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A NONCOMMUTATIVE UNITON THEORY

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Abstract. Harmonic two-spheres in the unitary group may be constructed and described in terms of unitons. We present an analogue of this theory for those solutions of the noncommutative $\mathrm{U}(1)$ sigma-model that may be represented as finite-dimensional perturbations of zero-energy solutions. In particular, we establish that the energy of every such solution is an integer multiple of 8π, describe all solutions of small energy and give many explicit examples of non-Grassmannian solutions.

1. Introduction

This paper is an extended version of the author's talk at the XXV workshop on geometric methods in physics (Bialowieza, Poland, 02–08 July 2006). We start by briefly recalling some aspects of the theory of harmonic maps from the two-dimenisonal sphere to the unitary group. Then we consider a noncommutative analogue of this sigma model and state the main results of the uniton theory. As an application, we describe all solutions of small energy and give explicit examples (apparently absent in the existing literature) of non-Grassmannian solutions of any admissible energy. Complete proofs will be given elsewhere.

2. Harmonic Maps From $\mathbb{S}^2$ to $\mathrm{U}(n)$

Consider the energy functional

$$E(\varphi) = \frac{1}{2}\int_{\mathbb{C}} |\varphi^{-1}\mathrm{d}\varphi|^2\,\mathrm{d}x\mathrm{d}y = 2\int_{\mathbb{C}} |\varphi^{-1}\overline{\partial}\varphi|^2\,\mathrm{d}x\mathrm{d}y \tag{1}$$

on the set of all smooth maps $\varphi : \mathbb{C} \to \mathrm{U}(n)$. Here $\overline{\partial} = (\partial_x + \mathrm{i}\partial_y)/2$ is the derivative with respect to $\overline{z}$, and $|A|^2 = \mathrm{tr}(AA^*)$ for any matrix A. The critical points of $E(\varphi)$ are called *harmonic maps from* $\mathbb{S}^2$ *to* $\mathrm{U}(n)$. They coincide with finite-energy solutions of the Euler–Lagrange equations $\partial(\varphi^{-1}\overline{\partial}\varphi) + \overline{\partial}(\varphi^{-1}\partial\varphi) = 0$,