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MOTION OF CHARGED PARTICLES FROM THE GEOMETRIC VIEW POINT

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Abstract. This is a review article on the motion of charged particles related to the author's study. The equation of motion of a charged particle is defined as a curve satisfying a certain differential equation of second order in a semi-Riemannian manifold furnished with a closed two-form. Charged particle is a generalization of geodesic. We shall oversee the geometric aspect of charged particles.

1. Introduction

Let F be a closed two-form and U a function on a connected semi-Riemannian manifold $(M, \langle\,,\rangle)$, where $\langle\,,\rangle$ is a semi-Riemannian metric on M. We denote by $\bigwedge^m(M)$ the space of m-forms on M. Denote by $\iota(X) : \bigwedge^m(M) \to \bigwedge^{m-1}(M)$ the interior product operator induced from a vector field X on M, and by $\mathcal{L} : T(M) \to T^*(M)$, the Legendre transformation from the tangent bundle $T(M)$ of M onto the cotangent bundle $T^*(M)$, which is defined by

$$\mathcal{L} : T(M) \to T^*(M), \quad u \mapsto \mathcal{L}(u), \quad \mathcal{L}(u)(v) = \langle u, v\rangle, \quad u, v \in T(M). \tag{1}$$

A curve $x(t)$ in M is called the *motion of a charged particle under electromagnetic field F and potential energy U*, if it satisfies the following second order differential equation

$$\nabla_{\dot{x}}\dot{x} = -\mathrm{grad}U - \mathcal{L}^{-1}(\iota(\dot{x})F) \tag{2}$$

where ∇ is the Levi-Civita connection of M. Here $\nabla_{\dot{x}}\dot{x}$ means the acceleration of the charged particle. Since $-\mathcal{L}^{-1}(\iota(\dot{x})F)$ is perpendicular to the direction $\dot{x}$ of the movement, $-\mathcal{L}^{-1}(\iota(\dot{x})F)$ means the Lorentz force. This equation originated in the theory of general relativity (see § 2 or [26]). When $F = 0$ and $U = 0$, then $x(t)$ is merely a geodesic. When M is a Kähler manifold with a complex structure J, then it is natural to take a scalar multiple of the Kähler form Ω defined