq 0$ (since $f(x_0) > \inf f = \varphi(0)$). We have that

$$\langle x - x_0, x^* \rangle + \langle y - y_0, y^* \rangle \leq f(x) \vee \varphi(\|y\|) - f(x_0) \vee \varphi(\|y_0\|)$$

for every $(x, y) \in X \times Y$ such that $f(x) \vee \varphi(\|y\|) < \gamma_0$; such (x, y) exist in our conditions. Considering $t \in [0, t_0[$, $x \in X$ with $f(x) \vee \varphi(t) < \gamma_0$, and arbitrary $y \in Y$ with $\|y\| = t$ from the above relation we get

$$\langle x - x_0, x^* \rangle + t \|y^*\| - \langle y_0, y^* \rangle \leq f(x) \vee \varphi(t) - f(x_0) \vee \varphi(t_0). \tag{3.10}$$

Since x_0 is not a local minimizer of f, there exists $(x_n)_{n \geq 1} \subset [f < f(x_0)]$ such that $(x_n) \to x_0$. From (3.10) we obtain that

$$\langle x_n - x_0, x^* \rangle + t \|y^*\| - \langle y_0, y^* \rangle < 0 \quad \forall t \in [0, t_0[.$$

Taking the limit for $n \to \infty$, we obtain that

$$t \|y^*\| \leq \langle y_0, y^* \rangle \leq \|y_0\| \cdot \|y^*\| \quad \forall t \in [0, t_0[.$$

Taking $t \uparrow t_0$ we get $\langle y_0, y^* \rangle = \|y_0\| \cdot \|y^*\|$. Now, from (3.10), we obtain that

$$\langle x - x_0, x^* \rangle + \|y^*\| (t - t_0) \leq f(x) \vee \varphi(t) - f(x_0) \vee \varphi(t_0) \tag{3.11}$$

whenever $f(x) \vee \varphi(t) < f(x_0) \vee \varphi(t_0)$, i.e. $(x^*, \|y^*\|) \in \partial^<(f \vee \varphi)(x_0, \|y_0\|)$. Fixing $x \in [f < \gamma_0]$ and taking $t \uparrow t_0$ in (3.11), we obtain that $x^* \in N([f < \gamma_0], x_0)$.

If $f(x_0) < \varphi(t_0)$, in (3.11) we may take $x = x_0$, obtaining that $\|y^*\| \in \partial^< \varphi(t_0)$. □

Corollary 3.21. *Let Y be a normed space, $f : X \to \mathbb{R} \cup \{\infty\}$ be such that* $\inf f = 0$ *and $M(x, y) = f(x) \vee \|y\|$. Suppose that $x_0 \in X$ is not a local minimizer of f and $y_0 \in Y$ is such that $f(x_0) = \|y_0\|$. Then* (3.9) *holds at* (x_0, y_0).

Proof. Consider $g : Y \to \mathbb{R}$, $g(y) = \|y\|$. As (3.8) holds, let $(x^*, y^*) \in \partial^< M(x_0, y_0)$. If $y^* = 0$ then, taking $y = 0$ and $x \in [f < f(x_0)]$ in the definition of $\partial^< M(x_0, y_0)$, we get that $x^* \in \partial^< f(x_0)$. Hence $(x^*, y^*) \in 1 \cdot \partial^< f(x_0) \times 0 \cdot \partial^< g(y_0)$. Suppose now that $y^* \neq 0$. Taking $\varphi(t) := t \vee 0$, from Proposition 3.20 we have that

$$(x^*, \|y^*\|) \in \partial^<(f \vee \varphi)(x_0, \|y_0\|), \quad \langle y_0, y^* \rangle = \|y_0\| \cdot \|y^*\|, \quad x^* \in N([f < f(x_0)], x_0).$$

If $\|y^*\| \geq 1$, $\|y^*\|^{-1} y^* \in \partial g(y_0)$, whence $y^* \in \partial^< g(y_0)$. It follows that $(x^*, y^*) \in 0 \cdot \partial^< f(x_0) \times 1 \cdot \partial^< g(y_0)$ in this case. Let $0 < \|y^*\| < 1$. By Lemma 3.18 we have that $(1 - \|y^*\|)^{-1} x^* \in \partial^< f(x_0)$, which shows that $(x^*, y^*) \in (1 - \|y^*\|) \cdot \partial^< f(x_0) \times \|y^*\| \cdot \partial^< g(y_0)$. Hence (3.9) holds. □

The following lemma is an important step toward the case of the sublevel-convolution.

Lemma 3.22. *Let $f, g : X \to \mathbb{R} \cup \{\infty\}$ and let $M, H : X \times X \to \mathbb{R} \cup \{\infty\}$, be given by $M(x, y) = f(x) \vee g(y)$ and $H(w, x) = M(x, w - x)$. Suppose that $r_0 := H(w_0, x_0) \in \mathbb{R}$. Then*

$$(w^*, x^*) \in \partial^< H(w_0, x_0) \Leftrightarrow (w^* + x^*, w^*) \in \partial^< M(x_0, w_0 - x_0).$$

In particular, for $y_0 := w_0 - x_0$, one has

$$(w^*, 0) \in \partial^< H(w_0, x_0) \Leftrightarrow (w^*, w^*) \in \partial^< M(x_0, y_0).$$

If $f(x_0) = g(y_0)$, then for $C := \bigcup_{\lambda \in [0,1]} (\lambda \partial^< f(x_0) \times (1 - \lambda) \partial^< g(y_0 - x_0))$ one has

$$\{(y^*, x^* - y^*) \mid (x^*, y^*) \in C\} \subset \partial^< H(x_0, y_0),$$

If $f(x_0) < g(y_0)$ then

$$\{(y^*, x^* - y^*) \mid (x^*, y^*) \in N([f < r_0], x_0) \times \partial^< g(y_0)\} \subset \partial^< H(x_0, y_0).$$

Proof. Let $A \in L(X \times X, X \times X)$ be given by $A(w, x) = (x, w - x)$. One gets easily that $A^t(w^*, x^*) = (x^*, w^* - x^*)$. As $H = M \circ A$ and A is an isomorphism, the conclusion follows from Propositions 3.2 and 3.14. □

It is known (see [11], [15, Cor. 2.7.6]) that for proper convex functions $f, g : X \to \mathbb{R} \cup \{\infty\}$, $x_0 \in \operatorname{dom} f$, $y_0 \in \operatorname{dom} g$, $w_0 = x_0 + y_0$ and $r_0 = (f \diamond g)(w_0)$, one has

$$\partial(f \diamond g)(w_0) = \begin{cases} \partial f(x_0) \diamond \partial g(y_0) & \text{if } f(x_0) = g(y_0) = r_0, \\ N(\operatorname{dom} f, x_0) \cap \partial g(y_0) & \text{if } f(x_0) < g(y_0) = r_0, \end{cases} \tag{3.12}$$

where, for subsets A and B of X

$$A\Diamond B := \left(\bigcup_{\lambda\in\,]0,1[} \lambda A\cap(1-\lambda)B\right)\cup(0\cdot A\cap B)\cup(A\cap 0\cdot B)\,; \tag{3.13}$$

here, as in [8], $0\cdot A = 0^+A$ with $0^+\emptyset=\{0\}$, unless A is a subdifferential in which case we use the convention (3.6).

