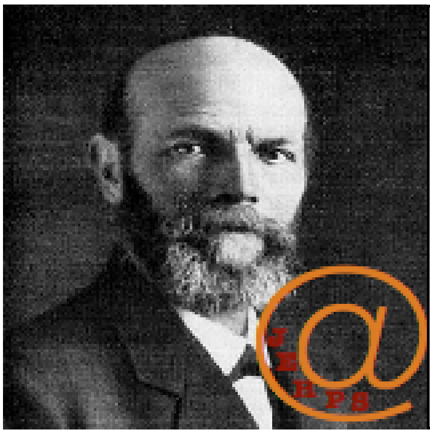

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# Dissertations on Probability in Paris in the 1930s

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**Abstract**

In the present paper, we comment on the six theses dealing with probability theory presented to the University of Paris in the 1930s. The students were trained and advised by Fréchet and Darmois in new directions of the theory.

## 1 Introduction

At the end of the 1930s, in the space of four years, six dissertations on probability were submitted to the Faculty of Science in Paris: by Daniel Dugué in 1937, Wolfgang Doeblin in 1938, Jean Ville, Robert Fortet and Gustave Malécot in 1939 and Michel Loève in 1941. Four years later, in 1945, the dissertation of André Blanc-Lapierre followed. As previous historical research has indicated, these students mostly followed the teaching of Fréchet and Darmois at the Faculty of Science. The circumstances in which the theses were submitted were completely different from those prevailing in the previous decade whether in France, or abroad. The academics with established positions in the Sorbonne Faculty of Science were Émile Borel (1871-1956) (though his influence was declining) and, especially, Maurice Fréchet (1878-1973) and Georges Darmois (1888-1960). It is no surprise then

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to find them on the committees for all seven theses.[2] Paul Lévy (1886-1971), a professor at the École Polytechnique, was not on any of the committees although his numerous publications in probability had brought him international recognition.[3]

Another development was the first international event dedicated to probability theory and its applications, a conference in Geneva in October 1937, chaired by Fréchet. The research of the seven doctoral students covered a broad range and reflected many of the themes represented in the conference[4]: statistics, foundations of probability, Markov chain theory and issues associated with the convergence of sequences of random variables. Some subjects were missing, notably Brownian motion and stochastic integrals, but these would only be taken up later by Lévy during the Second World War and so their absence from the work of the doctoral candidates is understandable.[5]

In this paper, which is only a part of the chapter devoted to probability in Leloup's thesis (2009), we shall only briefly comment on Doeblin's and Ville's works which have been thoroughly studied in other papers.

## 2 Dugué on the theory of estimation

The first of the probability dissertations to be submitted was Daniel Dugué's "Application des propriétés de la limite au sens du calcul des probabilités à l'étude de diverses questions d'estimation." The topic was on the border between statistics and probability. Dugué treated the problem of estimation from a theoretical point of view: he aimed to "clarify the mathematical questions" raised by the physical problem of ascertaining a quantity $M$ from the results of a series of measurements of this quantity [6]. The general problem was to estimate a probability distribution of known form but depending on certain parameters the values of which were unknown.

Dugué's starting point was the work of the statistician Ronald Aylmer Fisher: though Dugué spoke of his "many papers," he only referred explicitly to one, FISHER 1925.[7]. Dugué uses the notation and terminology of Fisher's paper, such

[2]Borel chaired the committee for the first three theses, of Dugué, Doeblin and Jean Ville. Fréchet chaired the committees for Malécot, Loève and Blanc Lapierre. The committee for Fortet was chaired by Henri Villat. Fréchet was rapporteur for four of the theses (Doeblin, Villle, Fortet and Loève) and Darmois rapporteur for the other three (Dugué, Malécot and Blanc-Lapierre.)

[3]Traditionally the committee was drawn from the Paris Faculty of Science.

[4]These themes figure in the account by de Finetti in DE FINETTI 1939.

[5]See Bernard Locker's thesis, LOCKER 2001 for a thorough study of the development of these theories. The introduction of BARBUT ET AL 2004 describes Paul Lévy's contribution to these fields.

[6]Cf. DUGUÉ 1937, p. 1.

[7]Cf. DUGUÉ 1937, p. 2.

as that of a "statistic" for the function expressing the results of experiments [8] and "consistent" statistics but Dugué also calls the latter "correct estimates." [9]

The first part of his work is devoted to considering "correct" estimates for the case of one, two or a countable number of parameters [10]. The proofs are based on a theorem proved by Khintchine in KHINTCHINE 1929, which states the law of large numbers for a random variable $x$ using only the assumption that the "mean [of this random variable] exists."[11]. Dugué also provides a proof of this.[12].

By means of this theorem Dugué is able to find correct estimates. First, he considers an elementary probability distribution $f(x, m)$ with

$$\int f(x, m)dx = 1$$

where $x$ is a random variable and $m$ is the parameter whose value is to be estimated. He assumes that the function $f(x, m)$ has a derivative with respect to $m$ in the entire range of variation of $x$ except possibly for a set of zero probability. In the case when $x$ is not confined to a bounded interval, he assumes that the integral $\int_0^C \frac{\partial f(x,m)}{\partial m} dx$ converges uniformly to a limit when $|C|$ increases indefinitely and that in every interval where $m$ varies, $\frac{1}{f(x,m)} \frac{\partial f(x,m)}{\partial m}$ is uniformly continuous with respect to $m$ for all values of $x$ with the exception of a set of probability zero. Under these conditions, Khintchine's theorem can be used to demonstrate that

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[8] In DUGUÉ 1937 p.2 Dugué forms functions $f_1(x_1), ..., f_n(x_1, x_2, ..., x_n)$ of the observations $x_1, x_2, ..., x_n$ . Fisher calls such functions "statistics."

[9] Cf. FISHER 1925, p. 702 and DUGUÉ 1937, pp. 2-3. According to Dugué, each function (see the preceding note) has a probability distribution. He puts $P^i_{\alpha,\beta}$ for the measure of the set of points in $i$-dimensions for which $\alpha \leq f_i(x_1, ..., x_i) \leq \beta$ . "Consistent" statistics are those which converge in probability to the quantity, $m$, we wish to estimate, i.e., those for which we can find a value $i$ for which $P^i_{m-\epsilon,m+\epsilon}$ is as close to unity as desired for arbitrary $\epsilon$.

[10] Cf. DUGUÉ 1937, p. 3.

[11] Cf. DUGUÉ 1937, p. 6. Khintchine in KHINTCHINE 1929, p. 477 states the theorem as follows: "Let $x$ be a random variable with expected value $a$ and let $x_1$, $x_2$;... be successive values of $x$ realised in an indefinite series of experiments. The law of large numbers states that the probability of the inequality

$$\left| \frac{1}{n} \sum_{k=1}^{n} x_k - a \right| > \epsilon$$

tends to 0 as $n$ tends to infinity for $\epsilon \neq 0$. The statement is usually proved under the condition that the expected value of $x^2$ is finite but it a general results valid for all cases if the expected value of $x$ exists.

[12] Based on one of those indicated by Khintchine in KHINTCHINE 1929. The characteristic function of the random variable is used to establish convergence in probability.

the solution of the following equation in $T$ (where $T$ depends on $n$):

$$\sum_{i=1}^{n} \frac{1}{f(x_i,T)} \frac{\partial f(x_i,T)}{\partial m} = 0$$

converges in probability to $m$. In his proof Dugué uses Khintchine's theorem, arguments of uniform continuity and differentiation under the integral sign [13]. He then find the "maximum of likelihood" estimator given, following Fisher, by a solution to the equation [14]

$$\sum_{i=1}^{n} \frac{\partial f(x_i,m)}{\partial m} = 0.$$

In the rest of the thesis Dugué gives results for Fisher's method as special cases of his own results, that he obtains in his memoir by means of a method for finding new consistent estimates in a broader framework [15]. To achieve this he generalises Khintchine's theorem to the case where the random variables under consideration do not have the same distribution but satisfy certain conditions [16]. He also offers various generalisations of his results by making less restrictive assumptions than, for example, the existence the expectation of the random variable.

In the second part of his thesis Dugué investigates the limiting distribution of the "correct" estimates he had found, deriving properties, such as the standard deviation [17]. He consider especially the estimates obtained by the method of maximum likelihood. He examines the behaviour of what he calls the "infinitely small probability" [18]: $f_n - m$ where $f_n$ is the estimate of the parameter $m$. Under the condition that $\sigma^2 = E\left(\left(\frac{1}{f}\frac{\partial f}{\partial m}\right)^2\right)$ exists, Dugué shows that for the method of maximum likelihood that the random variable $\sqrt{n}(T_n - m)$ converges in probability

[13]DUGUÉ 1937, pp. 9-10. Dugué also adds the following condition on the equation $\sum_{i=1}^{n} \frac{1}{f(x_i,m)} \frac{\partial f(x_i,m)}{\partial m} = 0$ : it must have a unique root in $m$ the interval under consideration.

