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Stochastic Processes from 1950 to the Present

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Doing "history of mathematics" about Probability Theory is an undertaking doomed to failure from the outset, hardly less absurd than doing history of physics from a mathematician's viewpoint, neglecting all of experimental physics. We can never say often enough, *Probability Theory is first of all the art of calculating probabilities*, for pleasure and for probabilists to be sure, but also for a large public of users: statisticians, geneticists, epidemiologists, actuaries, economists.... The progress accomplished in fifty years responds to the increasing role of probability in scientific thought in general, and finds its justification in more powerful methods of calculation, which allow us for example to consider the measure associated with a stochastic process as a whole instead of considering only individual distributions of isolated random variables.

It must be acknowledged from the beginning that the "history" below, written by a mathematician, not only ignores the work accomplished by non-mathematicians and published in specialized journals, but also the work accomplished by mathematicians deepening classical problems – sums of independent variables, maxima and minima, fluctuations, the central limit theorem – by classical methods, because daily practice continues to require that these old results be improved, the same way the internal combustion engine continues to be improved to build cars.

Probability has developed many branches in fifty years. The schematic description found here concerns only stochastic processes, understood in the restricted sense of random evolutions governed by time (continuous or discrete time). Moreover, we must leave aside (for lack of competence) the study of classes of special processes.

I have presented the parts of probability that I myself came in contact with, and their development *as it appeared to me*, trying at most to verify certain points by bibliographical research. In particular, saying that an article or an author is "important" signifies that they have aroused a certain enthusiasm among my colleagues (or in me), that they were the source of some other work, that they enlightened me on this or that subject. I feel especially uncomfortable presenting work that appeared in the East (Japan being part of the West on this occasion). In fact, not only was communication slow between the two political blocs, but probabilists worked in slightly different mindsets, with certain mental as well as linguistic barriers. Even in the West, we can distinguish smaller universes, each with its traditions, tastes and aversions. The balance between pure and applied probability, for example, was very different in the Anglo-Saxon countries, endowed with powerful schools of statisticians, than in France or Japan. The text that follows should therefore be considered as expressing personal opinions, not value judgments.

Probability around 1950

This initial date may be less arbitrary in probability than elsewhere. In fact, it is marked by two works that have reached a broad public, the first one summarizing two centuries of ingenuity, the second one providing tools for future development. First Feller's book *An Introduction to Probability Theory and Its Applications*, without a doubt one of the most beautiful mathematics book ever written, with technical tools barely exceeding the level of high school. Next Halmos' *Measure Theory*, the first presentation of measure theory, in the West, free of unnecessary subtleties, and well adapted to the teaching of probability according to Kolmogorov's axioms (until Loève (1960), for many years the standard reference). In fact, discussions on the foundations of probability, which had embroiled the previous generation, were over. Mathematicians had made a definitive choice of their axiomatic model, leaving it to the philosophers to discuss the relation between it and "reality". This did not happen without resistance, and a majority of probabilists (particularly in the United States) long considered the teaching of the Lebesgue integral not only waste of time, but also an offense to "probabilistic intuition".

Early developments. Note, just before the period at hand, a few mathematical events that seeded future developments. The first article published by Itô on the stochastic integral dates back to 1944. Doob worked on the theory of martingales from 1940 to 1950, and it was also in a 1945 article by Doob that the strong Markov property was clearly enunciated for the first time, and proven for a very special case. The theorem giving strongly continuous semigroups of operators their structure, which greatly influenced Markov process theory, was proven independently by Hille (1948) and Yosida (1948). Great progress in potential theory, which was also destined to influence probability, was achieved by H. Cartan in 1945 and 1946, and by Deny in 1950. In 1944, Kakutani published two brief notes on the relations between Brownian motion and harmonic functions, which became the source of Doob's work on this question and grew into a wide area of research. In 1949 Kac, inspired by the Feynman integral, presented the "Feynman-Kac formula", which remained a theme of constant study in various forms – we use this occasion to recall this extraordinary lecturer, originator of spontaneous ideas rather than author of completed articles. Finally, in 1948 Paul Lévy published an extremely important book, *Stochastic Processes and Brownian Motion*, a book that marshals the entire menagerie of stochastic processes known at the time. Like all of Lévy's work, it is written in the style of explanation rather than proof, and rewriting it in the rigorous language of measure theory was an extremely fruitful exercise for the best probabilists of the time (Itô, Doob). Another example of the depth probabilists reached working with their bare hands was the famous work of Dvoretzky, Erdős and Kakutani on the multiple points of Brownian motion in $\mathbb{R}^n$ (1950 and 1957). It took a long time to notice that although the result was perfectly correct, the proof itself was incomplete!

"Stochastic processes". Doob's book, *Stochastic Processes*, published in 1953, became the Bible of the new probability, and it deserves an analysis. Doob's special status (aside the abundance of his own discoveries) lies in his familiarity with measure theory, which he adopts as the foundation of probability without any backward glance or mental reservation. But the theory of continuous-time processes poses difficult measure theoretical problems: if a particle is subject to random evolution, to show that its trajectory is continuous, or bounded, requires that all time values be considered, whereas classical measure theory can only handle a *countable* infinity of time values. Thus, not only does probability depend on measure theory, but *it also requires more of measure theory than the rest of analysis*. Doob's book begins with an abrupt chapter and finishes with a dry supplement - between the two it adheres to a pure austerity accentuated by a typography that recalls of the great era of le *Monde*, but made pleasing by a style that is free of pedantry. From Doob on, probability, even in the eyes of Bourbaki, will be one of the respectable disciplines.

It is informative to enumerate the subjects covered in Doob's book: he starts with a discussion of the principles of the theory of processes, and in particular of the solution to the difficulty mentioned above (Doob introduces on this occasion the "separability" of processes); a brief exposition on sums of independent variables; martingale theory, in discrete and continuous time (work by Doob that was still fresh), with many applications; processes with independent increments; Markov processes (Markov chains, resuming Doob's 1945 work, and diffusions, presenting Itô's stochastic integral with an important addition for further work, and stochastic differential equations). It all appears prophetic now. On the other hand, three subjects are weakly addressed in Doob's book: Gaussian processes, stationary processes, and prediction theory for second order processes. Each of these branches is being called on to detach itself from the common trunk of process theory and to grow in an autonomous fashion – and we will not talk about them here.

We must comment on one aspect of Doob's book, crucial for the future.

Kolmogorov's mathematical model represents the events of the real world by elements of the sigma-algebra $\mathcal{F}$ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Intuitively speaking, the set Ω is a giant "urn" from which we pull out a "ball" ω, and the elements of $\mathcal{F}$ describe the various questions that one can ask about ω. Paul Lévy protested against this model, criticizing it for evoking *only one* random draw, whereas chance evidently enters at every moment in a random evolution. Doob resolved this difficulty in the following way: There is a single random draw, but it is "revealed" progressively. Time t (discrete or continuous) is introduced in the form of an increasing family $(\mathcal{F}_t)$ of sigma-algebras – what is currently called a *filtration*. The sigma-algebra $\mathcal{F}_t$ represents "what is known of ω up to time t". Let's then call T the moment where for the *first time* the random evolution shows a certain property – for the insurance company, the first fire of the year 1998, for example. It is a random quantity such that, to know if $T \leq t$, there is no need to look at the evolution beyond t – in mathematical language, the event $T \leq t$ belongs to $\mathcal{F}_t$ – in fact, to know if there was a fire in January 1998, there is no need to wait until the month of March. Compare this definition to that of the *last* fire of the year 1997: to know if it occurred in November, you need to know that a fire occurred in November, *and also that no fire occurred in December.* These "non-anticipatory" random variables are called today *stopping times*. The idea of non-anticipatory knowledge is implicit in French, where (normally) the declension of a word only depends on words coming before it, but not in German, where the whole meaning of the sentence depends on the final particle. The importance of the notion of stopping times comes surely from the work of Doob and of his disciple Snell (1952), but it must have a prior history, because it penetrates for example Wald's sequential statistical analysis.