Recall that when A, B are closed convex sets such that $\mathbb{P}A\cap\mathbb{P}B\neq\emptyset$ for $\mathbb{P}=]0,\infty[$, one has (see [8])

$$A\Diamond B = \operatorname{cl}\left(\bigcup_{\lambda\in\,]0,1[} \lambda A\cap(1-\lambda)B\right) = \operatorname{cl}(A\# B)\,.$$

From formula (3.12) one obtains that for f, g, x_0, y_0, w_0, r_0 as above one has

$$\partial^<(f\Diamond g)(w_0) = \begin{cases} \partial^< f(x_0)\Diamond\partial^< g(y_0) & \text{if} \quad f(x_0)=g(y_0)=r_0, \\ N(\operatorname{dom} f, x_0)\cap\partial^< g(y_0) & \text{if} \quad f(x_0)<g(y_0)=r_0. \end{cases} \tag{3.14}$$

A natural question is whether (3.14) is valid for arbitrary (quasi-convex) functions. In the next result we shall see that the inclusion $\supset$ always holds, while the converse one is true in very special cases. With respect to this point, some remarks are in order.

Given $f, g : X \to \mathbb{R}\cup\{\infty\}$, it is obvious that $\inf f\Diamond g = \inf f\vee\inf g$. Moreover, for $m := \inf f\vee\inf g$, $f_m := f\vee m$, $g_m := g\vee m$ we have $f\Diamond g = f_m\Diamond g_m$. Moreover, for every $\alpha\in\mathbb{R}$ and every $x_0\in\operatorname{dom} f$ such that $f(x_0)\geq\alpha$ one has $\partial^<(f\vee\alpha)(x_0)\supset\partial^< f(x_0)$. The above inclusion is obvious if $f(x_0)=\alpha$. Let $f(x_0)>\alpha$ and consider $x^*\in\partial^< f(x_0)$. If $(f\vee\alpha)(x)<(f\vee\alpha)(x_0)=f(x_0)$, $f(x)<f(x_0)$. Thus $\langle x-x_0, x^*\rangle\leq f(x)-f(x_0)\leq(f\vee\alpha)(x)-(f\vee\alpha)(x_0)$. Therefore $x^*\in\partial^<(f\vee\alpha)(x_0)$. Thus, it may be convenient to suppose that $\inf f=\inf g$ when considering $f\Diamond g$, especially when one wants to prove the inclusion $\partial^<(f\Diamond g)(w_0)\subset\partial^< f(x_0)\Diamond\partial^< g(y_0)$.

Proposition 3.23. *Let $f, g : X\to\mathbb{R}\cup\{\infty\}$, $h = f\Diamond g$ and $x_0\in\operatorname{dom} f$, $y_0\in\operatorname{dom} g$ and $w_0 = x_0+y_0$.*

(i) *If $h(w_0)=f(x_0)\vee g(y_0)$, x_0 is not a local minimizer of f and g is u.s.c. at y_0 then $f(x_0)\leq g(y_0)$.*
(ii) *If $f(x_0)=g(y_0)$ and $\partial^< f(x_0)\Diamond\partial^< g(y_0)$ is nonempty, then $h(w_0)=f(x_0)\vee g(y_0)$ and $\partial^< f(x_0)\Diamond\partial^< g(y_0)\subset\partial^<(f\Diamond g)(w_0)$.*
(iii) *If $f(x_0)<g(y_0)$ and $N([f<g(y_0)], x_0)\cap\partial^< g(y_0)\neq\emptyset$, then $h(w_0)=f(x_0)\vee g(y_0)$ and $N([f<g(y_0)], x_0)\cap\partial^< g(y_0)\subset\partial^<(f\Diamond g)(w_0)$.*

Proof. (i) Suppose that $f(x_0)>g(y_0)$. Since x_0 is not a local minimizer of f, there exists a net $(x_i)_{i\in I}\subset[f<f(x_0)]$ such that $(x_i)\to x_0$. Of course, because $(w_0-x_i)\to y_0$ and g is u.s.c. at y_0, there exists $i_0\in I$ such that $(w_0-x_i)_{i\succeq i_0}\subset[g<f(x_0)]$. We get the contradiction $h(w_0)\leq f(x_{i_0})\vee g(w_0-x_{i_0})<f(x_0)=h(w_0)$.

(ii), (iii) Consider $H : X\times X\to\mathbb{R}\cup\{\infty\}$, given by $H(w,x)=f(x)\vee g(w-x)$. We observe that h is the performance function associated to H.

Let $w^* \in \partial^< f(x_0) \Diamond \partial^< g(y_0)$ in case (ii) and $w^* \in N([f < g(y_0)], x_0) \cap \partial^< g(y_0)$ in case (iii); it is obvious that $(w^*, w^*) \in C$ (defined at Lemma 3.22) in the first case and $(w^*, w^*) \in N([f < g(y_0)], x_0) \times \partial^< g(y_0)$ in the second one. In both cases, by Lemma 3.22, $(w^*, 0) \in \partial^< H(w_0, x_0)$. By Lemma 3.1 we have that $h(w_0) = H(w_0, x_0)$ and $w^* \in \partial^< h(w_0)$. □

When g has the form $\varphi \circ \|\cdot\|$, a more special result can be given.

