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ON EXACTNESS OF LONG SEQUENCES OF HOMOLOGY SEMIMODULES

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(*communicated by George Janelidze*)

Abstract

We investigate exactness of long sequences of homology semimodules associated to Schreier short exact sequences of chain complexes of semimodules.

In [4], to define homology and cohomology monoids of presimplicial semimodules, we introduced a chain complex of semimodules (in particular, abelian monoids), its homology semimodules, and a $\pm$-morphism between chain complexes of semimodules. Next, in [5], we introduced a morphism between chain complexes of cancellative semimodules, defined a chain homotopy of morphisms and studied its basic properties. In this paper we investigate exactness of long sequences of homology semimodules associated to Schreier short exact sequences of chain complexes of semimodules.

The paper is divided into two sections. Section 1 contains the preliminaries. The main results are presented in Section 2.

1. Preliminaries

Recall [1] that a semiring $\Lambda = (\Lambda, +, 0, \cdot, 1)$ is an algebraic structure in which $(\Lambda, +, 0)$ is an abelian monoid, $(\Lambda, \cdot, 1)$ a monoid, and

$$\begin{aligned} \lambda \cdot (\lambda' + \lambda'') &= \lambda \cdot \lambda' + \lambda \cdot \lambda'', \\ (\lambda' + \lambda'') \cdot \lambda &= \lambda' \cdot \lambda + \lambda'' \cdot \lambda, \\ \lambda \cdot 0 &= 0 \cdot \lambda = 0, \end{aligned}$$

for all $\lambda, \lambda', \lambda'' \in \Lambda$. An abelian monoid $A = (A, +, 0)$ together with a map $\Lambda \times A \longrightarrow A$, written as $(\lambda, a) \mapsto \lambda a$, is called a (left) Λ-semimodule if

$$\begin{aligned} \lambda(a + a') &= \lambda a + \lambda a', \\ (\lambda + \lambda')a &= \lambda a + \lambda' a, \\ (\lambda \cdot \lambda')a &= \lambda(\lambda' a), \\ 1a = a, &\quad 0a = 0, \end{aligned}$$

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for all $\lambda, \lambda' \in \Lambda$ and $a, a' \in A$. It immediately follows that $\lambda 0 = 0$ for any $\lambda \in \Lambda$.

A map $f : A \longrightarrow B$ between Λ-semimodules A and B is called a Λ-homomorphism if $f(a + a') = f(a) + f(a')$ and $f(\lambda a) = \lambda f(a)$, for all $a, a' \in A$ and $\lambda \in \Lambda$. It is obvious that any Λ-homomorphism carries 0 into 0. A Λ-subsemimodule A of a Λ-semimodule B is a subsemigroup of $(B, +)$ such that $\lambda a \in A$ for all $a \in A$ and $\lambda \in \Lambda$. Clearly $0 \in A$. The quotient Λ-semimodule B/A is defined as the quotient Λ-semimodule of B by the smallest congruence on the Λ-semimodule B some class of which contains A. Denote the congruence class of $b \in B$ by $[b]$. Then $[b_1] = [b_2]$ if and only if $a_1 + b_1 = a_2 + b_2$ for some $a_1, a_2 \in A$.

Let N be the semiring of nonnegative integers. An N-semimodule A is simply an abelian monoid, and an N-homomorphism $f : A \longrightarrow B$ is just a homomorphism of abelian monoids, and A is an N-subsemimodule of an N-semimodule B if and only if A is a submonoid of the monoid $(B, +, 0)$.

Next recall that the group completion of an abelian monoid M can be constructed in the following way. Define an equivalence relation $\sim$ on $M \times M$ as follows:

$$(u, v) \sim (x, y) \Leftrightarrow u + y + z = v + x + z \quad \text{for some} \quad z \in M.$$

Let $[u, v]$ denote the equivalence class of (u, v). The quotient set $(M \times M)/\!\sim$ with the addition $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$ is an abelian group ($0 = [x, x]$, $-[x, y] = [y, x]$). This group, denoted by $K(M)$, is the group completion of M, and $k_M : M \longrightarrow K(M)$ defined by $k_M(x) = [x, 0]$ is the canonical homomorphism. If M is a semiring, then the multiplication $[x_1, y_1] \cdot [x_2, y_2] = [x_1x_2 + y_1y_2, x_1y_2 + y_1x_2]$ converts $K(M)$ into the ring completion of the semiring M, and k_M into the canonical semiring homomorphism. Now assume that A is a Λ-semimodule. Then $K(A, +, 0)$ with the multiplication $[\lambda_1, \lambda_2][a_1, a_2] = [\lambda_1 a_1 + \lambda_2 a_2, \lambda_1 a_2 + \lambda_2 a_1]$, $\lambda_1, \lambda_2 \in \Lambda$, $a_1, a_2 \in A$, becomes a $K(\Lambda)$-module. This $K(\Lambda)$-module, denoted by $K(A)$, is the $K(\Lambda)$-module completion of the Λ-semimodule A, and $k_A = k_{(A,+,0)}$ is the canonical Λ-homomorphism. Clearly, $K(A)$ is in fact an additive functor: for any homomorphism $f : A \longrightarrow B$ of Λ-semimodules, $K(f) : K(A) \longrightarrow K(B)$ defined by $K(f)([a_1, a_2]) = [f(a_1), f(a_2)]$ is a $K(\Lambda)$-homomorphism.