[14]Fisher presents his method in FISHER 1925, p. 708. Dugué does not give an explicit reference.

[15]The method is given in the first chapter.

[16]He requires that the mean values of these random variables are "equally convergent" to their limits. In other words, putting $x_i$ for the random variables, $F_i$ for the distribution function of $x_i$ one can find $N$ such that for $n > N$, $\int_{-\infty}^{n} x_i dF_i(x_i)$ is less than a certain $\alpha$. (at least in a neighbourhood of $+\infty$). If we put $a_i = E(x_i)$ where $E(.)$ represents the mathematical expectation (or mean) of the random variable and if $\frac{\sum_{i=1}^{n} a_i}{n}$ converges to $A$, then the arithmetic mean of $x_i$ converges in probability to $A$. Cf. DUGUÉ 1937, pp. 11-12.

[17]Cf. DUGUÉ 1937, p. 24.

[18]DUGUÉ 1937, p. 3

to the normal distribution with mean zero and standard deviation $\sigma$. [19] He then obtains a result that had already been stated and briefly demonstrated by Joseph Leo Doob in DOOB 1934, p. 774. Dugué explicitly acknowledges this in his introduction [20]. He present a more elaborate demonstration than that of Doob, based however, on similar ideas and using limiting arguments.

Dugué then studies the accuracy of various statistics: he considers the principal part of the quantity $T_n - m$, where $T_n$ is a statistic. Using the concept of a space of "Fisher statistics", he finds a result already stated by Fisher asserting the optimality of the method of maximum likelihood: "all asymptotically normal estimators such that their standard deviations converge to the standard deviation limit, have an accuracy bounded above by the method of maximum likelihood" [21]. He extends the treatment to the case of several parameters and to the case of a non-normal limiting distribution. He also examines the connection between different estimators of the same parameter.

Finally Dugué turns to "exhaustive" statistics, the sufficient statistics Fisher had discussed in FISHER 1925 and on which Darmois had published in DARMOIS 1935. Dugué is concerned to determine the distributions for which theorems can be formulated which will be valid for finite samples [22]. Dugué extended the work of Fisher and Darmois studying the estimators in different cases and showing that they achieve maximum accuracy.

Thus in the course of his thesis, Dugué returned to several results that had been stated by Fisher: he produced new proofs or proofs for theorems that lacked them and he summarised all known results on maximum likelihood estimation. He deepened and made precise certain issues of convergence by using theorems and tools from probability theory, such as those of Khintchine or Doob [23]. Dugué takes stock of the various tools of the theory of estimation theory by comparing estimators and by his extensive study of the method of maximum likelihood. He completes his study of consistent statistics with one of exhaustive statistics, incorporating the results of Fisher and Darmois on various issues. The subject studied by Dugué and the references he cites reflect the influence of Darmois. In 1930, Darmois was one of the few French mathematicians teaching and working

[19] DUGUÉ 1937, p. 27. He generalises the result to other statistics which he has explained in the first part of his work.

[20] Cf. DUGUÉ 1937, p. 3.

[21] Cf. DUGUÉ 1937, p. 4 and p. 44.

[22] The "finite samples" of FISHER 1918, p. 712.

[23] He also noted some recent results in the theory of probabilitiy. For example, DUGUÉ 1937, p. 23 mentions Kolmogorov's proof of the strong law of large numbers: in current terminology the arithmetic mean of independent and identically distributed random variables converges almost surely to their common expectation.

on statistical topics, in particular those raised by Ronald Fisher in England [24]. Moreover Dugué attended some of his lectures at the Institut Henri Poincaré [25]. The influence of Darmois on Dugué's thesis can be seen in the explicit reference to one of his publications and also in the acknowledgments at the end of the introduction: the author expresses his gratitude for having "pointed out the subject of this memoir and so kindly directing his work" [26]. Their subsequent careers would be linked. Dugué succeeded Darmois as director of the Institute of Statistics at the University of Paris in 1958 [27] and occupied the chair of mathematical statistics from the same date.

Dugué's dissertation is concerned essentially with the theory of estimation for independent experiments. He succeeds however, in extending some of his propositions to the case of experiments that are linked by simple or multiple chains. This is particularly so for the proposition that maximum likelihood achieves the lower bound for the precision of estimators with limiting normal distributions [28]. Dugué based this development on the work of Markov in 1907 and Hostinsky in 1929 [29]. These extensions to the case of chained events testifies to the position of Markov chains in French probability theory at the end of the 1930s. Dugué's memoir is not primarily about this theory and FISHER 1925 had not mentioned chained events. Yet the doctoral student emphasises these extensions. In 1938 and 1939 two theses on the theory were submitted in succession–by Wolfgang Doeblin and Robert Fortet.

# 3 Doeblin and Fortet on Markov chains

These two doctoral theses deal mainly with the theory of chains of probabilities, which was initiated by Markov at the beginning of the 20th century and which was rapidly developing in the 1930s, as is familiar from the historical literature.[30]

[24] As testified by the historical literature.

[25] See COLASSE and PAVÉ 2002, p. 89 and BENZECRI 1988. In the interview Bernard Bru states that Daniel Dugue also studied statistics with Fisher. However, I have found no other source confirming this information.

[26] Cf. DUGUÉ 1937, p. 5..

[27] Cf. COLASSE and PAVÉ 2002, p. 89.

[28] Cf. DUGUÉ 1937, pp. 52-55.

[29] Cf. MARKOV 1907; HOSTINSKY 1929. He uses their result on the existence of a limit of the expectation $E(\Psi(x_{n-1}, x_n))$ independent of $x_1$ such that the convergence in distribution of $\sqrt{n}\left(\frac{\sum_{i=1}^{n} x_i}{n} - a\right)$ to a normal distribution with mean $a$, the common mean of the variables in the case when the variables $x_i$ are linked in a chain.

[30] For a thorough development of Markov chain theory see the article by Eugene Seneta SENETA 1966. See also SHEYNIN 1989 and MAZLIAK 2007.

Recall, following MARKOV 1907, that a Markov chain is a sequence of random variables $(X_n)_{n\geq 0}$ such that knowledge of the random variable at time $n+1$ depends only on knowledge of the variable at time $n$. In other words, if the sequence $X_n$ takes values in a discrete set $I$, we have

$$P\left(X_{n+1}=j\,|X_n=i, X_{n-1}=i_{n-1},...,X_0=i_0\right)=P\left(X_{n+1}=j\,|X_n=i\right)=p_{i,j}.$$

The works–and lives–of Robert Fortet and especially of Wolfgang Doeblin have already been studied, mainly by Bernard Bru.[31] Doeblin's thesis, in particular, has already been analyzed in great detail by Bru and Mazliak. So I will give only some basic information, referring the reader to their work.[32] However a few words on the careers of Doeblin and Fortet will provide some context for their doctoral research.[33]

In 1938, five years after arriving in Paris, Doeblin published a paper, "Sur les propriétés asymptotiques de mouvements régis par certains types de chaînes simples." By 1935 he had "definitely begun to work alone on the theory of Markov chains which was known to be complete only for special cases; possible generalisations could only be glimpsed because the methods in use were not capable of significant extension."[34] Doeblin worked on his own and very quickly: he obtained his first results and submitted them to Fréchet on the latter's return from his trip to Eastern Europe in October and November 1935. Doeblin worked so intensively that by June 1936, he had the results for part of his thesis, DOEBLIN 1938b; they appear earlier in the *Bulletin mathématique de la société roumaine des sciences* of 1937.[35]

Like Doeblin, Robert Fortet soon began to work on probability chains. From 1935, he was attempting to elucidate the asymptotic behaviour of homogeneous Markov chains in discrete time with a countable set of states. In 1935 and 1936 Fortet published two notes in the *Comptes rendus de l'Académie des sciences.*[36] The first is concerned with the "study of the 'regular' case of countable chains by Markov's method" a method Fortet would simply cite in his thesis. The second contains important results from the thesis in a condensed form and without proof. Fortet says he will use "properties on the iteration of certain algebraic substitutions involving an infinite number of variables"[37]; from now on Fortet used the spectral

[31] For Doeblin, see BRU 1992, 1993, YOR 2002 and also MAZLIAK 2007; for Fortet see BRU 2002, and BRU and NEVEU 1998..

[32] See BRU 1992, 1993 and MAZLIAK 2007

[33] The present account is based on Bru's articles.

[34] Cf. BRU 1992 p. 10 and BRU 1993, p. 6.

[35] Cf. DOEBLIN 1937. In 1936 Doeblin had obtained results for a discrete state space; he still lacked the results on Smoluchowsky's equation.

[36] Cf. FORTET 1935, 1936. Bernard Bru provides a summary in BRU 2002, p. 21.

