JGSP **12** (2008) 97–100

JOURNAL OF
Geometry and Symmetry
in Physics

BOOK REVIEW

Compactifications of Symmetric and Locally Symmetric Spaces, by Armand Borel and Lizhen Ji, Birkhäuser, Boston, 2006, xx + 479pp., ISBN 100-8176-3247-6.

The book under review is a survey on the compactifications of the symmetric spaces and locally symmetric spaces, analyzing their properties, relationships and providing uniform methods for their construction. The basic ideas are extracted, formulated clearly and illustrated on various examples. The first part is devoted to the compactifications $\overline{X}$ of the symmetric spaces X, containing X as a dense open subset. The second one provides smooth compact manifolds M, containing finite disjoint unions of symmetric spaces, embedded as open but not dense subsets. The closure of each symmetric space in M is a manifold with corners. The third part deals with the compactifications of the locally symmetric spaces, their metric and spectral properties. The second part is mostly written by Armand Borel before his death on August 11, 2003. The rest of the book is worked out by Lizhen Ji.

Chapter 1 recalls the original motivations, constructions, properties and applications of the geodesic compactification $X \cup X(\infty)$; Karpelevič compactification $\overline{X}^K$; Satake compactifications $\overline{X}^S_\tau$, associated with certain irreducible projective representations $\tau : G \to P\mathrm{SL}(n, \mathbb{C})$ of the isometry group G of X; Baily-Borel compactification $\overline{X}^{BB}$; Frustenberg compactifications $\overline{X}^F_I$, corresponding to subsets I of simple roots of a fixed minimal parabolic subgroup $P_o \subset G$ with respect to its split component A_{P_o} and Martin compactifications $X \cup \partial_\lambda X$ for $\lambda \leq \lambda_o(X)$, where $\lambda_o(X)$ stands for the bottom of the spectrum of the Laplacian Δ on X.

The second chapter provides a uniform construction of the aforementioned compactifications. It is intrinsic in the sense that does not use closures under embeddings in compact spaces. This approach relates the compactifications of the symmetric spaces X with the compactifications of their corresponding locally symmetric spaces $\Gamma \setminus X$. For a suitable collection $\mathcal{C}$ of parabolic subgroups $P \subset G$, which is invariant under the adjoint action of G, one defines $\overline{X} = X \cup \coprod_{P \in \mathcal{C}} e(P)$ with boundary components $e(P)$, depending on the horospherical decomposition of X with respect to P or its refinements. The topology of $\overline{X}$ is given by the convergence of sequences from X or $e(P)$ to points from their boundaries. That allows explicit description of neighborhoods of the boundary points. Various kinds of Siegel sets are introduced and shown to have separation properties. That enables to prove that $\overline{X}$ is a compact Hausdorff space with continuous G-action.

The uniform method is illustrated on the construction of the aforementioned compactifications. In addition are considered the subgroup compactification $\overline{X}^{\text{sb}}$, the subalgebra compactification $\overline{X}^{\text{sba}}$ and Gromov compactification $\overline{X}^{G}$. Baily-Borel compactification $\overline{X}^{BB}$ of a Hermitian symmetric space of non-compact type, as well as Satake compactifications $\overline{X}^{S}_{\tau}$ and Martin compactifications $X \cup \partial_{\lambda} X$ are proved to be homeomorphic to a ball.

The second part constructs smooth compact manifolds, containing finitely many semi-simple symmetric spaces as open but not dense subsets. For an arbitrary manifold M with corners, whose boundary hypersurfaces admit a partition on N disjoint subsets, there exists a closed manifold $\widetilde{M}$, obtained by gluing 2^N copies of M along some boundary hypersurfaces. Any Lie group, acting on M extends to $\widetilde{M}$. The group $\mathbb{Z}_2^N$ acts on $\widetilde{M}$ and the quotient $\widetilde{M}/\mathbb{Z}_2^N \simeq M$. Oshima's compactification $\overline{X}^{O}$ is obtained from $\overline{X}^{S}_{\max}$ in the aforementioned way. For an arbitrary connected complex semi-simple group $\mathbf{G}$ of adjoint type and the fixed point set $\mathbf{H} = \mathbf{G}^{\sigma}$ of an involution $\sigma : \mathbf{G} \to \mathbf{G}$, De Concini-Procesi have constructed the wonderful compactification $\overline{\mathbf{X}}^{w}$ of $\mathbf{X} = \mathbf{G}/\mathbf{H}$, which is a smooth projective manifold, acted morphically by $\mathbf{G}$ and containing $\mathbf{X}$ as an open, Zariski dense subset. The real point set $\overline{\mathbf{X}}^{w}(\mathbb{R})$ of De Concini-Procesi compactification is a finite quotient of Oshima-Sekiguchi compactification $\overline{X}^{OS}$

The last part is on the compactifications of the locally symmetric spaces. Making use of the rational Langlands and horospherical decompositions, the book constructs the Satake compactifications $\overline{\Gamma \setminus X}^{S}_{\tau}$, the Borel-Serre compactification $\overline{\Gamma \setminus X}^{BS}$, the reductive Borel-Serre compactification $\overline{\Gamma \setminus X}^{RBS}$. In the special case of a Hermitian locally symmetric space $\Gamma \setminus X$ are considered Baily-Borel compactification by a normal projective variety $\overline{\Gamma \setminus X}^{BB}$ and the toroidal compactifications $\overline{\Gamma \setminus X}^{\text{tor}}_{\Sigma}$, resolving the singularities of $\overline{\Gamma \setminus X}^{BB}$ along its boundary. The toroidal compactifications are associated with the Γ-admissible families Σ of polyhedral decompositions of appropriate symmetric cones in the centers $U_{\mathbf{P}}$ of of the real loci $\mathbf{N}_{\mathbf{P}}(\mathbb{R})$ of the unipotent radicals $\mathbf{N}_{\mathbf{P}}$ of the rational parabolic subgroups $\mathbf{P}$.