Proposition 3.24. *Let X be a normed vector space and $h = f \Diamond (\varphi \circ \|\cdot\|)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is increasing on $\mathbb{R}_+$, with $\varphi(t) = 0$ for $t \leq 0$ and $f : X \to \mathbb{R} \cup \{\infty\}$ is such that $\inf f = 0$. Consider also $x_0 \in \operatorname{dom} f$, $y_0, w_0 \in X$ such that $w_0 = x_0 + y_0$ and $w^* \in X^*$.*

(i) *If*

$$(w^*, \|w^*\|) \in \partial^< (f \vee \varphi)(x_0, \|y_0\|), \quad \langle y_0, w^* \rangle = \|y_0\| \cdot \|w^*\|, \tag{3.15}$$

then $h(w_0) = f(x_0) \vee \varphi(\|y_0\|)$ and $w^* \in \partial^< h(w_0)$.

(ii) *If $h(w_0) = f(x_0) \vee \varphi(\|y_0\|)$, x_0 is not a local minimizer of f and φ is right-continuous at $\|y_0\|$ then $f(x_0) \leq \varphi(\|y_0\|)$. Moreover, if $w^* \in \partial^< h(w_0)$ then $w^* \neq 0$, $w^* \in N([f < h(w_0)], x_0)$ and* (3.15) *holds.*

(iii) *Under the conditions of* (ii)*, if $f(x_0) < \varphi(\|y_0\|)$ then*

$$\begin{aligned} \partial^< h(w_0) &= N([f < h(w_0)], x_0) \cap \partial^< (\varphi \circ \|\cdot\|)(y_0) \\ &= N([f < h(w_0)], x_0) \cap \partial^< \varphi(\|y_0\|) \cdot \partial^< \|\cdot\| (y_0). \end{aligned} \tag{3.16}$$

Proof. (i) By Proposition 3.19 we obtain that $(w^*, w^*) \in \partial^< M(x_0, y_0)$, where $g := \varphi \circ \|\cdot\|$, $M(x, y) = f(x) \vee g(y)$. Therefore, by Lemma 3.22, $(w^*, 0) \in \partial^< H(w_0, x_0)$, where $H(w, x) = f(x) \vee g(w - x)$. Since h is the performance function of H, using Lemma 3.1, we obtain that $h(w_0) = H(w_0, x_0) = f(x_0) \vee \varphi(\|y_0\|)$ and $w^* \in \partial^< h(w_0)$.

(ii) The first part follows from Proposition 3.23 (i). Since x_0 is not a local minimizer of f, we have that $h(w_0) \geq f(x_0) > \inf f = \inf h = 0$. Hence w_0 is not a minimizer of h. Let $w^* \in \partial^< h(w_0)$; it follows that $w^* \neq 0$. Since h is the performance function of H defined above we have, by Lemmas 3.1 and 3.22, that $(w^*, w^*) \in \partial^< (f \vee \varphi \circ \|\cdot\|)(x_0, y_0)$. By Proposition 3.20 the conclusion follows.

(iii) Suppose that $f(x_0) < \varphi(\|y_0\|)$ $(= h(w_0))$ and take $w^* \in \partial^< h(w_0)$. Of course, the conclusion of part (ii) holds; moreover, from Proposition 3.20, we have that $\|w^*\| \in \partial^< \varphi(\|y_0\|)$. It follows that $w^* \in N([f < h(w_0)], x_0) \cap \partial^< \varphi(\|y_0\|) \cdot \partial^< \|\cdot\| (y_0)$. As the last equality of (3.16) is given by Corollary 3.8, the conclusion follows from Proposition 3.23 (iii). □

In the case $\varphi(t) = t$ we obtain a more precise result using Corollary 3.21 and the preceding proposition.

Corollary 3.25. *Let X be a normed vector space and $h = f \Diamond \|\cdot\|$, where $f : X \to \overline{\mathbb{R}}$ is such that $\inf f = 0$. Consider also $x_0, y_0, w_0 \in X$ such that $w_0 = x_0 + y_0$. If x_0 is not a local minimizer of f and $h(w_0) = f(x_0) \vee \|y_0\|$ then $f(x_0) \leq \|y_0\|$. Moreover, if $f(x_0) = \|y_0\|$ then*

$$\partial^< h(w_0) = \partial^< f(x_0) \Diamond \partial^< \|\cdot\| (y_0), \tag{3.17}$$

i.e. (3.14) *holds.*

Proof. The first part follows by Proposition 3.24 (ii).

Suppose that $h(w_0) = f(x_0) = \|y_0\|$ and take $w^* \in \partial^< h(w_0)$. Using again Proposition 3.24 (ii) and Corollary 3.21, we obtain that $w^* \in \partial^< f(x_0) \Diamond \partial^< \|\cdot\| (y_0)$. Therefore the inclusion $\subset$ holds in (3.17). The converse inclusion is true, by Proposition 3.23, even without requiring that x_0 is not a local minimizer of f. □

4. A new class of functions

The lower subdifferential and the subdifferentials of normal type are both important but have distinct features. For instance, we observed that $\partial^\nu f(x_0)$ and $\partial^\circledast f(x_0)$ always contain 0 and $\partial^* f(x_0)$ is nonempty whenever f is u.s.c. while $\partial^< f(x_0)$ is often empty. Moreover, we noticed that the inclusions one gets in the calculus rules sometimes are in opposite directions. Thus it is of interest to find conditions under which these two types of subdifferential are related.

We first observe that, taking into account the convention (3.6), the relation

$$\partial^\circledast f(x_0) = \mathbb{R}_+ \partial^< f(x_0)$$

holds for any function f and any point x_0 where f is finite. Such a relation is useful when considering multipliers for constrained problems. A similar purpose justifies the introduction of a new class of functions: the class $\mathcal{C}(x_0)$ of functions f which are finite at x_0 and such that (for $\mathbb{P} =]0, \infty[$)

$$\partial^* f(x_0) = \mathbb{P} \partial^< f(x_0).$$

Since the inclusion

$$\partial^* f(x_0) \supset \mathbb{P} \partial^< f(x_0)$$

always holds, this class is characterized by the opposite inclusion. Of course, if x_0 is a minimizer of f then $f \in \mathcal{C}(x_0)$.