Principal themes: 1950-1965

Markov processes. The efforts of probabilists of the first half of the century had been mostly dedicated (the problem of foundations aside), to the study of independence: sums of independent random variables, and corresponding limit distributions. After independence, the simplest type of random evolution is Markovian dependence (named after A. A. Markov, 1906). An example of it is given by the successive states of a deck of cards that is being shuffled. For predicting the order of cards after shuffling, all useful information is included in (complete) knowledge of the current state of the deck; if this is known, knowledge of previous states does not bring more information about the accuracy of the prediction. Most examples of random evolution given by nature are Markovian, or become Markovian by a suitable interpretation of the words "current state" and "complete knowledge". The theory of Markov processes divides into sub-theories, depending on whether time is discrete or continuous, or whether the set of possible states is finite or countably infinite (we speak then of Markov *chains*[1]), or continuous. On the other hand, the classical theory of sums of independent random variables can be generalized into a branch of Markov process theory where a group structure replaces addition: in discrete time this is called *random walks*, and in continuous time *processes with independent increments*, the most notable of which is *Brownian motion*.

>From a probabilistic point of view, a Markov process is determined by its *initial law* and its *transition function* $P_{s,t}(x, A)$, which gives, if we observed the process in state x at time s, the probability that we find it at a later time t in a set A (if we exclude the case of chains, the probability of finding it *exactly* in a given state y is null in general). The transition function is a simple analytical object – and in particular, when it is *stationary*, meaning it only depends on the difference $r = t - s$, we obtain a function $P_r(x, A)$ to which the analytical theory of semigroups, in full flower since Hille-Yosida's theorem, applies. Hence the interest in Markov processes around the 1950s.

The main question we ask ourselves about these processes is that of their long term evolution. For example, the evolution of animal or human populations can be described by Markovian models assuming three types of limit behavior: extinction, equilibrium, or explosion – the latter one, impossible in the real world, nevertheless constitutes a useful mathematical model for a very large population. The study of various states of equilibrium where a stationary regimen is established is related to statistical mechanics.

Continuous-time and finite-state space *Markov chains*, well known for years, represent a model of perfectly regular random evolution, which stays in a state for a certain period of time (of known law) then jumps into another state drawn at random according to a known law, and so on and so

[1]Some authors call a Markov process in discrete time with any state space a Markov chain.

forth indefinitely. But as soon as the number of states becomes infinite, extraordinary phenomena can happen: it could be that jumps accumulate in a finite period of time (and afterwards the process becomes indescribably complicated), even worse, it could be that from the start each state is occupied according to a "fractal" set. The problem is of elementary nature, very easy to raise and not easy at all to resolve. This is why Markov chains have played the role of a testing ground for every later development, in the hands of the English school (Kingman, Reuter, Williams…) and of K. L. Chung, whose insistence on a probabilistic rather than analytic attack on the problems has had a considerable influence.

The other area of Markov process theory which was in full expansion was *diffusion theory*. In contrast to Markov chains, which (in simple cases) progress only by jumps separated by an interval of constant length, diffusions are Markov processes (real, or with values in $\mathbb{R}^n$ or a manifold) whose trajectories are *continuous*. We knew from Kolmogorov that the transition function is, in the most interesting cases, a solution to a parabolic partial differential equation, the Fokker-Planck equation (in fact of two equations, depending on whether we move time forward or backward). During the 1950s, we were willing to construct diffusions with values in the manifolds by semigroup methods, but the work that stood out is Feller's analysis of the structure of diffusions in one dimension. One of the themes of the following years would be the analogous problem in higher dimensions, where substantial, but not definitive, results would be obtained.

The ideas introduced by Doob (increasing families of sigma-algebras, stopping times) made it possible to give a precise meaning to what we call the *strong Markov property*: Given a Markov process whose transition function is known (for simplicity let us say stationary), the process considered from a random time T is again a Markov process with the same transition function, *provided T is a stopping time.* This had been used (well before the notion of stopping time was formulated) in heuristic arguments such as D. André's "reflection principle"[2] – and also in false heuristic arguments (in which T is not really a stopping time). In fact, the first case where the strong Markov property was rigorously stated and proved is found, it seems, in Doob's 1945 article on Markov chains, but Doob himself hides the question under a smoke screen in his great article of 1954. In the case of Brownian motion, the first modern and complete statement is due to Hunt (1956) in the West, while the Moscow school reached in parallel a greater generalization.

Development of Soviet probability. While probability was a marginal branch of mathematics in Western countries, it had always been among the strongest points of Russian mathematics, and it had grown with Soviet mathematics. Two generations of extraordinary quality would make of Moscow, then Kiev, Leningrad, Vilnius, probabilistic centers among the most im-

[2] Which allows the calculation of the distribution of the maximum of a Brownian motion.

portant of the world – before the post-Stalin wave of persecution (mostly antisemitic) brought this boom to a halt, and forced many major figures into internal or external exile (Dynkin himself left in 1976 for the United States). It would take a specialist to tell the whole story. In any case we can discern two dates, those of 1952 when Dynkin published his first article on Markov processes, and of 1956, the birth date of the journal *Teoriia Veroiatnostei*, which published in its first issue two still classic articles, by Prokhorov and Skorokhod, on narrow convergence[3] of measures on metric spaces (Skorokhod's classic book on processes, which extended this work, appeared in 1961).

Concerning the theory of Markov processes, which for many years was one of the principal themes (but not the only theme) of Soviet probability, the history of connections between the Russian school and "Western" probability (including the rich Japanese school!) is partly one of misunderstanding. This is probably due to the lack of structured research in the West, and to the systematic character, in contrast, of the publications of Dynkin's seminar, supporting each other, using a rather abstract common language, and giving prominence to Markov processes with nonstationary transition functions. The fact is that the main results on the regularity of trajectories and the strong Markov property have been proven twice: by Dynkin, Yushkevich, and by Hunt and Blumenthal. The situation was repeated much later, when many important Soviet works (on excursions, on "Kuznetsov measures") were understood late in the West, after being partially rediscovered.

After these generalities, we can examine various streams of ideas.

The great topics of the years 1950–1965

Classical potential theory and probability. In 1954, developing an idea of Kakutani's, dating from 1944 and taken up again in 1949, Doob published an article on the connection between classical potential theory in $\mathbb{R}^n$ and continuous-time martingale theory. The main idea is the link between the solution of Dirichlet's problem in an open set, and the behavior of Brownian motion starting from a point x of this open set: The first moment when a trajectory ω of Brownian motion meets the boundary depends on ω, it is therefore a "random variable". Let us call it $T(\omega)$; let $X(\omega)$ be the position of the trajectory at that moment. It is clear that it is a point on the boundary; so if f is a boundary function, $f(X)$ is a random quantity whose expected value (the integral) depends on the initial point x. Let us call it then $F(x)$: *this function on the open set solves Dirichlet's problem on the open set with boundary condition f.*

All of this had been known for a long time in the case of simple open sets like balls. But for arbitrary domains Doob had to resolve (relying on potential

[3]Narrow convergence is associated with the integration of bounded continuous functions.

theory) delicate problems of measurability, and most of all, he established a link between the harmonic and superharmonic functions of potential theory, and martingale theory: *if we compose a harmonic or superharmonic function with Brownian motion, we obtain a martingale or supermartingale with continuous trajectories*. Let us emphasize this continuity: superharmonic functions are not in general continuous functions, but Brownian trajectories "do not see" their irregularities. Doob uses this result, along with the theory of martingales, to study the behavior of positive harmonic or superharmonic functions at the boundary of an open set, a subject to which he will devote several articles.