[37] I explain below how Fortet details these properties and how he uses them in his thesis.

theory of quasi-compact operators.[38] So, like Doeblin, Fortet published two articles on topics he would treat in his thesis.

The two research students worked together in the context of the "Borel Seminar" but they also collaborated on the theory of completely connected chains, initiated by Onicescu and Mihoc in Bucharest, publishing a joint article in 1937, "Sur des chaînes à liaisons complètes", DOEBLIN and FORTET 1937a.[39] They also studied issues related to the Smoluchowsky equation in response to the work of Nicolas Kryloff and Nicolas Bogoliouboff, KRYLOFF and BOGOLIOUBOFF 1937b, a. Their joint note on this subject also appeared in the *Comptes rendus*, DOEBLIN and FORTET 1937b. The relations between the two students is also evident in the mutual references in their dissertations[40].

One last fact links the two doctoral students to each other and to the work of Fréchet. Among the works listed in FRÉCHET 1938, that generalise his study of the theory of probability in chains are the "theses in press by M. Doeblin and M. Fortet" as well as "a paper written by M. Doeblin in advance of his thesis." Doeblin and Fortet thus appear as "students" of Fréchet, working on a common theory under the direction of a professor of the Sorbonne[41].

## 4 Fortet on chains and linear operators

Fortet's dissertation also deals in part with the theory of chain probabilities, but the subject is approached from another side, that of the theory of linear operators and their iterations. The probabilistic methods of Doeblin are not used, instead

[38]Bernard Bru presents the theory in modern terminology.

[39]Letter 7 of DOEBLIN 2007, from Doeblin to Hostinsky (March 11, 1937), discusses the problem of the dependence of transition probabilities from one state to another and especially the problem of dependence with states away. Neither Onicescu nor Mihoc are cited in the letter. Laurent Mazliak suggests that it was in his reply to this letter that Hostinsky indicated the work of the Romanian mathematicians to Doeblin and we can assume that Doeblin had been previously unaware of it.

[40]Fortet cites Doeblin's thesis and his works on simple and multiple Markov chains with denumerable state-space (cf. FORTET 1939, p. 20) even if this is not reflected in the bibliography. Doeblin's bibliography refers to two notes by Fortet in the *Comptes rendus,* with the title "Sur les probabilités en chaîne." (cf. FORTET 1935, 1936 and DOEBLIN 1938, p. 122.

[41]A last proof in this direction can be found in the way the students thanked Fréchet. Doeblin expressed "respectful gratitude for the interest he has shown in the development of these works and for his unstinting advice both on the research and on its presentation." Cf. DOEBLIN 1938b, p. 5. The letters of Doeblin to Fréchet in the National Archives have been used by BRU 1992, 1993 in his work on the relationship of student and professor. As for Fortet, he writes "We cannot conclude this introduction without expressing our thanks to Professor Fréchet, our teacher, without whom this work would never have been: not only do we owe him the topic and the guiding ideas but he was good enough to follow our progress step by step and keep us on the right track." Cf. FORTET 1939, pp. 19-20.

heavy use is made of the theory of Fredholm kernels; Fréchet had brought out the analogies between that theory and the theory of substitutions [42].

Fortet took off from the theory of "completely continuous" linear operators[43] studied by Riesz and develops some points [44]. For example [45], Fortet solved a problem posed by Fréchet in FRÉCHET 1934a and FRÉCHET 1936, on the exact expression for the $n-th$ iterate $U^{(n)}$ of a continuous operator $U$. Fortet used results from Fréchet [46] to establish an asymptotic expression for $U^{(n)}$ in terms of $n$ [47].

The theory of linear operators and that of linear but not "completely continuous" operators appears naturally in the theory of chain probabilities in the way Fréchet approached it. This involves an application of these general theories to problems in probability.

In the introduction to the thesis Fortet outlines the two problems in the theory of probability chains that he will consider. The first focuses on simple Markov chains where there is a countable infinity of possible states. Fortet introduces the sequence of probabilities in the way that Doeblin had for the finite case. The difference is that the index of summations ranges up to infinity. The states are denoted $E_1, ..., E_i...$ $(i = 1, 2, ..., \infty)$; $p_{ik}$ denotes the probability of going from state $E_i$ to state $E_k$ in a trial and this probability is assumed to be independent of time. Following Fortet, we have on putting $P^n_{ik}$ for the probability that the system goes from $E_i$ to $E_k$ after $n$ periods the following relations: $P^1_{ik} = p_{ik}$, $P^n_{ik} \geq 0$, $\sum_{k=1}^{\infty} P^n_{ik} = 1$, $P^{m+n}_{ik} = \sum_{k=1}^{\infty} P^m_{ij} P^n_{jk}$ for $m$ and $n$ positive integers and also for $m = 1$. The problem is to study the asymptotic behaviour of $P^n_{ik}$.

The second problem described by Fortet relates to the case where a possible state of a stochastic system $S$ is represented by a point $E$ of a space with a finite number of dimensions. Fortet then assumed that the set of points $E$ is a measurable set $V$ and that a probability density $p(E, F)$ is defined on this set.

[42] See e.g. FRÉCHET 1936 to which Fortet refers, FORTET p. 204. According to Bru and Neveu it was Frechet who proposed the approach to Fortet.; cf. BRU and NEVEU p. 85.

[43] Compact in modern terminology.

[44] Cf. FORTET 1939, p. 15. Bru and Neveu, in BRU and NEVEU 1998, use current terms, referring to the theory as Riesz's spectral theory for compact operators.

[45] Cf. the fourth section of the thesis, FORTET 1939, pp. 203-229.

[46] In FRÉCHET 1934a.

[47] More precisely his fourth chapter, , cf.. FORTET 1939, p. 203, treats the special case of the problem: "Given in Hilbert space or $L^{(2)}$ space a linear algebraic substitution $A$ whose real or complex coefficients $a_{ik}$ $(i, k = 1, 2, ..., \infty)$ satisfy the condition: $\sum_{i,k} |a_{ik}|^2 < M$ to find an expression in terms of $n$ for the n-th iterate $A^n$ of $A$ $(= A^1)$" In addition, part of the fifth chapter (four of the five notes) is devoted to the study of problems of the theory of substitutions and their analogues in the theory of Fredholm kernels. Fortet was especially interested in the substitutions of Dixon, cf. FORTET 1939, p. 230.

This is a simple Markov chain, constant, with an uncountable infinity of states, under the hypothesis of the existence of a suitable probability density function. Fortet follows certain researches of Fréchet[48] and studies the asymptotic behaviour of functions $P^n(E,F)$ satisfying the following conditions: for $n$ and $m$ positive integers[49]: $P^1(E,F) = p(E,F), P^n(E,F) \geq 0, \int_V P^n(E,F)dF = 1$,

$$P^{m+n}(E,F) = \int_V P^m(E,G)P^n(G,F)dG, P^{n+1}(E,F) = \int_V p(E,G)P^n(G,F)dG.$$

In the first chapter of his memoir Fortet examines the two problems and studies the asymptotic behaviour of $P^n_{ik}$ and of $P^m(E,\omega)$ by a method he calls the "method of Markov" which is "only applicable to problems in probability"[50] and which does not involve the theory of linear operators. In the case of a countable infinity of possible states (problem 1), the method depends on considerations relating to the convergence of series and the limit of convergent series, which depends on a certain condition that must be satisfied by $P^n_{ik}$[51]. In the case of an uncountable infinity, the method uses similar tools and its application depends on a condition on $P^m(E,\omega)$ that Fortet called the "Markov condition" that must be checked[52]. Fortet generalises the works of Fréchet on the asymptotic behaviour of functions of points $P^n(E,F)$ and on the behaviour of set functions $P^m(E,\omega)$ where $\omega$ is an arbitrary measurable subset of $V$[53].

The rest of the thesis is based on the theory of linear operators and its application to solve the first problem described by Fortet. Indeed, to study the asymptotic behaviour of $P^n_{ik}$ . Fortet considers iterating the linear operator $P$ defined by the expression[54]

$$y_i = \sum_k p_{ik}x_k$$

As Fortet shows after introducing the theory of linear operators, the $P$ operator

[48]Fortet refers particularly to FRÉCHET 1933b, p. 179. In this article Fréchet repeats a part of the course on chain probabilities that he had given at the Sorbonne in the year 1931-2, cf. FRÉCHET 1933b, p. 176. Fortet indicates at several points that his study is a continuation of work by Fréchet, cf.. FORTET 1939, p.18, p. 47, p. 62.

[49]cf.. FORTET 1939, p.18. The integral used here is that of Lebesgue. cf.. FORTET 1939, p. 44. As he explains, that leads to conditions of measurability of $P^n(E,F)$ on for arbitrary $n$ and $E$ fixed and of $P^m(E,G)P^n(G,F)$ in $G$.

[50]Cf.. FORTET 1939, p.19.

