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THE UNIVERSAL KEPLER PROBLEM

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Abstract. For each simple euclidean Jordan algebra V, we introduce the analogue of hamiltonian, angular momentum and Laplace-Runge-Lenz vector in the Kepler problem. Being referred to as the universal hamiltonian, universal angular momentum and universal Laplace-Runge-Lenz vector respectively, they are elements in (essentially) the TKK (Tits-Kantor-Koecher) algebra of V and satisfy commutation relations similar to the ones for the hamiltonian, angular momentum and Laplace-Runge-Lenz vector in the Kepler problem. We also give some examples of Poisson realization of the TKK algebra, along with the resulting classical generalized Kepler problems. For the simplest simple euclidean Jordan algebra (i.e., $\mathbb{R}$), we give examples of operator realization for the TKK algebra, along with the resulting quantum generalized Kepler problems.

1. Introduction

Recall that, in the Kepler problem, the hamiltonian is

$$\mathrm{H} = \frac{1}{2}\mathbf{p}^2 - \frac{1}{r}\cdot \tag{1}$$

Here, r is the length of $\mathbf{r} \in \mathbb{R}^3_* := \mathbb{R}^3 \setminus \{\mathbf{0}\}$ and $\mathbf{p}$ is the (linear) momentum.

The hamiltonian H is clearly invariant under rotations of $\mathbb{R}^3$, thanks to Noether's theorem, the angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{2}$$

is conserved.

What is special about the Kepler problem is the existence of an additional conserved quantity, i.e., the Laplace-Runge-Lenz vector

$$\mathbf{A} = \mathbf{L} \times \mathbf{p} + \frac{\mathbf{r}}{r}\cdot \tag{3}$$

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