Maybe the most striking result of this probabilistic version of potential theory is the intuitive interpretation of the notion (relatively technical) of the *thinness* of a set, introduced in the study of Dirichlet's problem in an open set. We can always "solve" Dirichlet's problem in a bounded open set with a continuous boundary condition f, but we get a generalized solution that does not necessarily have f as limiting value *everywhere*, or have it (where it does have it) in the sense of the *ordinary* topology. There are bad points, and even at the good points one should not approach the boundary too quickly. The notion of thinness makes these two notions precise: "regular" points of the boundary, for example, are those where the complement of the open set is not thin. Now, the probabilistic interpretation of thinness is very intuitive: to say that a set A is thin at the point x means that a Brownian particle placed at the point x will take (with probability 1) a certain time before returning to the set A. (we say *re*turning to A rather than finding A, because, if the point x itself belongs to A, this encounter with A at moment 0 does not count). A certain number of delicate properties of thinness immediately become evident.

Even though it is not our subject, it is worth pointing out that this immediate post-war period, particularly fruitful in the area of probability, was also a fruitful one for potential theory. The very abundant and interesting production (never assembled) of mathematicians like M. Brelot and J. Deny bore fruit not only in potential theory and probability; few people know that distribution theory, for example, was born from a question posed to L. Schwartz on polyharmonic functions.

Theory of martingales. We will not give here the definition of martingales, even though it is simple, but only the underlying idea. The archetype of martingales is the capital of a player during a *fair* game: *on average*, this capital stays constant, but in detail it can fluctuate considerably; significant but rare gains can compensate for accumulations of small losses (or conversely). The notion of *super*martingale corresponds as well to an *un*favorable game (the "super" expressing the point of view of the casino). In continuous time, Brownian motion, meaning the mathematical model describing the motion of a pollen particle in water seen in a microscope, is also a pure fluctuation:

on average, the particle does not move: the two dimensional Brownian motion is a martingale.[4] If we add a vertical dimension, we lose the martingale property, because the particle will tend to go down if it is denser than water (in that case the vertical component is then a *super*martingale), and go up otherwise.

After a pre-history where the names of S. Bernstein (1927), of P. Lévy and J. Ville[5] (1939) stand out, the biggest name of martingale theory is that of Doob, who proved many fundamental inequalities, the first limit theorems, and linked martingales with the "stopping times" that we talked about above, these random variables that represent the "first time" that we observe a phenomenon. Doob gathered in his book so many striking applications of martingale theory that the probabilistic world found itself converted, and the search for "good martingales" became a standard method for approaching numerous probability problems. We have at our disposal a considerable number of results on martingales: conditions under which a martingale diverges to infinity. If it does not diverge, how to study its limit distributions if it does not diverge, and more importantly a set of very precise inequalities, allowing us to limit its fluctuation on characteristics we can observe. We will talk about this more below.

Markov processes and potential. It was clear that the results obtained by Doob for Brownian motion should extend to much more general Markov processes. Doob himself went from classical potential theory to a much less classical theory, that of the potential for heat.[6] But the fundamental work in this direction was accomplished by Hunt's very great article, published in three parts in 1957–58. This article (preceded by an article by Blumenthal that laid the foundation), contained a wealth of new ideas. The most important for the future, probably, was the direct use in probability (for lack of an already developed potential theory, which Doob already had in his first article) of Choquet's theorems on capacities. But Hunt also established (by a proof that is a real masterpiece) that *any* potential theory satisfying certain axioms stated by Choquet and Deny is susceptible of a probabilistic interpretation. This result unifying analysis and probability contributed to making the latter a respectable field.

The third part of Hunt's article is also very original, because it provides a substitute for the *symmetry* of Green's function in classical potential theory. The main role is no longer played by a single semigroup, but by a pair of transition semigroups that are "dual" with respect to a measure – in classical potential theory, the Brownian semigroup is its own dual with respect

[4] Brownian motion happens to be simultaneously a martingale and a Markov process, but these two notions are not related.

[5] Ville's remarkable book, which introduced the name martingale, by the way, became known in the USA only after the war.

[6] Of which the core is the elementary solution of the heat equation, that is, the Brownian transition function itself.

to Lesbegue measure. In this case, we can build a much richer potential theory, but (provisionally) duality remains devoid of probabilistic interpretation: folklore sees it as a kind of time reversal, but this interpretation is rigorous only in particular cases.

A second aspect of probabilistic potential theory concerns the study of the *Martin boundary*. This is a concept introduced in 1941 in a (magnificent) article by R. S. Martin, a mathematician who died shortly afterwards. On one hand, he interpreted the Poisson representation of positive harmonic functions as an integral representation by means of extreme positive harmonic functions; on the other hand, he indicated a method for constructing these functions in any open set: He "normalized" Green's function $G(x, y)$ by dividing it by a fixed function $G(x_0, y)$, then compactified the open set so that all these quotients are extended by continuity; all the extreme harmonic functions then are among these limit functions. This idea was picked up again and developed by Brelot (1948, 1956), and it was partly the origin of Choquet's research on integral representation in convex cones. It was again Doob who, in 1957, discovered the probabilistic meaning of these quotients of harmonic or superharmonic functions. A series of subsequent articles was meant to extend all of this to general Markov processes, by showing that "Martin's boundary" was a good replacement for the "boundaries" introduced earlier to capture the asymptotic behavior of Markov processes. Yet the most decisive step was to be accomplished by Hunt (1960) in a brief and schematic article – his last publication in this area – that introduced a new way to "reverse time" for Markov processes starting from certain random times, and so gave a very useful probabilistic interpretation of Martin's theory. Hunt's article, which concerned only discrete chains, was extended to continuous time by Nagasawa (1964), and by Kunita and T. Watanabe (1965). The result of this work is a rigorous probabilistic interpretation of the duality between Markov semigroups.

In two dimensions, Brownian motion is said to be *recurrent*: its trajectories, instead of tending to infinity, come back infinitely often to an arbitrary neighborhood of any point of the plane. It gives rise to the special theory of logarithmic potential. There exists a whole class of Markov processes of the same kind, whose study is related rather to ergodic theory. This is an opportunity to mention Spitzer's 1964 book on recurrent random walks, which has had a considerable influence. It opened an important line of research, linking probability, harmonic analysis and group theory (discrete groups and Lie groups). It merits a special study, which surpasses my own competence.

Work a little remote from this, which deserves to be cited because it concludes years of research on the regularity of trajectories of Markov processes, is an article of D. Ray from 1959. This article shows (using methods close to those of Hunt) that it is in part an artificial problem. *Any Markov process* can be rendered strongly Markovian and right-continuous by compactifying its state space by adding "fictitious states". Ray's article contained an error, corrected by Knight, but it is in fact a very fruitful method, also fated to rejoin Martin's theory of compactification. On this subject there was again a development parallel to the work of the Russian school, but the results are not directly comparable.

The classic book presenting Hunt's theory and it development (with the exception of Martin's boundary) is the 1968 book of Blumenthal and Getoor. Since we will return very infrequently to probabilistic potential theory, let us mention nevertheless that the subject has remained active up to the present time, mainly in the United States (Getoor, Sharpe). For modern presentations, see the books of Sharpe (1988) and of Bliedtner and Hansen (1986). For interactions between classical potential theory and Brownian motion, the reference is Doob's monumental treatise (1984). Yet the most active branch currently is that of *Dirichlet spaces*, which we will say a word about later on.