Proposition 4.1. *Any proper convex function $f : X \to \overline{\mathbb{R}}$ belongs to $\mathcal{C}(x_0)$ if x_0 is not a minimizer of f and $\mathbb{P}(\operatorname{dom} f - x_0) = X$.*

Proof. Let $x^* \in \partial^* f(x_0)$. Since x_0 is not a minimizer of f, one has $x^* \neq 0$ and $[f \leq f(x_0)]$ is contained in the closure of $[f < f(x_0)]$, so that $x^* \in N([f \leq f(x_0)], x_0) \backslash \{0\}$. By [8] Prop. 5.4 it follows that $x^* \in [0, \infty[\partial f(x_0)$. Let $\lambda \geq 0$ be such $x^* \in \lambda \partial f(x_0)$ with the convention of [8] reproduced in relation (3.6). Since $x^* \neq 0$ and $\mathbb{P}(\operatorname{dom} f - x_0) = X$, so that $N(\operatorname{dom} f, x_0) = \{0\}$, as observed in [8] Remark 5.1, we cannot have $\lambda = 0$. Thus $x^* \in]0, \infty[\partial f(x_0) =]0, \infty[([1, \infty[\partial f(x_0)) =]0, \infty[\partial^< f(x_0)$ by [5] Prop. 10. □

Example 4.2. The convex function with domain $[-1, 1]$ given by $f(x) = -\sqrt{1 - x^2}$ does not belong to $\mathcal{C}(x_0)$ for $x_0 = 1$. Here the assumption $\mathbb{P}(\operatorname{dom} f - x_0) = X$ is not satisfied. We also note that the assumption $\partial^< f(x_0) \neq \emptyset$ of the following result is not satisfied.

Proposition 4.3. *For $X = \mathbb{R}$, any function $f : X \to \overline{\mathbb{R}}$ finite at x_0 and such that $\partial^< f(x_0) \neq \emptyset$ belongs to $\mathcal{C}(x_0)$.*

The easy proof of this result relies on a distinction between the three cases $\partial^< f(x_0) \subset \mathbb{P}$, $\partial^< f(x_0) \subset -\mathbb{P}$ and $\partial^< f(x_0) = \mathbb{R}$.

Although not all quasiconvex functions belong to the class $\mathcal{C}(x_0)$, as the preceding example (or the example $x \mapsto x^3$) shows, the stability properties described below enrich this class.

Proposition 4.4. *Given locally convex spaces X, Y, $x_0 \in X$, $A \in L(X,Y)$ and $g \in \mathcal{C}(y_0)$, where $y_0 := Ax_0$, let $f := g \circ A$. If A is onto, if $R(A^t)$ is w^*-closed (in particular if X and Y are Banach spaces) and if y_0 is not a minimizer of g, then $f \in \mathcal{C}(x_0)$.*

Proof. By Proposition 2.6 we have $\partial^* f(x_0) = A^t\left(\partial^* g(y_0)\right)$ and by Proposition 3.2 we have $\partial^< f(x_0) = A^t\left(\partial^< g(y_0)\right)$. Taking products with elements of $\mathbb{P}$, we get the result by the very definition of $\mathcal{C}(y_0)$. □

Proposition 4.5. *Let $g : X \to \mathbb{R}\cup\{\infty\}$ and let $\varphi : \mathbb{R} \to \mathbb{R}\cup\{\infty\}$ be a nondecreasing function. Set $\varphi(\infty) = \infty$ and consider $x_0 \in X$ such that $\varphi(t_0) \in \mathbb{R}$, where $t_0 := g(x_0)$. Let $f = \varphi \circ g$. Suppose $g \in \mathcal{C}(x_0)$, $t_0 := g(x_0)$ is not a local minimizer of φ and $\partial^<\varphi(t_0) \neq \emptyset$. Then $f \in \mathcal{C}(x_0)$.*

Proof. Of course, $\partial^<\varphi(t_0) \subset \mathbb{P}$. By Proposition 2.7 we have $\partial^* f(x_0) = \partial^* g(x_0)$; also Proposition 3.5 (i) yields $\partial^< f(x_0) \supset \partial^<\varphi(t_0) \cdot \partial^< g(x_0)$. So,

$$\partial^* f(x_0) = \partial^* g(x_0) = \mathbb{P}\partial^< g(x_0) = \mathbb{P}\partial^<\varphi(t_0) \cdot \partial^< g(x_0) \subset \mathbb{P}\partial^< f(x_0),$$

whence $f \in \mathcal{C}(x_0)$. □

Note that the assumption $\partial^<\varphi(t_0) \neq \emptyset$ is crucial: taking $\varphi(t) = t^3$, $g(x) = x$, we see that $g \in \mathcal{C}(x_0)$ for each $x_0 \in \mathbb{R}$ but $\varphi \circ g \notin \mathcal{C}(x_0)$.

Let us turn to the supremum of two functions. For the sake of brevity we restrict our attention to the simple case of functions of independent variables.

Proposition 4.6. *Let $f : X \to \mathbb{R}\cup\{\infty\}$, $g : Y \to \mathbb{R}\cup\{\infty\}$ and $M : X \times Y \to \mathbb{R}\cup\{\infty\}$, $M(x,y) = f(x) \vee g(y)$. Consider $z_0 := (x_0, y_0) \in X \times Y$ such that $r_0 := f(x_0) = g(y_0) \in \mathbb{R}$ and x_0 and y_0 are not local minimizers of f and g, respectively. If $f \in \mathcal{C}(x_0)$ and $g \in \mathcal{C}(y_0)$ then $M \in \mathcal{C}(z_0)$.*

Proof. Let $z^* := (x^*, y^*) \in \partial^* M(z_0)$. Using Proposition 2.4 we may suppose $x^* \in \partial^{\circledast} f(x_0) = 0 \cdot \partial^< f(x_0)$ and $y^* \in \partial^* g(y_0) = \mathbb{P}\partial^< g(y_0)$. Setting $y^* = \mu v^*$ with $\mu \in \mathbb{P}$ and $v^* \in \partial^< g(y_0)$ we obtain, by Proposition 3.14,

$$\begin{aligned} z^* &= \mu\left(\mu^{-1}x^*, v^*\right) \in \mu\left(0 \cdot \partial^< f(x_0) \times 1 \cdot \partial^< g(y_0)\right) \\ &\subset \mathbb{P} \bigcup_{\lambda \in [0,1]} \left(\lambda \partial^< f(x_0) \times (1-\lambda)\partial^< g(y_0)\right) \subset \mathbb{P}\partial^< M(x_0, y_0). \end{aligned}$$

