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TOPOLOGY, GEOMETRY AND PHYSICS: BACKGROUND FOR THE WITTEN CONJECTURE I.

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Abstract. The profound, beautiful and, at times, rather mysterious symbiosis between mathematics and physics has always been a source of wonder, but, in the past twenty years, the intensity of the mutual interaction between these two has become nothing short of startling. Our objective here is to provide an introduction, in terms as elementary as possible, to one small aspect of this relationship. Toward this end we shall tell a story. Although we make no attempt to relate it chronologically, the story can be said to begin with the efforts of Yang and Mills to construct a nonabelian generalization of classical electromagnetic theory, and to culminate in a remarkable conjecture of Witten concerning the Donaldson invariants of a smooth four-manifold.

1. Instantons and four-Manifolds

The central characters in our story are all "classical gauge theories" and we will eventually introduce them in some generality (Section 3), but we would like to begin by getting to know a few of them personally. For this we first recall the construction of the **quaternionic Hopf bundle**

$$Sp\,(1) \hookrightarrow \mathbb{S}^7 \xrightarrow{\ \pi\ } \mathbb{HP}^1. \tag{1.1}$$

Here $Sp(1)$ is the Lie group of unit quaternions (those $g \in \mathbb{H}$ satisfying $|g| = 1$). As a manifold it is diffeomorphic to $\mathbb{S}^3$, but it is also isomorphic to the Lie group $SU(2)$ of 2×2 complex matrices U that are unitary ($U^{-1} = \bar{U}^T$) and satisfy $\det U = 1$. Indeed, every such U can be written in the form $U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$, where $\alpha, \beta \in \mathbb{C}$ satisfy $|\alpha|^2 + |\beta|^2 = 1$ and the map

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \longrightarrow \alpha + \beta \mathrm{j} &= \alpha^1 + \alpha^2 \mathrm{i} + \left(\beta^1 + \beta^2 \mathrm{i}\right) \mathrm{j} \\ &= \alpha^1 + \alpha^2 \mathrm{i} + \beta^1 \mathrm{j} + \beta^3 \mathrm{k} \end{aligned} \tag{1.2}$$