Special Markov processes. Hunt's general theory of Markov processes is only one of the branches of Markov process theory. The 1960s marked an extraordinary activity in the study of special processes. First, the very meticulous study of the trajectories of classical processes – Hausdorff dimensions, etc., what we would call today their fractal structure. Let us cite for example, other than the works of Dvoretzky-Erdős-Kakutani, those of S. J. Taylor. Then the study of Markov chains with little regularity, which provides an inexhaustible source of examples and counterexamples (Chung; Neveu (1962) – the latter according to Williams (1979) "the finest paper ever written on chains"). Finally a very rich production in the study of *diffusions*, which will find its Bible in the (too long awaited) book of Itô and McKean (1965). The main problem of concern here is the structure of diffusions in several dimensions, and in particular the possible behavior, at the boundary of an open set, of a diffusion whose infinitesimal generator is known in the interior. For example, take a problem dealt with by Itô and McKean in 1963: find all strongly Markovian processes with continuous trajectories on the positive closed half-line, which are Brownian motions in the open half-line – but of course the problem in several dimensions (studied by the Japanese school; we cite for example Motoo 1964) is much more difficult. It is a matter of making precise the following idea: the diffusion is formed from an interior process, describing the first trip to the boundary, then the subsequent excursions starting and ending on the boundary. An infinite number of small excursions happen in a finite amount of time, and we must manage to describe them and piece them back together. It is a difficult and fascinating problem.

Links between Markov processes and martingales. It is natural that martingales should be applied to Markov processes. Conversely, methods developed for the study of Markov processes have had an impact on the theory of martingales.

Probabilistic potential theory developed for a stationary transition function, i.e. for a *semigroup of transition operators* (P_t); the latter operates on positive functions, and functions that generalize superharmonic functions, here called *excessive functions*, are measurable positive functions f such that $P_t f \leq f$ for every t (and a minor technical condition). In classical potential theory, it is known how to describe these functions, which decompose into a sum of a positive harmonic function, and a Green potential of a positive measure μ. On the other hand, we can associate a Markov process (X_t) with the transition function, and the excessive functions are those for which the process $(f(X_t))$ is a *supermartingale*. In probabilistic theory, there are no measure potentials available, but Dynkin had stated the problem of representing an excessive function f as the potential of an *additive functional*: without getting into technical details, such a functional is given by a family of random variables (A_t) representing the "mass of the measure μ which is seen by the trajectory of the process between times 0 and t", and the connection between A_t and the function f is as follows: for a process starting from the point x, the expected value of A_∞ is equal to $f(x)$. The Russian school (Volkonskii 1960, Shur 1961) had obtained very interesting partial results. In the West, Meyer (who was working with Doob) was able to improve (1962) Shur's result by giving a necessary and sufficient condition for an excessive function to be representable in this way (a condition Doob formulated earlier in potential theory) and to study the uniqueness of the representation.

A little later, we noticed (Meyer 1962) that the methods that had just been used in the theory of Markov processes transposed without change to the theory of martingales, to solve an old problem raised by Doob: the decomposition of a *supermartingale* into a difference of a martingale and a process with increasing trajectories – an obvious result in discrete time. We knew that conditions were needed (Ornstein had shown an example where the decomposition did not exist), and the notion of "class (D)" answered the question precisely. From that moment on, methods that had succeeded with Markov processes would be grafted onto the general theory of processes, giving numerous results. In particular, capacitary methods would make their entry into the theory of processes. This is quite hard to imagine in an environment that was still balking at the Lebesgue integral ten years earlier! Whence a certain bad mood, quite noticeable particularly in the United States.

Before resuming the main flow of thought, a few remarks about a very important particular case of the problem of decomposition. The one dimensional Brownian motion does not admit positive superharmonic functions, but on the other hand plenty of positive subharmonic functions (the convex positive functions) were found, and the corresponding problem of representation had been solved by hand. One of the marvels of Lévy's work had been the discovery and study of the *local time* of Brownian motion at a point, which measures in a certain sense the time spent "at that point" (in all rigor, this time is zero, but the time spent in a small neighborhood, properly normalized, admits a nontrivial limit). Trotter had made a thorough study of it in 1958. In 1963, Tanaka made the link between local time and Doob's decomposition of the absolute value of Brownian motion, thus establishing what was henceforth called "Tanaka's formula". The construction of local times for various types of processes (Markovian, Gaussian…) has remained a favorite theme of probabilists. On local times one may consult the collection of Azéma-Yor (1978).

The problem of decomposition has had other important extensions. An article by Itô and Watanabe (1965), devoted originally to a Markovian problem, introduced the very useful notion of *local martingale*,[7] which allows us to treat the problem of decomposition without any restriction. On the other hand, an article by Fisk (1965), developing Orey's work, introduces the notion of *quasi-martingale*, corresponding somewhat to the notion of a function of bounded variation in analysis.

We could choose as the symbolic date to close this period the year 1966, during which the second volume of Feller's book appeared. Like the first one, it addresses the vast audience of probability users, and remains as concrete and elementary as possible. Like the first, it assembles and unifies an enormous mass of practical knowledge, but this time it uses measure theory. Moreover, the period preceding 1966 had been a time of synthesis and perfection, during which Dynkin's second book on Markov processes (1963), Itô-McKean's book on diffusions (1965), and the synthesis of recent works on the general theory of processes by Meyer (1966) were published.

The 1965–1980 period

The stochastic integral. Doob's book pointed out already that Itô's stochastic integral theory was not essentially tied to Brownian motion, but could be extended to some square-integrable martingales. As soon as the decomposition of the submartingale square of a martingale was known, this possibility was opened in complete generality (Meyer 1963). Thus, two branches of probability were brought back together. We have already talked about martingales; we must go back to talk about the stochastic integral.

A stochastic process X can be considered a function of two variables $X(t,\omega)$ or $X_t(\omega)$, where t is time, and ω is "chance", a parameter drawn randomly from a giant "urn" Ω. The *trajectories* of the process are functions of time $t \longmapsto X_t(\omega)$. In general they are irregular functions, and we cannot define by the methods of analysis an "integral" $\int_0^t f(s)dX_s(\omega)$ for reasonable functions of time, which would be limits of "Riemann sums" on the interval $(0,t)$

$$\sum_i f(s_i)\,(X_{t_{i+1}} - X_{t_i}),$$

[7] Technical definition weakening the integrability condition for martingales.

where s_i would be an arbitrary point in the interval (t_i, t_{i+1}). This is all the more impossible if the function $f(s, \omega)$ itself depends on chance. Yet Itô had studied since 1944 the case where X is Brownian motion, and f a process such that *at each instant* t, $f(t, w)$ *does not depend on the behavior of the Brownian motion after the instant* t, and where s_i is the *left* endpoint of the interval (t_i, t_{i+1}). In this case, we can show that the Riemann sums converge – not for each ω, but as random variables on Ω – to a quantity that is called the stochastic integral, and that has all the properties desired for an integral.

All this could seem artificial, but the discrete analog shows that it is not. The sums considered in this case are of the form

$$S_n = \sum_{i=1}^{n} f_i\,(X_{i+1} - X_i).$$

Set $X_{i+1} - X_i = x_i$, and think of S_n as the capital (positive or negative!) of a gambler passing his time in a casino, just after the nth game. In this capital, f_i represents the *stake*, whereas x_i is a normalized quantity representing the gain of a gambler who stakes 1 franc at the ith game. That f_i only depends on the past then signifies that *the gambler is not a prophet.* Instead of using the language of games of chance, we can use that of financial mathematics, in which the normalized quantities X_t represent *prices*, of stocks for example – and we know this is how Brownian motion made its appearance in mathematics (Bachelier 1900).