So $M \in \mathcal{C}(z_0)$. □

5. Subdifferentials associated to derivatives

In the present section, we consider the subdifferentials associated to some generalized derivatives. In particular, when f is finite at x, we use the *contingent derivative* (or lower Hadamard derivative) given by

$$f'(x,v) := \liminf_{(t,u)\to(0_+,v)} \frac{f(x+tu) - f(x)}{t}, \qquad v \in X,$$

and the *incident* (or adjacent, or upper epi-) *derivative* given by

$$f^i(x,v) := \sup_{U\in\mathcal{N}(v)} \limsup_{t\searrow 0} \inf_{u\in U} \frac{f(x+tu)-f(x)}{t}, \qquad v\in X,$$

where $\mathcal{N}(v)$ is the family of neighborhoods of v. These derivatives are such that their epigraphs are the *contingent cone* $T(E_f, x_f)$ and the *incident cone* $T^i(E_f, x_f)$ to the epigraph E_f of f at $x_f := (x, f(x))$ given respectively by

$$T(E_f,x_f) := \limsup_{t\searrow 0} t^{-1}(E_f - x_f), \qquad T^i(E_f,x_f) := \liminf_{t\searrow 0} t^{-1}(E_f - x_f).$$

A pleasant situation occurs when these two derivatives coincide; then f is said to be *epi-differentiable* at x. Such a condition is less stringent than requiring that $f'(x,\cdot)$ coincides with the *upper derivative*

$$f^\sharp(x,v) := \limsup_{(t,u)\to(0_+,v)} \frac{f(x+tu)-f(x)}{t}$$

since $f'(x,\cdot) \le f^i(x,\cdot) \le f^\sharp(x,\cdot)$. When f is *directionally steady at* x in the sense that for each $v\in X\backslash\{0\}$ one has

$$\lim_{(t,u)\to(0_+,v)} \frac{f(x+tu)-f(x+tv)}{t} = 0,$$

then one has

$$f^i(x,v) = f^\sharp(x,v) \qquad \forall v\in X\backslash\{0\}$$

and $f'(x,\cdot)$ and $f^i(x,\cdot)$ coincide with their radial counterparts on $X\backslash\{0\}$. Observe that f is directionally steady at x whenever f is directionally Hadamard differentiable at x, i.e. when $f'(x,\cdot) = f^\sharp(x,\cdot)$, or when X is a normed space and f is Lipschitzian on a neighborhood of x or, more generally, when f is *directionally Lipschitzian* at x in the sense: for any $v\in X\backslash\{0\}$ there exist a neighborhood V of v and $\varepsilon>0$, $c>0$ such that $|f(x+tv')-f(x+tv'')| \le ct\,\|v'-v''\|$ for any $v',v''\in V$, $t\in[0,\varepsilon]$.

In the sequel we say that f is *calm* at x if $f'(x,0)=0$; in fact it would be enough to suppose f is *incidently calm* in the sense $f^i(x,0)=0$ (note that $f^i(x,0)$ and $f'(x,0)$ are either 0 or $-\infty$).

For f finite at x we set

$$\partial^{\prec} f(x) := \partial^{<} f'(x,\cdot)(0), \qquad \partial^{\prec i} f(x) := \partial^{<} f^i(x,\cdot)(0).$$

If the first (resp. second) set is nonempty, f is calm (resp. incidently calm) at x. The subdifferential $\partial^{\prec i} f(x)$ is more important than $\partial^{\prec} f(x)$ because it coincides with the Fenchel subdifferential at 0 of the function $p_<$ associated to $p := f^i(x,\cdot)$ via a variant of the Crouzeix's decomposition of p described in [5]. Here $p_<$ is defined by $p_<(v) := p(v)$ whenever $v\in \operatorname{cl} D_p \cup \{0\}$, $+\infty$ else, where $D_p := p^{-1}(]-\infty,0[)$. Note that $\partial^{\prec i} f(x)$ is nonempty whenever f is quasiconvex and p does not take the value $-\infty$ (or $p(0)=0$, i.e. f is (incidently) calm at x_0) because $p_<$ is l.s.c. and sublinear (see [5]).

Let us start with calculus rules related to order. We first observe that both subdifferentials are homotone, so that easy consequences can be derived.

Proposition 5.1. *Let f and g be quasiconvex functions finite and calm at x_0, with $f(x_0) = g(x_0)$, and let $h := f \vee g$. Suppose $h^i(x_0, u) < 0$ for some $u \in X$ and g is directionally steady at x_0. Then*

$$\partial^{\prec i} h(x_0) = \overline{\text{co}} \left(\partial^{\prec i} f(x_0) \cup \partial^{\prec i} g(x_0) \right). \tag{5.1}$$

The proof relies on the following lemma.

Lemma 5.2. *Suppose $h := f \vee g$ where f and g are finite and calm at x_0, with $f(x_0) = g(x_0)$ and g is directionally steady at x_0. Then $h^i(x_0, v) = f^i(x_0, v) \vee g^i(x_0, v)$ for each $v \in X$.*

Proof. The inequalities $h^i(x_0, v) \geq f^i(x_0, v)$, $h^i(x_0, v) \geq g^i(x_0, v)$ are obvious, so that $h^i(x_0, v) \geq f^i(x_0, v) \vee g^i(x_0, v)$. As $h^i(x_0, 0) \leq 0$, equality holds for $v = 0$ when f and g are finite and calm at x_0. On the other hand, it is not difficult to prove that

$$h^i(x_0, v) \leq f^i(x_0, v) \vee g^\sharp(x_0, v).$$

Thus, when $g^\sharp(x_0, v) = g^i(x_0, v)$ for $v \in X \backslash \{0\}$, equality holds. □

Proof of Proposition 5.1. Let us first observe that, setting $p := f^i(x_0, \cdot)$, $q := g^i(x_0, \cdot)$, $r := h^i(x_0, \cdot)$, so that $r = p \vee q$ by the preceding lemma, one has $\text{cl}\, D_r = \text{cl}\, D_p \cap \text{cl}\, D_q$ where $D_p := p^{-1}(]-\infty, 0[)$, with a similar notation with q and r. In fact, given $u \in D_r = D_p \cap D_q$, for any $v \in \text{cl}\, D_p \cap \text{cl}\, D_q$ and any $t \in]0, 1[$ one has $p((1-t)v + tu) < 0$ as $p\,|_{\text{cl}\, D_p}$ is sublinear; similarly, $q((1-t)v + tu) < 0$, so that $(1-t)v + tu \in D_r$ and $v = \lim_{t \searrow 0}((1-t)v + tu) \in \text{cl}\, D_r$. Extending p (resp. q, r) by $+\infty$ outside of $\text{cl}\, D_p \cup \{0\}$ (resp. $\text{cl}\, D_q \cup \{0\}$, resp. $\text{cl}\, D_r \cup \{0\}$) into a sublinear function $p_<$ (resp. $q_<$, $r_<$) we get

$$r_<(v) = p_<(v) \vee q_<(v) \qquad \text{for each } v \in X.$$

Therefore, by a well-known rule of convex subdifferential calculus,

$$\partial^{\prec i} h(x_0) = \partial r_<(0) = \overline{\text{co}} \left(\partial p_<(0) \cup \partial q_<(0) \right) = \overline{\text{co}} \left(\partial^{\prec i} f(x_0) \cup \partial^{\prec i} g(x_0) \right).$$

□

The case of a supremum of two functions of independent variables does not require the assumptions of the general case.