The uniform construction of the compactifications of the locally symmetric spaces $\Gamma \setminus X$ is an original result of the authors. It starts by a partial (i.e., non-compact)

Hausdorff compactification ${}_{\mathbb{Q}}\overline{X}$ of X. While the boundary components of the compactifications $\overline{X}$ were labeled by real parabolic subgroups $P \subset G$, the boundary components of ${}_{\mathbb{Q}}\overline{X}$ correspond to proper rational parabolic subgroups $\mathbf{P}$ of the linear algebraic groups $\mathbf{G} \subset \mathrm{GL}(n, \mathbb{C})$ with real loci $\mathbf{G}(\mathbb{R}) = G$. One chooses ${}_{\mathbb{Q}}\overline{X} = X \cup \coprod_{\mathbf{P}} e(\mathbf{P})$ for some boundary components $e(\mathbf{P})$, depending on the rational Langlands or horospherical decompositions. If the Γ-action on X extends continuously to ${}_{\mathbb{Q}}\overline{X}$ and the quotient $\Gamma \setminus {}_{\mathbb{Q}}\overline{X}$ is a compact Hausdorff space, then it is a compactification of $\Gamma \setminus X$. The topology of ${}_{\mathbb{Q}}\overline{X}$ is defined in terms of converging sequences. That enables to provide explicit neighborhoods of the boundary points in ${}_{\mathbb{Q}}\overline{X}$ and $\Gamma \setminus {}_{\mathbb{Q}}\overline{X}$. The reduction theory of arithmetic groups is used for proving that the quotient $\Gamma \setminus {}_{\mathbb{Q}}\overline{X}$ is Hausdorff and compact.

By the means of the uniform method are constructed Borel-Serre compactification $\overline{\Gamma \setminus X}^{BS}$, the reductive Borel-Serre compactification $\overline{\Gamma \setminus X}^{RBS}$, the maximal Satake compactification $\overline{\Gamma \setminus X}^{S}_{\max}$ and Tits compactification $\overline{\Gamma \setminus X}^{T}$.

Some of the compactifications of the locally symmetric spaces $\Gamma \setminus X = \Gamma \setminus G/K$ can be obtained as K-quotients of appropriate compactifications of the homogeneous spaces $\Gamma \setminus G$. The corresponding partial compactifications ${}_{\mathbb{Q}}\overline{G}$ of G are constructed by the means of the horospherical decompositions of G with respect to the rational parabolic subgroups $\mathbf{P} \subset \mathbf{G}$. This procedure is applied to $\overline{\Gamma \setminus G}^{BS} = \Gamma \setminus {}_{\mathbb{Q}}\overline{G}^{BS}$ and $\overline{\Gamma \setminus G}^{RBS} = \Gamma \setminus {}_{\mathbb{Q}}\overline{G}^{RBS}$. If $\mathrm{rk}_{\mathbb{Q}}(\mathbf{G}) = r$ then an appropriate gluing of 2^r copies of $\overline{\Gamma \setminus X}^{BS}$ produces the Borel-Serre-Oshima compactification $\overline{\Gamma \setminus X}^{BSO}$. In the case of an arithmetic maximal discrete subgroup $\Gamma \subset G$, one can define the subgroup compactifications $\overline{\Gamma \setminus G}^{\mathrm{sb}}$ and $\overline{\Gamma \setminus X}^{\mathrm{sb}}$. Identifying $\mathrm{SL}(n, \mathbb{Z}) \setminus \mathrm{SL}(n, \mathbb{R})$ with the space of the unimodular lattices in $\mathbb{R}^n$, one obtains its lattice compactification, sublattice compactification and flag compactification.

The last four sections discuss some metric properties of the compactifications of $\Gamma \setminus X$. After introducing the notions of hyperbolic compactification and hyperbolic reduction of a compactification, $\overline{\Gamma \setminus X}^{RBS}$ is shown to be the hyperbolic reduction of $\overline{\Gamma \setminus X}^{BS}$ and $\overline{\Gamma \setminus X}^{BB}$ is proved to be the hyperbolic reduction of $\overline{\Gamma \setminus X}^{\mathrm{tor}}_{\Sigma}$ of a Hermitian locally symmetric space $\Gamma \setminus X$. As a generalization of the hyperbolic compactifications are considered the asymptotic compactifications. The book ends with a discussion on the relationships among the continuous spectrum of the Laplacian on a locally symmetric space $\Gamma \setminus X$, its local boundary symmetric spaces and scattering geodesics.

The book can be of great use for researchers, graduate students and anyone, who is interested in the geometric and topological properties of the compactifications of the symmetric and locally symmetric spaces. The classical results of the subject are exposed comprehensively, emphasizing on their basic ideas. It is easier and more efficient to study them by the book of Borel and Ji, instead of following the original publications. Together with the basics, the book provides a lot of details, revealing the specific features of the techniques. Systematizing a lot of compactification procedures by dwelling on their common background and individual diversity, the monograph paves the way for further original achievements in the area.

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