[51]Cf.. FORTET 1939, pp.29-30: Fortet assumes that there exists an index $j_0$ and an index $\upsilon$ such that $P^{\upsilon}_{ij_0} > \eta > 0$. He refers to work by Hostinsky HOSTINSKY 1931, p.14 and generalises the method of Markov in the case of a finite number of states, a method that Hostinsky uses.

[52]cf.. FORTET 1939, pp.47-8: "It is necessary that there exists a subset $\Omega$ of $V$, measurable and of positive measure, such that at all points of $F$ of $\Omega$, we have $\min_{E\in V} p^{\upsilon}(E,F) \geq \eta$."

[53]This generalisation is also noted explicitly by Fréchet in his report on the thesis, cf also FORTET 1939, p. 62.

[54]Cf. FORTET 1939, p.17.

is a linear substitution in the space $D_\omega$ of Fréchet[55] which is only exceptionally "completely continuous"[56].

Fortet then studies the linear substitutions in $D_\omega$ and their iterations and generalises his results to the case of arbitrary linear operations [57]. Substitutions are no longer assumed to be "completely continuous" [58]. He introduces concepts that generalising the theory of Fredholm kernels [59] such as the "resolvent" of a substitution which he borrows from Riesz, and which gives certain properties, such as holomorphy on the domain of definition [60]. Fortet also introduces the "polar radius" of the substitution[61]. He then applies the results directly obtained in the theory of polar radius to the first problem where the polar radius is $> 1$. He then deduced in some cases the asymptotic behaviour of $P^n_{ik}$, drawing explicitly on certain methods of Fréchet [62]. He examines too another special case where the polar radius of the corresponding substitution is not strictly greater than 1 and applies the results found to the study of the asymptotic behaviour of $P^n_{ik}$.[63]

Besides these two problems, Fortet also considered a more general problem of chain theory: the chains "à liaisons complètes" discussed by Onicescu and Mihoc in ONICESCU and MIHOC 1935. This problem leads to a type of linear operator

[55] Fortet refers to Fréchet's *Les espaces abstraits*, FRÉCHET 1928. The space $D_\omega$ is the space of points $x$ whose coordinates $x_1, ..., x_k, ... (k = 1, 2, ...., \infty)$ are such that $|x_k|$ remains bounded as $k$ varies. This space is endowed with a distance between two points, $x$ and $x'$ denoted by $|x - x'|$ , which is the upper limit of $|x_k - x'_k|$ when $k$ varies. Cf. FORTET 1939, pp. 63-64.

[56] Cf. FORTET 1939, p.17.

[57] According to the definition given by Fortet, FORTET 1939, p. 66, a "linear substitution" is a point transformation that is single-valued, distributive and continuous.

[58] According to the definition given by Fortet, FORTET 1939, p. 66, a substitution is called "completely continuous" if it transforms every bounded set of points of $D_\omega$ into a compact set.

[59] According to Fréchet in his report on the thesis. The report emphasises these two notions.

[60] If $A$ is a linear substitution and if $\lambda \in \mathbb{C}$ is such that $E - \lambda A$ has an inverse (or when $E$ is the identity transformation) the resolvent is the function of $\lambda$ defined by: $A(0) = A$ if $\lambda = 0$ and by $A(\lambda) = \frac{1}{\lambda}\left[(E - \lambda A)^{-1} - E\right]$ if $\lambda \neq 0$, cf. FORTET 1939, p. 71.

[61] Cf. FORTET 1939, p.79. The polar radius $P$ of $A(\lambda)$ is "the largest of the positive numbers $\rho$ such that in the region completely inside the circle $|\lambda| \leq \rho$, the singular points of $A(\lambda)$ comprise only a finite number of poles of finite rank." The generalisation of the Fredholm theory is illustrated in the following theorem stated by Fortet, FORTET 1939, p. 79: "Given the substitution $A$ with polar radius $P$, such that $|\lambda| < P$, one can apply to the equation

$$x - \lambda A(x) = y$$

the classical theorems of Fredholm."

[62] Cf. FORTET 1939, p. 99 and FRÉCHET 1934b. The case is that where the substitution considered relative to $p_{ik}$ has a polar radius $> 1$ , which Fortet characterises not only from the point of view of spectral analysis but also from that of probability, cf. FORTET 1939, p. 98. For the statement of certain properties, cf. e.g. FORTET pp. 110-111.

[63] This is the case of homogeneous finite substitutions investigated in FORTET 1939, pp. 115-172.

that is more complex than those involved in the first problem. Fortet proposes some additions to the results already obtained by the two Romanian mathematicians. He described the problem of chains "à liaisons complètes" of Onicescu and Mihoc, still called "OM-chains": consider a random system $S$ that can take one or other of the states $E_1, ..., E_m, E_{m+1}$ following successive experiments numbered $1, 2, ..., n, ...$ Fortet assumes that the actual probabilities (not a priori) of the states $E_j$ to the $(n-1)$th trial are respectively $x_1(n-1), ..., x_j(n-1), ..., x_{m+1}(n-1)$. If the $(n-1)$th trial has realised another state $E_i$ then the probabilities $x_k(n)$ of the states $E_k$ to the $n$th trial are determinate functions of $x_j(n-1)$ and of $i$ with $x_k(n) = \phi_{ik}[x_1(n-1), ..., x_j(n-1), ..., x_{m+1}(n-1)]$ where $k = 1, ..., m$ and $i = 1, ..., m+1$. The chain is then determined by $\phi_{ik}$ and the probabilities $x_j(1) = x_j$ relative to the first trial. Fortet studies the asymptotic behaviour of the probability $P_s^{(n)}(x_1, ..., x_m)$ that the $n$th trial results in state $E_k$.

Fortet took up the method used by Onicescu and Mihoc in ONICESCU and MIHOC 1935. He also uses some results on points of attraction of certain transformations to show the convergence of iterates of functions defined by a recurrence relationship. He also compares this method with his own theory based on linear operators and their iterations in a particular case and thus he completes some results established by the Romanian mathematicians.

This is the topic on which Doeblin and Fortet collaborated in DOEBLIN and FORTET 1937a.[64] The two doctoral students study some special cases of the chains à liaisons complètes taking over some of the concepts described by Doeblin in his own thesis (such as that of "groupes finaux"). The method used is not the same as Fortet used in his thesis for OM-chains which is probably why he does not refer to this earlier publication.[65]

The intellectual link between Maurice Fréchet and Robert Fortet is clear throughout his thesis.[66] Fortet followed Fréchet's linear operator approach and tackled some problems identified by him. Some of the questions he studied were common to Doeblin but the approaches and methods of the two students were very different. Fortet emphasised above all the method and not the results on the two problems he attacked. In the introduction to his thesis he states that his results have been overtaken by those found by Doeblin in his thesis and by Kolmogorov in parallel and simultaneous work.[67] The approach based on operator theory seems therefore less suited to the study of chains than the probabilistic method of Doe-

[64]As noted above at the beginning of this section.

[65]At the end of the article is, however, a note on the resolution of a functional equation using Riesz's theory of linear operations, cf. DOEBLIN and FORTET and FORTET 1937a, p. 142-148. There Fortet announces the method he will use in his thesis.

[66]Confirming what Bru writes of the intellectual relations between the two in BRU 2002.

[67]Cf. FORTET 1939, p. 20.

blin.[68] Nevertheless Fortet can state for the case of countable chains a simple regularity condition, which Doeblin generalised to the case of a continuous state space.[69] Fréchet characterises Fortet's method, of which he was one of the creators, as "more general" [70] because it "can be applied to problems, not only unrelated to the theory of probabilities, but of *a more general mathematical nature.*" Fréchet's description of the advantage attached to the method used by Fortet underlines the link between the works of the two. Bernard Bru and Jacques Neveu emphasise the broad range of applications of this method in their discussion of Fortet's work when he was preparing his thesis. They show the links that Fortet makes between various theories, such as probability theory, integral equations, substitution in an infinite number of variables, and thus confirm what Fréchet had said in his report on the thesis. They write that the case of countable chains first studied by Fortet naturally led on to processes with values in unbounded domains of which Fortet could be considered a precursor.[71]

# 5 Loève under the influence of the works of Lévy

Michel Loève took many courses in the Sorbonne Faculty of Science in the early 1930s and was initially orientated towards theoretical physics and actuarial science (on which he worked with Darmois) before going on to prepare a dissertation on probability theory, supervised by Maurice Fréchet.[72]

Loève's thesis, *Étude asymptotique des sommes de variables aléatoires liées*,

[68] Cf. also BRU 2003, p. 171.

[69] See DOEBLIN 1938b.

[70] Cf. FRÉCHET 1934a, p. 69.