Another question of great practical importance involving the stochastic integral is the modeling of the *noise* that disturbs the evolution of a mechanical system. Here we should mention a stream parallel to the purely probabilistic developments: the efforts devoted to this problem by applied mathematicians close to engineers, and we should cite the name of McShane, who has devoted numerous works to diverse aspects of the stochastic integral. The only one of these aspects that has a properly mathematical importance is Stratonovich's integral (1966), which possesses the remarkable property of being the limit of deterministic integrals when we approach Brownian motion by differentiable curves. Whence in particular a general principle of extension from ordinary differential geometry to stochastic differential geometry.

Itô's most important contribution is not to have defined stochastic integrals – N. Wiener had prepared the way for him – but to have developed their calculus (this is the famous "Itô's formula", which expresses how this integral differs from the ordinary integral) and especially to have used them to develop a very complete theory of *stochastic differential equations* – in a style so luminous by the way that these old articles have not aged.

There is still a lot to say about Itô's differential equations properly speaking, and we will mention them again in connection with stochastic geometry. Here, we will talk about generalizations of this theory.

The theory of the stochastic integral with respect to a square-integrable martingale is the subject of the still famous article by Kunita and Watanabe

(1967), oriented by the way to applications to Markov processes: it is related to an article by Watanabe (1964) that gives a general form to the notion of *Lévy system*, which governs the jumps of a Markov process, and to an article of Motoo-Watanabe (1965). Kunita-Watanabe's work was taken up again by Meyer (1967) who added complementary ideas, such as the *square bracket* of a martingale (adapted from a notion introduced by Austin in discrete time), the precise form of dependence only on the past of the integrated process (what are now called *predictable* processes), and finally a still imperfect form of the notion of a *semimartingale* (see below).

This theory would very quickly extend to martingales that are not necessarily square-integrable, on one hand by means of the notion of a local martingale (Itô-Watanabe 1965), which leads to the final notion of semimartingale (Doléans-Meyer), and on the other hand by means of new martingale inequalities, which will be discussed later (Millar 1968). It would be useless to go into details. Let us consider instead the general ideas.

From a concrete point of view, a semimartingale is a process obtained by superposing a *signal* – that is to say, a process with regular trajectories, say of bounded variation, satisfying the technical condition of being *predictable* – and a *noise*, that is, a meaningless process, a pure fluctuation, modeled by a local martingale. The decomposition theorem, in its final form, says that under minimal integrability conditions (absence of very big jumps), the decomposition of the process into the sum of a signal and a noise is unique: knowing the law of probability we can filter the noise and recover the signal in a unique manner. Yet this reading of the signal depends not only on the process, but also on the underlying *filtration*, which represents the knowledge of the observer.

We can extend to all semimartingales the fundamental properties of Itô's stochastic integral, and most of all develop a unified theory of stochastic differential equations with regard to semimartingales. This was accomplished by Doléans (1970) for the *exponential equation*, which plays a big role in the statistics of processes, and by Doléans (1976) and Protter (1977) for general equations (Kazamaki 1977 opened the way for the case of continuous trajectories). The study of stability (with respect to all parameters at the same time) was carried out in 1978 by Emery and by Protter. We can equally extend to these general equations a big part of the theory of *stochastic flows*, which developed after the "Malliavin calculus".

The theory of stochastic differential equations therefore ends up being in complete parallelism with that of ordinary differential equations. Like the latter theory, it can be approached by two types of methods: for the variants of the Lipschitzian case, Picard's method leading to results of existence and uniqueness, and for existence without uniqueness, methods of compactness of Cauchy's type. However, there is a distinction specific to the probabilistic case, the distinction between uniqueness of trajectories and *uniqueness in law*. We limit ourselves here to mentioning the work of Yamada and Watanabe (1971).

The possibility of bringing several distinct driving semimartingales, in other words several different "times", into a stochastic differential equation in several dimensions makes them resemble equations with total differentials more than ordinary differential equations, with geometric considerations (properties of Lie algebras) that enter in Stroock-Varadhan's article (1970) before reaching their full development in the "Malliavin calculus".

Let us come back for a moment to Itô's integral. We can say that it is not a "true" integral, trajectory by trajectory, but it is one in the sense of *vector measures*. M. Métivier was one of the rare probabilists to know the world of vector measures, and he devoted (with J. Pellaumail) part of his activity to the study of the stochastic integral as a vector measure with values in L^2, then in L^p, then in the non-locally convex vector space L^0 (finite random variables with convergence in measure). Métivier and Pellaumail suspected that semimartingales were *characterized* by the property of admitting a good theory of integration (see Métivier-Pellaumail 1977). This result was established independently by Dellacherie and Mokobodzki (1979) and by Bichteler (1979), who started from the other end, that of vector measures.

It is impossible to take account here of the abundance and the variety of the work related to semimartingales. It is indeed a class of processes large enough to contain most of the usual processes, and possessing very good properties of stability. In particular, if we replace a law on the space Ω by an equivalent law[8] without changing the filtration, the semimartingales for the two laws are the same (whereas their decompositions into "signal plus noise" change). This remarkable theorem is due, in its final form, to Jacod and Mémin (1976), but it has a long history (which relates it in particular to *Girsanov's theorem* (1960) in the particular case of Brownian motion). It opens the way to a general form of the statistics of stochastic processes. Indeed, statistics seeks to determine the law of a random phenomenon from observations, and we do not know a priori what this law is. The search for properties of processes that are invariant under changes in the law is therefore very important. See for example Jacod-Shiryaev (1987).

The rapid evolution of ideas in probability resulted – this a general phenomenon in mathematics – in the multiplication of informal publications, such as the volumes of the Brelot-Choquet-Deny seminar on potential theory. The birth of Springer's *Lecture Notes* series led to the international distribution of publications of this type, which were at first "in house". In probability, we find the series *Séminaires de Probabilités* (1967), then the lecture notes of l'Ecole d'Eté de St Flour (1970), and finally the *Seminar on Stochastic Processes* in the United States (1981).

Markov processes. During this whole period, the general theory of Markov processes remained extremely active, but it was no longer the dominant subject in probability as it had been in the preceding period.

[8]Two laws are said to be equivalent if they have the same sets of null measure.

We can distinguish a few themes particularly studied.

In the beginning of the theory of Markov processes, various classes of processes had been introduced axiomatically: Dynkin's "standard processes", and "Hunt" processes, which allowed them to develop probabilistic potential theory. An article by C. T. Shih (1970) is at the origin of a movement of ideas that identified a class of Markov processes, the *right processes*, that possess remarkable stability properties. We will limit ourselves here to mentioning the essential role of Ray's compactification in these questions, and to referring to two books of synthesis: Getoor (1975) and Sharpe (1988).

A second important theme is the *duality* of Markov processes with respect to a measure. Here we start not from a given pair of Markov semigroups in duality with respect to a measure, but rather from a single semigroup, for which we want to build a dual semigroup. The most important article on this question is that of Chung and Walsh (1969).

For the understanding of duality, Mitro's articles (1979) had a great impact. They give a construction for adjoining the forward trajectories of one of the Markov processes with the backward trajectories of its dual process, in order to make it into a *stationary* process arising at a random moment (possibly $-\infty$) and disappearing at a random moment (possibly $+\infty$). In fact, all of this had already been discovered, and under a more general form, in two articles by Dynkin (1973) and Kuznetsov (1974), whose discovery (after Dynkin's arrival in the USA!) generated a good number of papers. The importance of these results for potential theory (excessive measures) has been progressively recognized (Fitzsimmons-Maisonneuve 1986, see the 1990 book of Getoor). The connection with the strange processes constructed by Hunt (1960), completing the understanding of this article, was given by Fitzsimmons (1988).