Proposition 5.3. *Let $f : X \to \mathbb{R} \cup \{\infty\}$, $g : Y \to \mathbb{R} \cup \{\infty\}$ and $M : X \times Y \to \mathbb{R} \cup \{\infty\}$, $M(x, y) = f(x) \vee g(y)$. Consider $(x_0, y_0) \in X \times Y$ such that $M(x_0, y_0) \in \mathbb{R}$. If $f(x_0) < g(y_0)$, if f is u.s.c. at x_0 and if g is l.s.c. and calm at y_0 then $\partial^{\prec i} M(x_0, y_0) = \{0\} \times \partial^{\prec i} g(y_0)$ when $0 \notin \partial^{\prec i} g(y_0)$ and $\partial^{\prec i} M(x_0, y_0) = X^* \times Y^*$ when $0 \in \partial^{\prec i} g(y_0)$. If $f(x_0) = g(y_0)$ then*

$$\partial^{\prec i} M(x_0, y_0) \supset \bigcup_{\lambda \in [0,1]} \left(\lambda \partial^{\prec i} f(x_0) \times (1 - \lambda) \partial^{\prec i} g(y_0) \right). \tag{5.2}$$

Equality holds in the preceding relation if f and g are quasiconvex and calm at x_0 and y_0 respectively and if either $0 \notin \partial^{\prec i} f(x_0)$, $0 \notin \partial^{\prec i} g(y_0)$ or $0 \in \partial^{\prec i} f(x_0)$, $0 \in \partial^{\prec i} g(y_0)$.

Proof. Under the assumptions of the first assertion we have $M(u,v) = g(v)$ for (u,v) in a neighborhood of (x_0, y_0). Thus $M^i(x_0, y_0, x, y) = g^i(y_0, y)$ for any $x \in X,\ y \in Y$. The assertion then follows from Corollary 3.4.

Now let us suppose $f(x_0) = g(y_0)$. We first observe that $M^i(x_0, y_0, x, y) = f^i(x_0, x) \vee g^i(y_0, y)$ for any $x \in X,\ y \in Y$. In fact, for any $t > 0$ and any neighborhoods U, V of x and y respectively we have

$$\inf_{(u,v)\in U\times V} r_t(u,v) = \inf_{(u,v)\in U\times V} p_t(u) \vee q_t(v) = \left(\inf_{u\in U} p_t(u)\right) \vee \left(\inf_{v\in V} q_t(v)\right),$$

with

$$r_t(u,v) := t^{-1}\left(M(x_0 + tu, y_0 + tv) - M(x_0, y_0)\right),$$

$p_t(u) := t^{-1}\left(f(x_0 + tu) - f(x_0)\right)$ and $q_t(v) := t^{-1}\left(g(y_0 + tv) - g(y_0)\right)$, hence, by the continuity of the operation $(r,s) \mapsto r \vee s$, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0_+} \sup_{0<t<\varepsilon} \inf_{(u,v)\in U\times V} r_t(u,v) &= \lim_{\varepsilon\to 0_+} \left(\sup_{0<t<\varepsilon} \inf_{u\in U} p_t(u) \vee \sup_{0<t<\varepsilon} \inf_{v\in V} q_t(v)\right) \\
&= \left(\lim_{\varepsilon\to 0_+} \sup_{0<t<\varepsilon} \inf_{u\in U} p_t(u)\right) \vee \left(\lim_{\varepsilon\to 0_+} \sup_{0<t<\varepsilon} \inf_{v\in V} q_t(v)\right)
\end{aligned}$$

so that, taking the supremum on U and V we get the announced equality. Setting $p := f^i(x_0, \cdot)$, $q := g^i(x_0, \cdot)$, $r := M^i(x_0, y_{0,} \cdot, \cdot)$, the inclusion follows from Proposition 3.14.

Now let us suppose f and g are quasiconvex. As $r = p \vee q$, we have $D_r = D_p \times D_q$, whence $\operatorname{cl} D_r = \operatorname{cl} D_p \times \operatorname{cl} D_q$. Therefore, when $0 \notin \partial^{\prec i} f(x_0)$, $0 \notin \partial^{\prec i} g(y_0)$, the sets D_p and D_q are nonempty and one has $\operatorname{cl} D_r \cup \{0\} = \operatorname{cl} D_r = (\operatorname{cl} D_p \cup \{0\}) \times (\operatorname{cl} D_q \cup \{0\})$ and $r_<(x,y) = p_<(x) \vee q_<(y)$ so that, as $p_<$ and $q_<$ are sublinear and l.s.c., by usual rules of convex analysis,

$$\partial r_<(0,0) = \overline{\operatorname{co}}\left((\partial p_<(0) \times \{0\}) \cup (\{0\} \times \partial q_<(0))\right).$$

Since $0 \in \partial^{\prec i} f(x_0)$ (resp. $0 \in \partial^{\prec i} g(y_0)$) means that $f^i(x_0, \cdot) \geq 0$ (resp. $g^i(y_0, \cdot) \geq 0$), both sides of relation (5.2) are $X^* \times Y^*$ when $(0,0) \in \partial^{\prec i} f(x_0) \times \partial^{\prec i} g(y_0)$. □

Let us turn to performance functions.

Proposition 5.4. *Let p be the performance function associated with a perturbation $F : W \times X \to \overline{\mathbb{R}}$ by $p(w) = \inf_{x\in X} F(w,x)$. Suppose that $p(w_0) = F(w_0, x_0) \in \mathbb{R}$. If $w^* \in \partial^{\prec i} p(w_0)$ then $(w^*, 0) \in \partial^{\prec i} F(w_0, x_0)$. Moreover, when $p^i(w_0, .) = q(.)$, where $q(w) := \inf_{x\in X} F^i(w_0, x_0, w, x)$, one has $w^* \in \partial^{\prec i} p(w_0)$ if $(w^*, 0) \in \partial^{\prec i} F(w_0, x_0)$.*

The proof depends on the following lemma of independent interest which also gives a sufficient condition ensuring the assumption of the last assertion. This condition is a variant of [4] Proposition 3.5.