[71] Cf. BRU and NEVEU 1998, p. 85-86: "He treats the behavior of iterates of operator defined on a suitable Banach space which is generally not compact. One can then contemplate an appropriate generalization of Riesz's spectral theory of compact operators as Fortet does in his thesis. He thus extends the classical theory of quasi-compact operators of Kryloff and Bogoliouboff. "

On the impact of Fortet's thesis, see Bru's discussion in BRU 2002. The writers of treatises do not seem to have initially associated Fortet's name with these results. As for research related to Kryloff and Bogoliouboff, Bru indicates that the names of Russian mathematicians and of the Japanese mathematicians Yosida and Kakutani are more likely to be linked to it. The latter published their theory only in 1941. Bru states that the whole theory was known to Fortet and Doeblin in 1937, cf. BRU 2002, p. 22, but they do not refer to it in their dissertations.

[72] Cf. BRU 1992, p. 43. The obituary UNIVERSITY OF CALIFORNIA (SYSTEM) ACADEMIC SENATE 1980 confirms this. Another relevant fact is that Loève was awarded the title of actuary by the University of Lyoin in 1936.

One also sees signs of his early research interests in the notes he published in the *Comptes Rendus*: two are mentioned in the bibliography of his thesis, LOÈVE 1941a, p. 69: they are from 1934: "Sur l'intégration des équations de Dirac", *Comptes Rendus*, **198**, (1934). 799-801 and "Sur les moyennes de la théorie de Dirac", *Comptes Rendus*, **198**, (1934) 1303-1305.

was the last of the six submitted between 1937 and 1941 and marked the end of the first wave of probability research. [73] In the thesis Loève generalises and develops some results obtained during the first half of the twentieth century, on the convergence of sums of random variables. He draws attention to this in the Introduction to the thesis: he recalls, "the most important results of probability theory concerning the asymptotic properties of sequences of random events and sums of independent random variables."[74] Loève works in a broader framework than that of independent events and random variables, for he studies how these properties change when dependent events and variables are considered.

On the subject of dependence, Loève indicates the work that has already been done on simple Markov chains, and mentions the contributions of Fréchet and "his students (Doeblin, Fortet)".[75] But he does not place his own work in that tradition. At no point in his thesis does he consider Markov chains and he cites neither the theses of Doeblin and Fortet nor the publications of Fréchet on this subject. He places his work in the research direction suggested by Fréchet in his introductory lecture at the international symposium in Geneva in 1937. Referring to recent developments in this area, Fréchet says: "It is first of all an attempt [. . . ] to get free of the independence condition under which the fundamental classical properties have been obtained"[76]. It is to this task that Loève devotes himself. There are references throughout his work to the classical works on probabilities, such as those by Poincaré, Borel, Cantelli, Chebyshev, Bienaymé, Kolmogorov, etc.., which he interprets as special cases of his own results.[77] His wish to generalise the classical proposition also extends to the case of independence. [78]

Loève's starting point for his research is the work of Serge Bernstein and Paul Lévy on the study of dependence to which he makes frequent reference. In the introduction to his thesis he refers to Bernstein's extension in 1922 "of the theorem of Liapounoff to variables that he calls almost independent" [79], and also to the 1935-1936 work of Lévy on chained variables[80]. It is these kinds of result that he

[73]A later thesis that of André Blanc-Lapierre was submited in 1945.

[74]Cf. LOÈVE 1941a, p. 1. That is, for the two subjects the law of large numbers, the strong law of large numbers the "central tendency" the last being called the "central limit theorem" in works on probability theory.

[75]Cf..LOÈVE 1941a, p. 1.

[76]Cf. FRÉCHET 1939.

[77]This is recognised in Fréchet's report on the thesis. This describes how Loève has succeeded "in encompassing in several theorems generalisations of previously obtained results which become simple cases of its own analysis."

[78]Loève writes in the introduction to his theis: "Even in the case of independence we have tried to obtain more general propositions than the classical ones."LOÈVE 1941a, p. 2.

[79]Liapounoff's theorem is equivalent to the theorem that is today called the central limit theorem. Cf. LOÈVE 1941a, p. 1.

[80]In the course of the thesis, Loève refers to LÉVY 1935, 1936. See also MAZLIAK 2009a

seeks.

One of the main ideas of the thesis is to impose on the random variables asymptotic assumptions in order to study the asymptotic behaviour of their sum. To this end, he introduces new concepts inspired by the work of Kolmogoroff, such as the notions of order of magnitude, of infinitesimal order in probability or almost surely, of the stability of a sequence of random variables, of residual fluctuation.[81] With these new concepts, Loève reformulates and finds complements to several additions theorems including those of Bernoulli and Poisson. These theorems appear as special cases of more general propositions giving necessary and sufficient conditions for the stability of a sequence of events.[82] Michel Loève also defines the concepts of independence in mean for random variables and asymptotic independence in mean and establishes criteria for then when there are infinitely many events. He then shows how the theorems of Borel and Cantelli are special cases of these criteria [83].

Michel Loève then uses a similar approach to generalize the law of large numbers and the law of "central tendency". He examines various proofs of these theorems and considers the existing conditions on the random variables. He then makes more general assumptions which no longer rely on the assumption of independent variables and shows that the conclusions of the theorems are still valid.

For the law of large numbers, Loève explains that one of the points of the classical proof using the method of Tchebycheff [84] rests on "Bienaymé's equality":

$$\sigma^2(S_n) = \sigma^2(X_1) + \sigma^2(X_2) + ... + \sigma^2(X_n)$$

where $\sigma$ represents the standard deviation of a variable and $S_n = X_1 + ... + X_n$ and the $X_i$ are mutually independent random variables.

Loève then describes the generalisations already achieved by Kolmogoroff in KOLMOGOROFF 1933 and by Lévy in LÉVY 1935, 1936 who replaces the hypothesis of independence of variables by uncorrelatedness. Lévy also considers the

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where Laurent Mazliak analyses some of Lévy's work on chained variables.

[81]Cf. LOÈVE 1941a, pp. 4-6. He refers especially to Kolmogoroff's definition of stable sequences in KOLMOGOROFF 1933 to which he refers explicitly.

[82]Cf. LOÈVE 1941a, p. 9. The theorem of Bernoulli which he cites on p. 5 is the following: "a sequence of independent events of constant probability $p$ is stable." Poisson established the same proposition but assuming that the events do not have the same probability. According to Loève, a sequence of events $(A_i)$ is said to be stable if for every $\epsilon > 0$,. $P\left(\left\|\frac{R_n}{n} - \frac{E(R_n)}{n}\right\|\right) \to 0$ with $\frac{1}{n}$ where $R_n$ represents the number of the first $n$ events that occur (or expressed otherwise $R_n = \sum_{i=1}^{n} 1_{A_i}$). cf. LOÈVE 1941a, pp.4-5

[83]These theorems of Borel state results for the realisation of an infinite number of events assumed independent based on the convergence of the series $\sum P(A_i)$. In the case when this series is convergent Cantelli extended Borel's result on the zero probability of achieving an infinite number of events to the case when these events are not independent

[84]Cf. LOÈVE 1941a, p.20.

mean $M_{i-1}(X_i)$ which represents the mean of the variable $X_i$ evaluated when one knows the realised values of $X_1, ..., X_{i-1}$ ; his assumption is written $M_{i-1}(X_i) = 0$ for $i = 1, 2, ..$ whatever the values realised.

Loève generalises this problem by considering [85] first of all sequences of variables with double indices $(X_{n,i})_{n\geq 1, 1\leq i\leq n}$ and of sums with double indices $S_{n,v} = X_{n,1} + X_{n,2} + ... + X_{n,v}$ and assuming that all the variables have zero mean. The following two theorems are typical.[86]

**Theorem D** When for $n \to \infty$

$$\sum_{i=1}^{n} \sup |M'(X_{n,i})| \to 0 \quad \text{and} \quad \sum_{i=1}^{n} \sup \left| M\left(X_{n,i}^2\right)\right| \to 0$$

the sequence $\{S_{n,n}\}$ is stable.

**Theorem H** When $a_n$ are chosen such that

$$0 < a_n < a_{n+1}$$

and there exists a sequence $a_{n_i}$ such that $\frac{a_{n_i}}{a_{n_{i+1}}} \to \alpha$ a constant when $i \to \infty$;

$$\begin{aligned} \frac{1}{a_n} \sum_{i=1}^{n} \sup |M'(X_{n,i})| &\to 0 \text{ with } \frac{1}{n} \text{ (the } (X_i) \text{ being such that } M(X_i) = 0 \\ \text{for } i &= 1, 2, ..) \\ \sum_{i=1}^{\infty} \frac{\sigma_i^2}{a_i^2} &< \infty, \text{ writing } \sigma_i^2 = M\left(X_i^2\right) \end{aligned}$$

then $\frac{S_n}{a_n} \to 0$ with $\frac{1}{n}$ almost surely.