It is impossible to do more here than name other important subjects: "Lévy systems" of general Markov processes; local times of Markov processes; the various transformations preserving the Markov property (which constituted an essential element of Dynkin's program from the beginning). It is better to devote a little time to a particularly fascinating theme, excursion theory.

The fundamental idea of excursion theory is to study the behavior of a Markov process "around" a fixed state a. The simplest example is that of discrete time Markov chains; there the process's successive passage times in a state constitute what is called a *renewal process*, and the structure of these processes (which have countless applications) has long been known. Between successive passages through a, the chain makes "excursions", which (in the most interesting case where the chain returns to a an infinite number of times) are independent and have the same law. In continuous time, the situation is much more complicated. The model is Lévy's in-depth study of the *passages of Brownian motion at* 0. The set of these passages is a perfect set of null measure, riddled with small holes during which Brownian motion makes its excursions. How to enumerate them, how to compare them with one another, in what sense to consider them as independent and equally distributed? The problem arises in fact for all Markov processes (and it is especially interesting for continuous-time Markov chains that are not regular, a case studied by Chung). It is even more difficult to describe when the Markov process is not studied in the neighborhood of a point, but in the neighborhood of a whole "boundary", because then the impact point moves on the boundary, and we must describe how.

Concerning the encounters with a single state, the axiomatic characterization of the random sets that can be interpreted as the moments of a Markov process's return to a fixed state was the work (after the preliminary studies of Kingman) of Krylov and Yushkevich (1965), in a difficult article, taken up and greatly simplified by Hoffman-Jørgensen (1969). On excursions themselves, the new idea that clarified the problem came from Itô (1971), certainly one of the big conceptual achievements of probability, because the excursion, which is a trajectory, is treated like a point, and the succession of excursions is treated like a new random process that has a simple description. Finally, on boundary problems, we must limit ourselves to citing a remarkable article by Dynkin (1971), which has been read too little in the West (but see El Karoui-Reinhard 1975), and Maisonneuve's work (1974).

We must finally mention an important development for the future: the construction of reversible Markov processes (also called symmetric) by the Hilbertian method of *Dirichlet forms*. Introduced in potential theory by Beurling and Deny (1959), this method came into probability with Silverstein (1974) and Fukushima (1975 for the Japanese edition of his book). It has become one of the most powerful tools for building Markov processes in infinite dimensions, a subject very much alive because of its possible applications to physics.

General theory of processes. The "general theory of processes" is the development of one of the subjects initiated by Doob's book, that of *families of increasing sigma-algebras* (called today *filtrations*) and of stopping times. This development took place in constant interaction with Markov process theory on one hand, martingale theory on the other hand, and the division is therefore somewhat artificial.

The beginnings of the general theory of processes are dominated by the notion of stopping times, and in particular by the numerous results proven around the strong Markov property, and around the "slightly less strong" variants discovered in Markov chain theory by Chung. Numerous results on filtrations are given as lemmas in articles on Markov processes, Dynkin's in particular. One of the first articles dedicated entirely to the general theory is that of Chung and Doob (1965), and the first – or only – book completely dedicated to it is that of Dellacherie (1972), which introduces the notion of the *dual projection* of an increasing process, particularly useful for applications. Another motivation for the study of the general theory of processes is provided by the study of transformations of Markov processes.

A chapter of the general theory of processes that deserves mention, because it is particularly attractive from the viewpoint of the philosophy of probability, is that of *enlargement of filtrations*, whose starting point is a theorem proven independently, in 1978, by Barlow and by Yor, and whose strongest results are due to Jeulin (1980). It can be presented as follows. Doob's fundamental idea in martingale theory had been to express mathematically the fact that players are not prophets, using the notion of *filtration.* Can we also describe mathematically what a prophet is? – needless to say, we are concerned with a mathematical abstraction, analogous to conditioning by the value of a random variable, which does not suppose that we really know this value. We describe "the universe plus the prophet" by a second filtration, bigger than the first since we have more knowledge at every moment. The theorems that we establish take therefore the following form: martingales of the small filtration (or only a few of them) become *semi*martingales in the big filtration, which we know how to decompose into "signal plus noise".

The set of all these topics – martingale inequalities, general theory, stochastic integral, enlargement – constitute what we call *Stochastic Calculus.* But the tree carries yet more branches; let us mention some. The use of martingale methods to deal with problems of narrow convergence of process laws, a subject illustrated by Aldous' long article (1978), published only in part and followed by numerous authors; the generalization of martingale convergence theorems in the form of *asymptotic martingales* or *amarts*, in discrete or continuous time (the reader wishing to pursue this question can consult Edgar-Sucheston 1992); the extension of known results on martingales to certain *multidimensional* time processes (Cairoli 1970, Cairoli-Walsh 1975). Finally, let us note the development of a "prediction theory" by F. Knight (1979, 1992), which shows the tight links uniting the most general possible theory of processes with Markov processes.

Stochastic calculus, despite its relatively abstract character, rapidly found applications. The first ones came from *electrical engineering* laboratories (transmissions of signals in the presence of noise). But the most recent and widest ones concern "financial mathematics", thus going back to the very sources of Brownian motion theory (Bachelier 1900). These mathematical problems even resuscitated, in the 1990s, branches of stochastic calculus that seemed asleep since 1970.

Inequalities of martingales and analysis. I do not pretend to deal here with all the relations between probability and analysis (harmonic analysis, Banach spaces, fractals. . .), a subject on which I lack competence.

Relations between martingales and analysis were already present in the work of Doob, who applied martingale convergence theorems to derivation theory on one hand, and to the behavior of harmonic functions at the boundary on the other. This subject remains partly open, by the way, especially with respect to its extensions to multivariable complex functions. But the most fruitful interactions start with the discovery of *Burkholder's inequalities* (1964, 1966). These inequalities establish a norm equivalency in L^p, for $p > 1$, between two random variables associated with the martingale: one is the absolute upper bound of the trajectory, and the other a quantity of Hilbertian type (the square root of a sum of squares), easy to define in discrete time, more delicate in continuous time. These equivalencies of norms have been extended by Burkholder and Gundy (1970) to spaces other than L^p, and by Davis to L^1 norms, thus initiating the theory of the H^1 space of martingales. For a synthesis of various types of inequalities, see Lenglart-Lépingle-Pratelli (1980).

On the other hand, the period around 1970 was marked in analysis by a return to direct methods of real variables – rather than abstract methods of functional analysis, or classical methods of complex analysis (inherently limited to the plane). The theory of singular integrals and the theory of $H^p(\mathbb{R}^n)$ spaces were developing very fast. Equivalencies between a "maximum" norm and various "quadratic" norms played an important role in these theories. In 1970 Stein's treatise appeared, followed by a small volume on Littlewood-Paley's theory, which called directly for martingale methods. The development of H^1 space theory (Fefferman-Stein 1972), H^p spaces for $p > 1$, the duality between H^1 and BMO (John-Nirenberg 1961, Fefferman 1971), the "atomic" approach to H^p theories (Coifman 1974), all this would have consequences and parallels in probability, under the form of H^p spaces of martingales imported by Herz in discrete time, then extended to continuous time. The H^1 space in particular took on great importance in stochastic integral theory.