Lemma 5.5. *Let F, p, w_0, x_0 be as above. Then for each $w \in W$ one has*

$$p^i(w_0, w) \leq q(w) := \inf_{x\in X} F^i(w_0, x_0, w, x).$$

If W and X are normed spaces, if F is continuous, differentiable with respect to its first variable with a partial derivative $D_W F$ jointly continuous in (w,x) and if for any sequences $(u_n) \to w_0$, $(\varepsilon_n) \to 0_+$ there exists some $(x_n) \to x_0$ such that $F(u_n, x_n) \leq p(u_n) + \varepsilon_n$ then $p^i(w_0, \cdot) = q$.

This last condition is satisfied if x_0 is the unique minimizer of $F(w_0, \cdot)$ and if there exist a neighborhood W_0 of w_0 and a real number $r > \sup_{w \in W_0} p(w)$ such that $\{x \in X \mid \exists\, w \in W_0,\ F(w,x) < r\}$ is relatively compact. When X is a finite dimensional space, this last condition amounts to a coercivity condition.

Proof. Let $w_0 \in W$ and $x_0 \in X$ such that $p(w_0) = F(w_0, x_0) \in \mathbb{R}$. For any neighborhoods U and V of $w \in W$ and $x \in X$, respectively, we have for any $t > 0$ that

$$\inf_{w' \in U,\, x' \in V} \frac{F(w_0 + tw', x_0 + tx') - F(w_0, x_0)}{t} \geq \inf_{w' \in U} \frac{p(w_0 + tw') - p(w_0)}{t},$$

whence $F^i(w_0, x_0, w, x) \geq p^i(w_0, w)$. Hence $q \geq p^i(w_0, \cdot)$.

In order to prove the last assertion, let $w \in W$ and let $r > p^i(w_0, w)$. For any sequence $(t_n) \to 0_+$ there exists a sequence $(w_n) \to w$ such that

$$r > \limsup_n \frac{1}{t_n}(p(w_0 + t_n w_n) - p(w_0)).$$

Taking $u_n := w_0 + t_n w_n$, $\delta_n := p(w_0) + t_n r - p(w_0 + t_n w_n) > 0$ for n large enough, $\varepsilon_n := \min(\delta_n, 2^{-n})$ and picking $(x_n) \to x_0$ such that $F(u_n, x_n) \leq p(u_n) + \varepsilon_n$, using the mean value theorem we get

$$\begin{aligned} D_W F(w_0, x_0) w &\leq \limsup_n \frac{F(w_0 + t_n w_n, x_n) - F(w_0, x_n)}{t_n} \\ &\leq \limsup_n \frac{p(w_0 + t_n w_n) + \delta_n - F(w_0, x_n)}{t_n} \\ &\leq \limsup_n \frac{p(w_0) + t_n r - F(w_0, x_0)}{t_n} = r, \end{aligned}$$

so that $D_W F(w_0, x_0) w \leq p^i(w_0, w)$. On the other hand, for any sequence $(t_n) \to 0_+$ one has

$$F^i(w_0, x_0, w, 0) \leq \limsup_n \frac{F(w_0 + t_n w, x_0) - F(w_0, x_0)}{t_n} = D_W F(w_0, x_0) w.$$

Therefore, $p^i(w_0, w) \leq F^i(w_0, x_0, w, 0) \leq D_W F(w_0, x_0) w \leq p^i(w_0, w)$ and equality holds. □

Proof of Proposition 5.4. As $p^i(w_0, \cdot) \leq q$ and $p^i(w_0, 0) = F^i(w_0, x_0, 0, 0)$ when $\partial^{\prec i} p(w_0)$ is non empty, applying Lemma 3.1 we get that for any $w^* \in \partial^{\prec i} p(w_0) = \partial^{<} p^i(w_0, \cdot)(0) \subset \partial^{<} q(0)$ we have $(w^*, 0) \in \partial^{<} F^i(w_0, x_0, \cdot, \cdot)(0,0) = \partial^{\prec i} F(w_0, x_0)$. Moreover, when $p^i(w_0, \cdot) = q$, any w^* such that $(w^*, 0) \in \partial^{<} F^i(w_0, x_0, \cdot, \cdot)(0,0)$ belongs to $\partial^{\prec i} q(0) = \partial^{<} p^i(w_0, \cdot)(0) = \partial^{\prec i} p(w_0)$. □

Now let us consider the case of the composition with a nondecreasing function. More refined assumptions could be used, but we prefer to give a simple statement.

Proposition 5.6. *Let $g : X \to \mathbb{R} \cup \{\infty\}$ and let $\varphi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a nondecreasing function. Set $\varphi(\infty) = \infty$ and consider $x_0 \in X$ such that φ is finite and differentiable at $t_0 = g(x_0)$ and $g'(x_0, \cdot)$ and $g^i(x_0, \cdot)$ are finite.*

Then, if $\varphi'(t_0) = 0$ one has $\partial^{\prec}(\varphi \circ g)(x_0) = \partial^{\prec i}(\varphi \circ g)(x_0) = X^*$ while if $\varphi'(t_0) \neq 0$

$$\partial^{\prec}(\varphi \circ g)(x_0) = \varphi'(t_0) \cdot \partial^{\prec} g(x_0), \qquad \partial^{\prec i}(\varphi \circ g)(x_0) = \varphi'(t_0) \cdot \partial^{\prec i} g(x_0).$$

Proof. Let us set $f := \varphi \circ g$. When $\varphi'(t_0) = 0$, one easily sees that $f'(x_0, \cdot) = 0$ and $f^i(x_0, \cdot) = 0$, so that 0 is a minimizer of $f'(x_0, \cdot)$ and $f^i(x_0, \cdot)$ hence $\partial^{\prec} f(x_0) = X^*$, $\partial^{\prec i} f(x_0) = X^*$. Suppose now that $\varphi'(t_0) \neq 0$, so that $\varphi'(t_0) > 0$. Then $f'(x_0, \cdot) = \varphi'(t_0) \cdot g'(x_0, \cdot)$ and $f^i(x_0, \cdot) = \varphi'(t_0) \cdot g^i(x_0, \cdot)$. The result follows from the relation $\partial^{<}(rh)(0) = r\partial^{<} h(0)$ for $r > 0$, h arbitrary. $\square$

Let us turn to composition with a continuous linear map.