The "weak" (convergence in probability) and "strong" (almost sure convergence) laws of large numbers then appear as special cases of the two theorems [87]

Loève proceeds in the same way to demonstrate a generalized version of the theorem on the convergence of the distribution of sums of random variables to the de Moivre-Laplace or Gaussian law–he uses both names. He modifies the two methods of proof of the theorem in the case of independent random variables: the first due to Liapounoff and generalised by Lindeberg, also called the "method of moments" by Loève[88], the second using the characteristic function for random variables and which Lévy and Feller helped improve. Loève also shows as special cases of these results those obtained by Bernstein for the case of the random

[85] Cf. LOÈVE 1941a, p.21.

[86] These are on p. 25 and p. 35 of Loève's thesis.

[87] Such as the Glivenko-Cantelli theorem on the distribution functions of random variables.Cf. LOÈVE 1941a, p.34.

[88] Fréchet uses the same term in his report on the thesis.

variables he called "almost independent"[89], and by Lévy in 1935-1936. This result is a "fundamental theorem" [90], which includes the central limit theorem under as general assumptions as possible. This is the only theorem that Fréchet mentioned explicitly in the report on the thesis and he insisted strongly on its importance.[91]
**Fundamental Theorem** The distribution of $S_{n,n}$ tends towards the de Moivre-Laplace distribution with mean 0 and variance $\sigma^2$ when for $n \to \infty$ and for all $\varepsilon > 0$

$$\sum_{i=1}^{n} \sup \int_{|\xi| \leq \varepsilon} dF'_{n,i}(\xi) \rightarrow 0,$$
$$\sum_{i=1}^{n} \sup \int_{|\xi| \leq \varepsilon} \xi dF'_{n,i}(\xi) \rightarrow 0,$$

and for each value of $\varepsilon$, if there exists quantities $\sigma^2_{n,i}$ such that

$$\sum_{i=1}^{n} \sup \left| \int_{|\xi| \leq \varepsilon} \xi dF'_{n,i}(\xi) - \sigma^2_{n,i} \right| \rightarrow 0,$$
$$\sum_{i=1}^{n} \sigma^2_{n,i} \rightarrow \sigma^2.$$

Thus Michel Loève's thesis of 1941 provides a contribution to the asymptotic study of dependent variables in the tradition of Bernstein and Paul Lévy on the subject. The various concepts he introduced also enabled him to generalise the theorems obtained for the case of independence and in this framework he presents the three results described in the introduction to his memoir as "the most important results in probability theory "[92]: the law of large numbers for events or dependant random variables, the strong law of large numbers for dependent random variables, the "central tendency" for dependent random variables [93]. Loève also establishes some propositions for moments of sums of dependent random variables. He applies some to study a concept introduced by Serge Bernstein, that of the "rayon d'activité de la liaison" between random variables [94]. In his bibliography, Loève not mention any of Bernstein's publications from the 1930s. We can then assume

[89]Bernstein assumes the existence of the first three moments of the random variables, cf. LOEVE 1941a, p. 37.

[90]Thus described by Loève and Fréchet.

[91]Loève's fundamental theorem and its different forms are stated on p. 42 with the same notations as before $F'_{n,i}(x)$ is the probability $P\prime(X_{n,i} < x)$ for which $X_{n,i} < x$ evaluated for the category of trial where the realised value of $S_{n,i-1}$ is known.

[92]Cf. LOÈVE 1941a, p.1.

[93]These three results also provide the material for three notes in the *Comptes rendus*, presented in the year Loève submitted his thesis. They are included in the bibliography of his thesis and Fréchet refers to them in his rerport.:. LOÈVE 1941b,c,d..

[94]This notion was introduced by Loève, LOÈVE 1941a, pp.60-1: it means the difference be-

that his results were communicated by Maurice Fréchet, who, despite the political situation in Russia, continued to correspond with Russian mathematicians. He seems thus to have played a role in disseminating some of their mathematical results. [95].

Loève's approach may be compared with that of Jacques Dufresnoy who also submitted a dissertation in 1941 on function theory. Dufresnoy used a new method based on a topological approach to the theory of functions of a complex variable, introduced by Lars Ahlfors in 1935, to re-prove classic theorems of the theory from the late nineteenth and early twentieth centuries, such as Picard's third theorem on the exceptional values of meromorphic functions or the theorems of Landau and Schottky of 1904, as expounded by Montel and Valiron, etc.. Dufresnoy, like Loève, used new ideas and new concepts to reprove results considered classical at the time.

Bernard Bru describes Michel Loève as being "directed" by Maurice Fréchet but it seems that his influence on the student's work was weak and that his role was more that of an advisor.[96] The works of his that Loève cites, such as FRÉCHET 1937, 1938, are essentially surveys of topics in probability that report the results of other mathematicians.[97] By contrast, the intellectual influence of Paul Lévy is greater in the number of references to his works made by Loève. According to Bernard Locker [98], Lévy was pleased to call him "My student and my friend," presumably after the war for, as Locker points out, Loève has not been his student in an academic sense of the term. Incidentally the correspondence between Lévy and Fréchet shows that in 1941 Lévy did not yet know the doctoral student's work. In letter 45 dated August 6, 1943, Lévy told Fréchet thus he had "finally received Loève's thesis" and that his "first impression is very favorable "[99]. In letter 46 of August 27, 1943 [100] Lévy responded to the first fifty pages that he had read and discussed the novelty and significance of the theorems presented by Loève, remarking on the "central tendency" towards his own work. Lévy was thus unaware of Loève's research during its formative period and his influence was

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tween variables from which the influence between the corresponding variables weakens; in other words it assumes a weakening of the relationship between the variables $X_i$ and $X_j$ for $1 \leq i, h \leq m$ :and $|i - h| > d_n$ where $d_n$ is the "rayon d'activité de la liaison."

[95]For example on p. 20 Loève evokes a result of Bernstein explained by Fréchet in FRÉCHET 1937.

[96]Also shown by the address Loève thanks at the end of his introduction to
Frechet: he thanked for "his interest in [its] contention, [. . . ], For his comments and advice to [it] was also useful for research than for writing and for the many
instructive conversations that [he] got the most benefit, "cf. Loeve 1941a, p. 3.

[97]cf. LOÈVE 1941a, p. 11, p. 20..

[98]cf. LOCKER 2001, p. 12.

[99]cf. BARBUT ET AL. 2004, p. 188..

[100]cf. BARBUT ET AL. 2004, p. 191.

exerted through his writings [101]. The other writers of probability theses refer to some of Lévy's work but his influence on Loève's thesis was unique.

# 6 Malécot on probability and the modelling of heredity

Gustave Malécot was a student at the École Normale Superieure from 1932 to -35 and, according to Maxime Lamotte, while there he was noticed by Georges Darmois. [102] Thomas Nagayaki reports that Malécot subsequently had a fellowship for four years at the Institut Henri Poincaré, where he worked with Darmois[103]. In 1939 after four years he submitted his thesis, *Théorie mathématique de l'hérédité mendélienne généralisée.* The subject was on the border between statistics, probability and genetics and the work reflected the influence of Darmois, as we will see.

Malécot's subject was the theory of heredity and at the end of the nineteenth century there had been two distinct approaches to this theory: that of Mendel and that of Francis Galton and Karl Pearson. According to Malécot [104], Mendel's laws assume that inheritance is "particulate" and that children depend only on their parents. By contrast, the results of the English biometric school [105] "reflect a blended inheritance in which there is a relationship between the average value of some continuous quantity for children and the average for parents and various ancestors." Pearson and R. A. Fisher had already investigated the apparent divergence of the two theories and Malécot took as his starting point a 1918 article by Fisher [106], "The correlation between relative on the supposition of Mendelian inheritance," FISHER 1918, which he describes as "fundamental"[107]. This side

[101] We also found a trace in the manuscript of the thesis available at the library
Mathematics-Research Institute of Mathematics of Jussieu. A handwritten inscription written by
Michel Loève it appears on the cover: "To Professor Paul Levy" magician-probe "which
the magnificent work was and remains the basis of the work of the author.

[102] Cf. LAMOTTE 1999a.

[103] See NAGAYAKI 1989, p. 254. Nagayaki states that Malécot's research was guided by Darmois. However, as there was no such official role at the time, I do not know what exactly Nagayaki means. Lamotte in LAMOTTE 1999b, p. 59 states that Malécot worked as Darmois' research assistant for those four years.

[104] Cf. MALÉCOT 1939, p. 1.

[105] Darmois used the term "English biometric school" in his report on the thesis but it is also used by NAGYLAKI 1989, EPPERSON 1999, GILLOIS 1999.

[106] He is explicit about this in the introduction to his thesis, cf. MALÉCOT 1939, p. 2.

[107] In EPPERSON 1999, p. 477, Bryan K. Epperson writes, "Malécot told me how he had spent 2 years reading and mastering (no doubt in rigorous mathematical detail) Fisher's article."

of Fisher's research was quite distinct from that studied by Daniel Dugué and Malécot makes no reference to his thesis.