Another aspect, that of Littlewood-Paley theory: under its classical form, it concerns harmonic or holomorphic functions in the unit disc, but it is also used in the half-space $\mathbb{R} \times \mathbb{R}_+$, then extends to $\mathbb{R}^n \times \mathbb{R}_+$, and at last (Stein made the step) to $E \times \mathbb{R}_+$, where E is an abstract space with a Markovian semigroup (P_t). We can therefore introduce martingale methods (Meyer 1976, Varopoulos 1980). These probabilistic methods have applications in group theory (Varopoulos), and to analysis of semigroups in infinite dimension, where they allow us to define notions which in Riemannian geometry (where the semigroup is that of Brownian motion) correspond to Ricci's curvature. This is a large subject, which we cannot take up here, but for which we will direct the reader to the bibliography of Bakry's lectures (1994).

This subject is related to *hypercontractivity*, for which on the contrary it is probability that has influenced analysis: the starting point is Nelson's (1973) probabilistic proof (through Wiener chaos), motivated by quantum field theory. Gross' famous article (1976) gave the problem its definitive status, and it is still a subject very much alive.

Martingale problems. The universality of martingales during the period we are studying translates into a new notion, that of *martingale problems*, introduced in diffusion theory by Stroock and Varadhan (1969), then used in many other areas, like that of point processes (see Jacod 1979).

Stroock and Varadhan's idea consists in characterizing the law of a stochastic process by a family of processes that we require to be martingales (eventually local martingales). In the case of diffusions (or more generally Markov processes), these processes are constructed in a simple manner from the infinitesimal generator. The unknown of the martingale problem is therefore a probability law, for which we must discuss existence and uniqueness – and for existence, it is quite natural to use a method of *narrow convergence*. Stroock and Varadhan used this method to handle the problem of diffusions with coefficients that are assumed only to be continuous, which seems to resist functional analysis methods.

The research undertaken in order to apply the method is even more important than the method itself: these tools are narrow compactness criteria using "local characteristics" of semimartingales, problems of constructing *all* martingales from a given family of martingales by means of stochastic integrals. Here again, space is insufficient to develop these themes, which have a considerable practical importance. Let us only mention articles by Jacod-Yor (1977) and Yor (1978).

"Stochastic mechanics". This stream of ideas has been relatively narrow in volume of publication, but constant up to the present time. Since the beginning of Quantum Mechanics, the Copenhagen interpretation has encountered opponents, some of whom were determinists, while others sought a classical probabilistic interpretation. We can trace the latter tendency to Schrödinger himself. The majority of physicists have ignored these moods, but Nelson (1967) made a fascinating presentation of them to probabilists: To any wave function $\psi(t, x)$ of quantum mechanics, a solution of a classical Schrödinger equation, Nelson associates a natural diffusion admitting at each moment the probability density $|\psi(t, x)|^2$, predicted by the Copenhagen interpretation. Loosely speaking, the wave function, which is complex and satisfies a linear equation, "codes" two transition functions, one forward, the other backward, which satisfy two coupled nonlinear equations. Nagasawa (1980) managed to present Nelson's equations, no longer as another interpretation of quantum mechanics, but as general model of equilibrium of "populations" of similar individuals, in which each individual interacts with the population density, that is with its own probability of presence. Thus it was becoming possible to be interested in these equations without taking sides in a theological quarrel. On the other hand, Nelson's book was incomplete from a mathematical point of view: in Nelson's two diffusions generators, the first order term explodes on "the nodal set" where the wave function becomes null, and Nelson could treat rigorously only the case where this set is empty. It is only in the works of Nelson (1985), Zheng (1985) and especially Carlen (1984) that this difficulty was resolved in a satisfying manner. Since then, the publication of books on these questions has never ceased, and we will not carry out an inventory.

Relations to physics. Yet Nelson's book had a completely different legacy, in the direction of "orthodox" quantum physics. This is a matter of Euclidian field theory, which we can try to describe this way: Despite its extraordinary practical successes, relativistic quantum field theory found itself in a state of intellectual confusion. One of the procedures considered for remedying this consisted of constructing "models" of nontrivial quantum fields satisfying a certain number of axioms that are natural from a physical point of view. Nelson (1973) showed how, by an analytic continuation, these constructions can be reduced to that of *probability measures* on a space of distributions $S'(\mathbb{R}^n)$ having a Euclidian invariance (rather than relativistic invariance) property, and a form of the Markov property. The is a subject on which we will say very little (by pure ignorance) except:

1) The method succeeds perfectly in dimension 2, where it has stimulated in an extraordinary way the meticulous study of planar Brownian motion, and especially the study of multiple points of the Brownian curve. Let us mention in passing that this is one of the subjects to which Dynkin devoted himself after his departure from Russia in 1976.

2) The problems related to quantum field theory have also motivated much research on measure construction in infinite dimensions.

Let us mention books by Simon (1974[9], 1979), and by Glimm and Jaffe (1981).

After 1980

As with Markov processes, stochastic calculus also began to fade, following the same scheme: the trunk does not continue to develop, a few branches stay very alive. For stochastic calculus in particular, we see it descend from the sky after 1980, reflect itself in remarkable textbooks, become a concrete working tool that allows us to continue the work of Paul Lévy and calculate countless laws of processes. The general direction being less clear, I will try to cite a few important directions, in which I myself became more or less seriously interested.

The "Malliavin calculus". The probabilistic theory of diffusions has always appealed to theorems borrowed from the theory of partial differential equations, which permit us to assert that the transition function of a diffusion

[9] The introduction of this book contains one of the most beautiful tributes paid to probability by a non-probabilist.

with a given generator has a sufficiently regular density. This was known in the case of elliptic generators, and also in some degenerate cases (hypoelliptic), thanks to a famous theorem of Hörmander. This did not lack concrete applications, in the study of stochastic control problems for example.

In 1976, P. Malliavin presented at a conference in Kyoto a *purely probabilistic* method to establish the existence of very regular densities for solutions of some stochastic differential equations with coefficients C^∞. This work was of extraordinary originality, and it took probabilists many years to understand it. Before talking about its content, we must say a word about the "tradition" in which this article appeared, at the junction of many streams about which we have said nothing so far.

Malliavin was known foremost as an analyst and a geometer, but in probability he was self-taught, educated by reading Itô's and Doob's books. He became interested in problems of vanishing cohomology – i.e., the nonexistence of nontrivial harmonic functions of certain degrees on certain compact manifolds. Now the nonexistence of nontrivial harmonic *functions* is related to the asymptotic behavior of Brownian motion on the manifold. Can we do the same for forms? This question had been taken up by Bochner and Yano. From a probabilistic point of view, the problem is related to two others: the construction of Brownian motion (i.e., construction of the diffusion whose generator is the Laplace-Beltrami operator), and how we can "follow" a differential form along Brownian motion trajectories. This takes us into the vast field of *stochastic differential geometry.*

Starting in 1963, Itô had studied the parallel transport of vectors along Brownian motion trajectories, a problem taken up again by Dynkin (1968). The generator of this operation on forms is nevertheless not the most interesting Laplacian (de Rham), but another Laplacian called *horizontal*, which differs from the former by a first-order term. After Itô and Dynkin, we can mention the works of the English school (Eells, Elworthy), then those of Malliavin himself. One of the results of these efforts was a probabilistic construction of Brownian motion on Riemannian manifolds by stochastic differential equations, without recollement, by lifting to the frame bundle. Malliavin had all these techniques, unknown to the majority of probabilists.

Second ingredient, the stochastic "calculus of variations", that is the variation of solutions of the equation as a function of the initial conditions. Here again, on this widely studied question, Malliavin brought a new tool (although it is found in part in a little known 1961 article of Blagoveschenskii-Freidlin): an Itô stochastic differential equation with C^∞ coefficients on $\mathbb{R}^n$ defines a "flow of C^∞ diffeomorphisms" on $\mathbb{R}^n$. That was sure to generate plenty of work on stochastic flows.