Proposition 5.7. *Let $g : Y \to \overline{\mathbb{R}}$, $A \in L(X, Y)$ and $x_0 \in X$ be such that $g(y_0) \in \mathbb{R}$ for $y_0 := Ax_0$. Then, for $f := g \circ A$,*

$$\partial^{\prec} f(x_0) \supset A^t \left(\partial^{\prec} g(y_0)\right), \qquad \partial^{\prec i} f(x_0) \supset A^t \left(\partial^{\prec i} g(y_0)\right).$$

If A is onto, if X and Y are Banach spaces and if $\inf g'(y_0, \cdot) < 0$, *respectively* $\inf g^i(y_0, \cdot) < 0$, *then*

$$\partial^{\prec} f(x_0) = A^t \left(\partial^{\prec} g(y_0)\right), \qquad \partial^{\prec i} f(x_0) = A^t \left(\partial^{\prec i} g(y_0)\right).$$

Proof. As the epigraph E_f of f and the epigraph E_g of g are related by $E_f = B^{-1}(E_g)$, where $B \in L(X \times \mathbb{R}, Y \times \mathbb{R})$ is given by $B(x, r) = (Ax, r)$, we have $T(E_f, x_f) \subset B^{-1}(T(E_g, y_g))$ for $x_f := (x_0, f(x_0))$, $y_g := (y_0, g(y_0))$. It follows that

$$f'(x_0, u) \geq g'(y_0, Au) \qquad \forall\, u \in X,$$

and similarly

$$f^i(x_0, u) \geq g^i(y_0, Au) \qquad \forall\, u \in X.$$

Therefore $f^i(x_0, \cdot) > -\infty$ when $g^i(x_0, \cdot) > -\infty$ and

$$\partial^{\prec} f(x_0) = \partial^{<} f'(x_0, \cdot)(0) \supset A^t \left(\partial^{<} g'(y_0, \cdot)(0)\right) = A^t \left(\partial^{\prec} g(y_0)\right), \tag{5.3}$$

and similarly

$$\partial^{\prec i} f(x_0) = \partial^{<} f^i(x_0, \cdot)(0) \supset A^t \left(\partial^{<} g^i(y_0, \cdot)(0)\right) = A^t \left(\partial^{\prec i} g(y_0)\right).$$

When A is onto and X and Y are Banach spaces one has $T(E_f, x_f) = B^{-1}(T(E_g, y_g))$ and $f'(x_0, u) = g'(y_0, Au)$ for each $u \in X$ (see [3] for example). If 0 is not a minimizer of $g'(y_0, \cdot)$ we can apply Proposition 3.2 and we get equality in (5.3). The case of $\partial^{\prec i} f(x_0)$ is similar. $\square$

Let us finish with the sublevel-convolution.

Proposition 5.8. *Let $f,\ g : X \to \mathbb{R} \cup \{\infty\}$, $h = f \Diamond g$ and $x_0 \in \operatorname{dom} f$, $y_0 \in \operatorname{dom} g$, $w_0 = x_0 + y_0$ with $h(w_0) = f(x_0) \vee g(y_0)$.*

(i) *If $g(y_0) < f(x_0)$ and g is u.s.c. at y_0 then w_0 is a local maximizer of h and x_0 is a local minimizer of f, so that $\partial^{\prec i} f(x_0) = X^*$.*

(ii) *If $f(x_0) = g(y_0)$, if f and g are quasiconvex and calm at x_0 and y_0 respectively and if $(0,0) \in \partial^{\prec i} f(x_0) \times \partial^{\prec i} g(y_0)$ or if $0 \notin \partial^{\prec i} f(x_0)$, $0 \notin \partial^{\prec i} g(y_0)$ then $\partial^{\prec i}(f \Diamond g)(w_0) \subset \partial^{\prec i} f(x_0) \Diamond \partial^{\prec i} g(y_0)$.*

Proof. (i) Let V be a neighborhood of 0 such that $g(y_0 + v) < f(x_0)$ for each $v \in V$. Then, for $v \in V$ we have $h(w_0 + v) \leq f(x_0) \vee g(y_0 + v) = f(x_0) = h(w_0)$. It follows that $h^i(w_0, \cdot) \leq 0$. Moreover, for $v \in V$ we have $f(x_0 - v) \geq f(x_0)$ as otherwise we would have $f(x_0 - v) \vee g(y_0 + v) < f(x_0) = h(w_0)$, a contradiction. Thus $f^i(x_0, \cdot) \geq 0$ and $\partial^{\prec i} f(x_0) = X^*$

(ii) Consider $H : X \times X \to \mathbb{R} \cup \{\infty\}$, given by $H(w, x) = f(x) \vee g(w - x)$. We observe again that h is the performance function associated with H and $H(w, x) = M(x, w - x) = M \circ A(w, x)$, where $A(w, x) = (x, w - x)$ and $M(x, y) = f(x) \vee g(y)$. As A is an isomorphism and $A^t(u^*, v^*) = (v^*, u^* - v^*)$ we have

$$(w^*, x^*) \in \partial^{\prec i} H(w_0, x_0) \Leftrightarrow (w^*, w^* + x^*) \in \partial^{\prec i} M(x_0, w_0 - x_0).$$

In particular,

$$(w^*, 0) \in \partial^{\prec i} H(w_0, x_0) \Leftrightarrow (w^*, w^*) \in \partial^{\prec i} M(x_0, y_0).$$

Let $w^* \in \partial^{\prec i} h(w_0)$. Then $(w^*, 0) \in \partial^{\prec i} H(w_0, x_0)$ hence $(w^*, w^*) \in \partial^{\prec i} M(x_0, y_0)$. Taking Proposition 5.3 into account we can find $\lambda \in [0, 1]$, $x^* \in \partial^{\prec i} f(x_0)$, $y^* \in \partial^{\prec i} g(y_0)$ such that $w^* = (1 - \lambda) x^* = \lambda y^*$. Thus $w^* \in \partial^{\prec i} f(x_0) \Diamond \partial^{\prec i} g(y_0)$. □

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Added in proof:

[L-V] D. T. Luc, M. Volle: Level sets under infimal convolution and level addition, J. Optim. Th. Appl. 94(3) (1997) 695–714.