Malécot's objective was to elucidate Fisher's entire argument, to make rigorous and generalize the reconciliation he had effected between the results of the English biometric school and the legacy of Mendel [108]. He sought to "develop systematically hypotheses and methods which would allow him to extend the laws of Mendel to explain the modes of 'blended' inheritance. "[109].

The way the subject is presented in Malécot's thesis reflects the influence of Darmois [110]. Darmois also reported on the thesis. Moreover, in the introduction[111] Malécot refers to one of his publications describing the results of the English biometric school.[112]. Finally, according to Lamotte [113], it was Darmois who led Malécot to the work of Ronald Fisher.

In his study of the theory of heredity, Malécot considers a measurable character of an individual, which generally reflect the action of "Mendelian" hereditary factors [114], denoted by $x$. Malécot's hypothesis regarding the operation of hereditary factors in the general framework is that the different pairs of genes that make up the hereditary structure of the individual contribute additively. Malécot associates with each factor a contribution which he interprets as a random variable can take a finite number of values according to the state of the couple of associated genes.[115] For example, if one takes $H$ the contribution associated with a pair of genes, individuals are divided into three categories according to whether they are carry the pair $AA$, the pair $Aa$ or the pair $aa$, $H$ then takes the three respective values $i$, $j$ and $k$. Malécot denotes by $P$, $2Q$, $R$, the frequency in the population of these 3 categories. He then interprets these as probability and describes the contribution $H$ as a random variable "which can take values $i, j, k$ with the probabilities $P$, $2Q$, $R$."

Under these assumptions, for an individual taken at random, the action of Mendelian factors $x$ is a random variables, a sum of random variables that Malécot

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[108] Cf. NAGAYAKI 1989, p. 254.

[109] Cf MALÉCOT 1939.

[110] As I have already pointed out, in the 1930s Darmois was the only French mathematician and professor in the faculty of science in the Sorbonne who taught and did research on statistics. He was most familiar with the English work of Pearson and Fisher.

[111] Cf. MALÉCOT 1939, p. 1.

[112] Malécot cites the following : G. Darmois, 1932, "La méthode statistique dans les sciences d'observation", *Annales de l'Institut Henri Poincaré* 3 191-228. In his introduction Malécot also thanks Henri Eyraud a eelative, who was at the University of Lyon at the end of the 1930s and turning his attention to mathematical finance.Cf. RITTER to appear in 2009.

[113] Cf. LAMOTTE 1999b, p. 59.

[114] Cf. MALÉCOT 1939, p. 3..

[115] Cf. MALÉCOT 1939, p. 4.

called "of the 3rd order"[116]. Malécot distinguishes two cases for the "stochastic relationship between the various factors "[117], i.e. the correlation between the different random variables. The first is that of random mating. There is then stochastic independence between the characters of the two individuals. The probability of a particular descendent is then the product of the probabilities of the two gametes that are or the sum of such products if it can be made up in several different ways [118]. The second case is that of "homogamy" or assortative mating[119]: the mates resemble each other than if they were chosen at random from the population. To model this second case, Malécot incorporates the assumption made by Fisher on the expression of probabilities of association of various possible states (probabilities of association factors): he introduces coefficients of association $f_{lm} > -1$ which vary with the pair of factors that are considered and which are zero in the case of independence[120]. In both cases, Malécot also looks within a given population the frequency of association of genes [121].

In his memoir Malécot studied under two assumptions (random mating and assortative mating), the change in genetic composition over time, from generation to generation, to see if they tend towards an equilibrium distribution. Given the genetic constitution of one generation, he derives the constitution of the next according to Mendel's laws on the two assumptions he has made about mating. By modelling the factors as random variables, Malécot intends to further the reconciliation already made by Fisher between the Mendelian theories of heredity and the theories of the English biometric school. For example, under the assumption of random mating, Malécot found the distribution of hereditary characters in blended inheritance of which height is the typical example. Galton had shown that the distribution of these characters is Gaussian. Malécot established this by supposing that these characteristics result from adding the effects of a large number $n$ of independent Mendelian factors.[122]

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[116]Cf. MALÉCOT 1939, p. 4. That is to say, they can take 3 values.

[117]Cf. MALÉCOT 1939, p. 5.

[118]Cf. MALÉCOT 1939, p. 11.

[119]Cf. MALÉCOT 1939, p. 6.

[120]Cf. MALÉCOT 1939, pp. 6-7. Malécot considers the different possible states $HK$ where $H$ takes the values $i, j, k$ with probabilities $P, 2Q, R$ and $K$ the values $i', j', k'$ with probabilities $P', 2Q', R'$. Then the probability that states $i, i'$ occur simultaneously is represented by $(11) = PP'(1 + f_{11})$. $f_{(11)}$ is the coefficient of association of states $(i, i')$. Malécot gives the details in the course of his work, Cf. MALÉCOT 1939, pp. 21-23.

[121]A factor is a pair of genes, the results are different but they depend on one another, the probability of association of genes dependent on the probability of association of factors. Cf. MALÉCOT 1939, pp. 8-9.

[122]Cf. MALÉCOT 1939, pp. 16-17. In the proof he uses Liapounoff's theorem, referring to the book by Lévy, LÉVY 1937, where several proofs are presented. Malécot uses the following formula due to Lindeberg, cf. LÉVY 1937, p. 241.. He puts $\sigma_x$ for the standard deviation of the

Malécot also works out the correlations between relatives[123] distinguishing not only the cases of random and assortative mating, but also formulating hypotheses about the genes that make up the couple and the dominance of one form of the gene over another. In case of no dominance, the contribution of a factor corresponding to a pair of genes is calculated as the sum of the effects of two genes which constitute it.[124] By contrast, in the case of dominance, they do not add up. The formula for the contribution of these two genes is obtained by adding to the effect of these two genes a residual. In Malécot's model, which incorporates that of Fisher, the values of the residual are obtained by the method of least squares.[125]

Malécot's dissertation proposes a probabilistic model theory of heredity. According to Bryan K. Epperson, the treatment "foreshadowed his stochastic process approach to other problems".[126] Epperson, further notes that in his later work Malécot considers genetic evolution as a Markov process: he is thus able to derive the distribution of gene frequencies in small populations and invents notions of identical genes and new mutant genes.[127] His thesis is thus the beginning of a research direction that will continue afterwards.

Gustave Malécot's thesis was the first to treat the theories of the English statisticians. It is evident from accounts of the period that only Darmois had published on this subject. The submission of such a thesis shows how the new mathematicians in France were interested in these theories and how studying them could lead to the award of a doctorate of science. This could be a sign of the institutional

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sum of the contributions of the independent Mendelian factors. He assumes that the maximum value taken by each of these contributions is strictly less than $\epsilon\sigma_x$. From the formula established by Lindeberg in his proof of Liapounoff's theorem he derives the following inequality:

$$\left| P\left(\frac{x}{\sigma_x} < v\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{t^2}{2})dt \right| < 6\epsilon^{1/4}$$

Thus the distribution differs little from that of Gauss.

[123]By not restricting itself to not only study the correlation between parent and child, but considering also the most ancestral correlations: small children, etc.. or the correlations between siblings

[124]Cf. MALÉCOT 1939, p.4.

[125]This is to minimise the expected value of the squared residual. cf. Cf. MALÉCOT 1939, Chapters III and IV. Fisher then resuming Malécot generalizes this model to adjust the additive contributions

any number of couples to the overall effect: the residue is considered globally to all couples. Cf. MALÉCOT 1939, pp.61-64.

[126]Cf. EPPERSON 1999, p. 477.

[127]According to Michel Gillois, Malécot reinterprets some coefficients as the coefficients of kinship and inbreeding as probabilities associated with random draws from genes, cf.

GILLOIS 1999, p. 2. For more information on Malécot's probability innovations in his theory of population genetics, see NAGYLAKI 1989, EPPERSON1999; GILLOIS 1999.

weight Darmois had on the mathematical scene where from the 1920s Borel had been promoting probability theory as a field where different scientific fields could meet. Darmois was able to support research on a topic that would not have been admissible for a thesis in mathematical science ten years earlier.

# 7 Conclusion: teachers and students

Probability theory changed dramatically during the inter-war period and the doctoral dissertations underline the change. The 1920s were marked by the omnipresence, institutional and intellectual, of Borel who was promoting this emerging field. In that decade only one thesis was submitted, on geometric probability, and it bore traces of his influence. In the 1930s and 1940s his influence diminished and he seemed no longer to have an intellectual role. No student based his work on Borel's research or acknowledged his influence. He was in the background, part of the landscape, and his main role was institutional, chairing the committees for several of the candidates–Daniel Dugué, Wolfgang Doeblin and Jean Ville[128]. Borel's place in probability was taken by the people he had brought in to help promote it, Fréchet, Darmois and also Lévy. A study of the dissertations submitted at the end of the 1930s brings out the intellectual contribution of these three and shows their particular spheres of influence.