Third ingredient, Wiener's (1938), or rather Itô's (1951) use of chaos expansion to introduce a Laplacian in infinite dimensions, the *Ornstein-Uhlenbeck Laplacian,* which is a self-adjoint operator relative to Wiener measure, and according to which Malliavin defines *Sobolev spaces in infinite dimensions*, his principal tool being an integration by parts formula for Wiener measure. Here again, he had forerunners: the L^2 space of the Brownian measure is isomorphic, when we use the chaos expansion, to the physicists' *Fock space* (Segal 1956), one of the basic objects of quantum field theory, and the idea of defining weakly differentiable functions and Sobolev spaces on Fock space had been widely studied outside of probability (on this subject, there were rich works by P. Krée in particular). But this had all remained rather abstract, whereas Malliavin made a very efficient tool out of it. As to the way Malliavin put these various elements together to establish Hörmander's theorem, it was simultaneously a mathematical tour de force and, for the probabilistic public, a shower of novelties to assimilate.

In these conditions, it is fair to mention the work of Stroock (1981), who (aside from his own original contributions) put all this within reach of probabilists. The 1980 Durham colloquium with its introduction by Williams also played a big role in the diffusion of the "Malliavin calculus".

Among developments that followed, we will mention only (for lack of space) the work of Bismut (1981), who modified and completed Malliavin's tools, established the complete form of Hörmander's theorem, extended it to diffusions with boundary conditions, carried out a fusion of Malliavin's calculus and large deviations methods – and especially, found a new outlet for them with his probabilistic proof of the Atiyah-Singer *index theorem.* But the strongest influence of the "Malliavin calculus" on probability properly speaking probably comes from a relatively secondary aspect of his technique: the use of Wiener chaos and the Ornstein-Uhlenbeck process on Wiener space. That has produced wide interest in analysis in infinite dimensions, coming back to certain concerns of theoretical physics.

Perhaps we should also mention an entirely different probabilistic approach to the existence of densities: that of Krylov (1973). Here there are deep results that remain isolated.

Stochastic differential geometry. This subject is prior to the "Malliavin calculus", but it profited from its growth. Here is a sample of problems dealt with during this period: how can we read the local geometry of a Riemannian manifold from the behavior of its Brownian motion over short periods of time? Or conversely, its global behavior from the asymptotic behavior of the Brownian? What is the behavior of trajectories of a process whose generator is the sum of a first-order term and a small second-order term? (This problem is related to the quasi-classical approximation of quantum mechanics when Planck's constant "approaches zero", and to large deviation problems.)

Another aspect of stochastic geometry, the study of *semimartingales on manifolds*, inaugurated by L. Schwartz (1980), and resting on the fact that the class of semimartingales is invariant under class C^2 applications. It is possible in particular to define *continuous martingales* with values in the manifolds (and of which the Brownian motion of a Riemannian manifold is an example). Here again, we meet an extension of the relation between Brownian motion and harmonic functions, in which the Brownian motion takes place in a curved space, and the harmonic functions become harmonic maps between Riemannian manifolds. For lack of space, let us refer to Emery's book (1989).

Distributions and white noise. The popularity of the "Malliavin calculus" brought into mainstream probability a subject that had diverged from it quite early. This is the subject of *distributions in infinite dimensions*, whose history will force us to go back.

First, there are the ideas of Gelfand (1955) and Minlos (1958) on random distributions, according to which the most natural way to consider the trajectories of a stochastic process is to regard them as distributions. The law of the stochastic process is then a measure on a space of distributions – and the main spaces of distributions being nuclear, they possess excellent properties from a measure theoretical point of view. We go from there to the study of a particularly interesting random distribution, that of *white noise*, which is the derivative of Brownian motion in the sense of distributions, developed starting in 1967 by Hida (see his 1980 book). Here the essential point is the expansion of functionals of Brownian motion into Wiener chaos, and the definition of classes of *generalized functionals* by non-convergent expansions. Hida's distributions are of interest to physicists, because they provide a rigorous way to understand the analogies between Brownian motion and the Feynman integral, the latter appearing as a distribution on Wiener space. All this has produced a stream of sustained publication, but a little on the fringe of the main streams of probability.

The "Malliavin calculus" renewed interest in these problems by introducing a whole family of Sobolev spaces of differentiable functionals, whose duals are quite naturally distribution spaces. This point of view is due primarily to Ikeda and Watanabe. We will not go into details here, but "Wiener analysis" is currently a flourishing branch.

Large deviations. I will do nothing but cite Shilder (1966) and the fundamental work of Donsker-Varadhan (1976). This subject deserves to be treated separately.

Noncommutative probability. The axioms of Quantum Mechanics developed by von Neumann in 1932 (thus two years before Kolmogorov's axioms!) were in fact – if we exclude the problem of quantification of classical mechanics – probability axioms, where random variables are called self-adjoint operators, probability laws are called positive self-adjoint operators of unit trace, etc. Later, there appeared the possibility of addressing probability in a more general setting, C^*-algebras for example. Quite naturally, one sought to develop a noncommutative measure theory, at least in the relatively simple case of a tracial law (Segal 1953, Nelson 1974). Again quite naturally, one posed problems of probabilistic nature, like the validity of the martingale convergence theorem in von Neumann algebras. But the absence of interesting examples, and the impossibility of defining conditioning in sufficient generality, left this stream of research marginal for a long time – among probabilists, because physicists needed models of *quantum noise.* It is just that mathematicians are hardly interested in anything but fundamental physics, whereas here it is rather a matter of applied physics (quantum optics), and so the fields of research remained nearly disjoint. We can point to Cushen and Hudson's (1971) definition of a noncommutative Brownian motion, and to Accardi, Frigero and Lewis' (1982) article on the general definition of noncommutative stochastic processes. A good reference for this period is Davies' book (1976).

Yet the situation changed completely with the development of a noncommutative form of stochastic calculus, with Streater's and Barnett-Streater-Wilde's (1983) articles on fermionic Brownian motion and the corresponding theory of the stochastic integral, and especially Hudson and Parthasarathy's (1984) article on bosonic Brownian motion. Independently of the value of this article, the reason it had so much impact is that it is accessible: unlike the others, it does not require a heavy background in functional analysis, and it connects much more directly to the classical Itô calculus and to the theory of Wiener chaos.

We will not comment further on this recent trend, except to mention that it had "repercussions" in classical probability, by raising beautiful problems about martingales that give rise to a chaotic representation ("Azema's martingales" for example). A good reference is Parthasarathy's (1992) book.

Omissions. The theory of stochastic processes is not all of probability, and I am far from having taken up all aspects of stochastic process theory, or even Markov process theory or martingale theory. I had to omit not only works on which I was poorly informed, but also works I know well and I admire. I hope the reader has taken pleasure in the preceding account, and I ask him to be indulgent.

References

The citations are limited to the minimum needed to locate them: the name of the author (without initials if there is no risk of confusion), the name of the journal, the year and the volume (without pages). The titles are sometimes shortened, and the following abbreviations are used: MP, PM (Markov process); MB, BM (Brownian motion); SG (semigroup); SP (stochastic process); EDS, SDE (stochastic differential equation), PDE (partial differential equation). For the journals:[10] *LN (Lecture Notes in Math.), SP (Séminaire de Probabilités), ZW (Zeitschrift für W-Theorie) and its successor PT*

[10]There are a few exceptions, for typographical reasons.

(Probability Theory and Related Fields), TV (Teoriia Verojatnostei), CRAS (Comptes Rendus de Paris), TAMS (Transactions AMS), JFA (J. of Functional Analysis).

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