The influence of Darmois is in evidence in the two dissertations devoted to the study of articles or results of R. A. Fisher: Daniel Dugué's on the theory of estimation and Gustave Malécot's on the theory of Mendelian inheritance, drawing respectively on FISHER 1925 and FISHER 1918. In the 1920s Borel had urged that probability be applied to other fields of science and Malécot's thesis reflects this tendency. By the late 1930s Darmois seems to have replaced Borel as the advocate of this tendency. His assumption of this role is confirmed by the thesis of André Blanc-Lapierre, an example of the application of probability to physics. This thesis also shows the emergence of a new generation for Robert Fortet helped Blanc-Lapierre with his research. The importance of Darmois cannot be measured by number of citations but rather it is shown by the directions of research that he encouraged. He directed his students to the work of Fisher and emphasised the role of applications. Dugué, like Malécot, acknowledged the part he played in the development of their work (the choice of subject, guidance and advice in the course of the research). His importance for Blanc-Lapierre was more in encouraging and facilitating the submission of his thesis to the Paris Faculty of Science. In 1945 Fortet, as a professor at the University of Caen, did not have the same institutional weight.

[128] It is in every way to the end of his academic career since he left office in January 1940, cf. e.g. MAZLIAK and SHAFER 2008, p. 4.

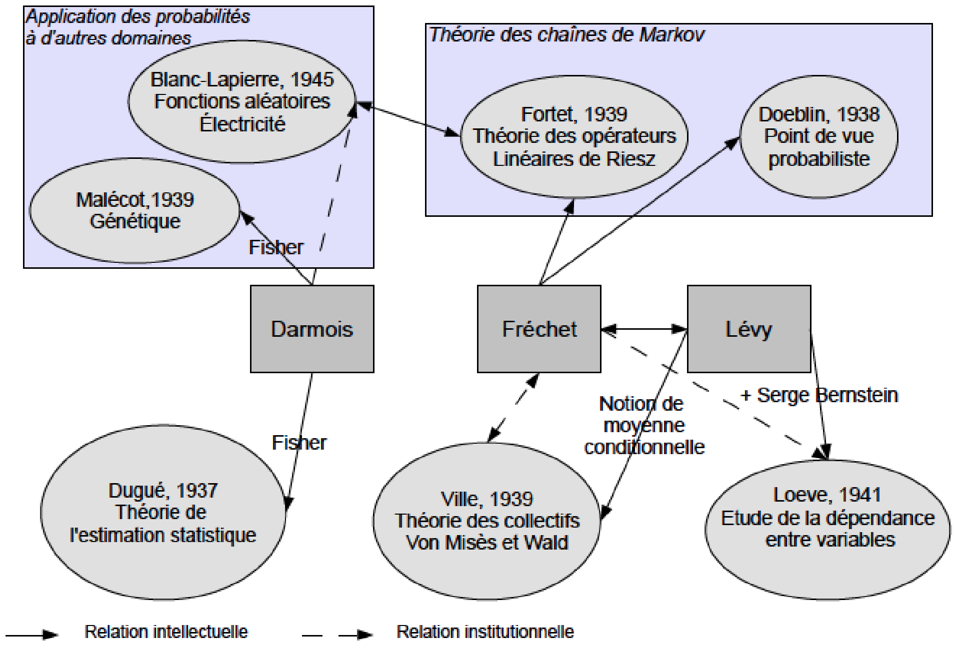
The influence of Darmois did not, however, go beyond the framework of these three theses. He appears not to have been involved with the other probability theses. His sphere of influence was quite separate from that of Fréchet, just as the fields in which they published were separate. This separation is illustrated in a letter Fréchet wrote to Doeblin on September 8, 1938.[129] Fréchet was writing about Michel Loève, who in 1938 was still interested in actuarial science: "You speak of Loève. Is it true that he is currently working with Darmois? In this case, I would rather not ask him, for it is natural that he will not divert from studies, research or work that M. Darmois has given him ." Even if their relations were cordial, Darmois and Fréchet appeared not to be very close.

Four of the doctors came within Fréchet's sphere of influence: Fortet, Doeblin, Ville and Loève. In current historiography, notably in Bernard Bru's work, they are sometimes called "students of Fréchet." Fortet and Doeblin worked on Markov chains, as Fréchet did, but Fréchet contributed in other ways to the work of Ville and Loève. Fréchet's publications were sources for known and classic results in probability and Fréchet introduced them to the work of other mathematicians such as Paul Lévy, whose position at the École Polytechnique denied him any role in the Paris Faculty of Science.[130]

The probability theses of the late 1930s testify to the emergence of probability as a field in French mathematics. In the theses a problem is addressed from a different point of view and there are references to different mathematical works. With the exception of Fortet, who refers to Doeblin's results on Markov chains–without using them–no student cites the work of any other. The dissertations are constructed independently. This does not mean that the students were isolated. Each took a different theme representing a direction of research in probability and incorporating new theories, developed in France or abroad. The seminar that brought them together at the end of the 1930s at the Institut Henri Poincaré (at least Ville, Fortet, Doeblin and Loève participated in some way) also shows that the students knew each other and knew each other's research topics and could see the subjects evolve, integrating different advances. There were collaborations between research students as well as between the students and teachers. André Blanc-Lapierre's thesis, coming after a gap in 1945, cites works of the students beyond those described in their theses. Thus his thesis shows the rapid evolution of these topics and of work in this developing field.

[129] Bru 1993, p. 26. According to the notes than Bernard Bru (p.52), this letter is filed Archives of Marbach, ref. : D. Döblin.C.D. Wolfgang Döblin

[130] A diagram summarizing the relationship between mathematicians and mathematicians and PhD in probability at the end of this chapter.




# 8 Epilogue: the students after the Second World War

The Second World War affected probability theory and limited its development. Doeblin died in the war and, from the works he left, it may be assumed that he would have had an important role [131]. Loève was imprisoned at Drancy during the German occupation. Then, from 1944 to 1946, he was chargé de recherches at the Institute Henri Poincaré; in 1946 he left France for the University of London and worked there until 1948. He then went to Berkeley via Columbia University.[132] One may also speculate that the war contributed to the limited impact of the theories and results contained in some of the theses, especially those of Fortet and Ville.

Of the seven probability students from the close of the inter-war period, only three went on doing probability in France: Loève was in Berkeley, Blanc-Lapierre worked on random functions but from the viewpoint of physics [133], while Malécot

[131]Witness also the adjectives attributed to him, such as "brilliant" or Paul Levy, who compares it to Abel and Galois in Levy 1955

[132]Cf. University of California (System) Academic Senate 1980.

[133]From what he says in Blanc-Lapierre et al. 1997, p. 3

specialised in probability and genetics. This leaves Daniel Dugué, who specialised in statistics, Jean Ville, who worked in industry, becoming Professor of Econometrics at the Faculty of Paris in 1958 [134], and Robert Fortet. Fortet was the one who most rapidly came to occupy a powerful institutional position (from 1939, he took over some of Darmois' duties at the Faculty of Sciences in Paris while being chargé de recherche and then professor at Caen). He was interested in probability theory and equally in its applications to other areas, as his involvement in the thesis of Blanc-Lapierre shows. In fact, he extended its zone of influence over arithmetic and algebra, geometry and analysis, the classical areas of mathematics. The research Ville did after his thesis was also on the margin of probability as it was understood in France. His interests included operational research, game theory applied to economics, areas which after the Second World War were more cultivated in the United States. Ville had no direct involvement in establishing probability as a classical field in French mathematics in the second half of the twentieth century. In terms of institutional weight, the probability group was weak. Perhaps the presence of Doeblin would have changed the balance of forces, especially as a group of young mathematicians involved in renewing and developing this area had begun to form in the late 1930s.

One can not fail to make an association between this group of mathematicians that came into existence at the end of the 1930s and the Bourbaki group which originated earlier in the 1930s. Ville, like some members of Bourbaki, found the sources for his research abroad, notably in Germany. The research students in probability were in a field that was not much cultivated in the academic mathematical milieu of the inter-war period. A Borel seminar was organised, echoing the Julia seminar. However, probability did not have the same success, at least in the decades after the Second World War. It would be interesting to understand this difference better and, in particular, the role of Bourbaki in forming a strong group of mathematicians devoted to the same cause and united in publishing a book, that symbolised their identity. For the probabilists there was nothing comparable.

[134] See BRU 1992, p. 41, footnote 44 also D'ORGEVAL 1992, p. 389. Several of Ville's results, especially on game theory, were known at the Liberation for industrial applications. According to Bernard d'Orgeval, Ville became, through Roger Julia, scientific advisor to the Société Alsacienne de Constructions Atomiques, de Télécommunications et d'Electronique, later CIT Alcatel where he would become a vice-president in 1962. He also collaborated with CEA but Orgeval does not go into details. For further biographical information on Ville during the war and on the course of his mathematical research, see BRU et al. 1999, 229-232.

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