A Λ-semimodule A is said to be cancellative if whenever $a + a' = a + a''$, $a, a'a'' \in A$, one has $a' = a''$. Obviously, A is cancellative if and only if the canonical Λ-homomorphism $k_A : A \longrightarrow K(A)$ is injective. Also note that A is a cancellative Λ-semimodule if and only if A is a cancellative $C(\Lambda)$-semimodule, where $C(\Lambda)$ denotes the largest additively cancellative homomorphic image of the semiring Λ. ($C(\Lambda) = \Lambda/\!\sim$, $\lambda_1 \sim \lambda_2$, $\lambda_1, \lambda_2 \in \Lambda \Leftrightarrow \lambda + \lambda_1 = \lambda + \lambda_2$, $\lambda \in \Lambda$. $\mathrm{cl}_\sim(\lambda_1) + \mathrm{cl}_\sim(\lambda_2) = \mathrm{cl}_\sim(\lambda_1 + \lambda_2)$, $\mathrm{cl}_\sim(\lambda_1) \cdot \mathrm{cl}_\sim(\lambda_2) = \mathrm{cl}_\sim(\lambda_1 \cdot \lambda_2)$.)

A Λ-semimodule A is called a Λ-module if $(A, +, 0)$ is an abelian group. One can easily see that A is a Λ-module if and only if A is a $K(\Lambda)$-module. Consequently, if A is a Λ-module, then $K(A) = A$ and $k_A = 1_A$.

1.1. Definition ([7, 2, 8, 3]). A sequence $E : A \rightarrowtail^{\lambda} B \xrightarrow{\tau} C$ of Λ-semimodules and Λ-homomorphisms is called a Schreier extension of A by C (some authors would say "C by A") if the following conditions hold:

1. λ is injective, τ is surjective, and $\lambda(A) = \mathrm{Ker}(\tau)$.

2. For any $c \in C$, $\tau^{-1}(c)$ contains an element u_c such that for any $b \in \tau^{-1}(c)$ there exists a unique element $a \in A$ with $b = \lambda(a) + u_c$.

The elements u_c, $c \in C$, are called representatives of the extension $E : A \rightarrowtail^{\lambda} B \xrightarrow{\tau} C$.

The following four properties of Schreier extensions of Λ-semimodules are easy to verify.

1.2. Let $E : G \rightarrowtail B \twoheadrightarrow C$ be a Schreier extension with G a Λ-module. Then any $b \in B$ is a representative of the extension E.

1.3. Let $E : A \xrightarrow{\lambda} B \xrightarrow{\tau} C$ be a Schreier extension with A a cancellative Λ-semimodule. If $\lambda(a) + b_1 = \lambda(a) + b_2$, $a \in A$, $b_1, b_2 \in B$, then $b_1 = b_2$.

1.4. If $E : A \rightarrowtail B \twoheadrightarrow C$ is a Schreier extension of Λ-semimodules, then B is cancellative if and only if A and C are both cancellative.

1.5. If $E : A \xrightarrow{\lambda} B \xrightarrow{\tau} C$ is a Schreier extension of Λ-semimodules, then $K(E) : 0 \to K(A) \xrightarrow{K(\lambda)} K(B) \xrightarrow{K(\tau)} K(C) \to 0$ is a short exact sequence of $K(\Lambda)$-modules.

1.6. A homomorphism $\varphi : A \longrightarrow B$ of Λ-semimodules is said to be *normal* (or kernel-regular in the sense of [9]) if whenever $\varphi(a_1) = \varphi(a_2)$, $a_1, a_2 \in A$, one has $\kappa_1 + a_1 = \kappa_2 + a_2$ for some $\kappa_1, \kappa_2 \in \mathrm{Ker}(\varphi)$. It is easy to see that φ is normal if and only if $\varphi : A \longrightarrow \varphi(A)$ is a cokernel of the inclusion $\mathrm{Ker}(\varphi) \hookrightarrow A$ (i.e., $\varphi : A \longrightarrow \varphi(A)$ is a normal Λ-epimorphism).

1.7. Any Λ-homomorphism $\varphi : G \longrightarrow B$ with G a Λ-module is evidently normal. Moreover, any Λ-homomorphism $\varphi : A \longrightarrow B$ with $\varphi(A)$ a Λ-module is normal. Consequently, if a sequence of Λ-semimodules and Λ-homomorphisms $A \xrightarrow{\alpha} G \xrightarrow{\beta} B$ with G a Λ-module is exact, then α and β are both normal.

1.8. Let $G \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be a sequence of Λ-semimodules and Λ-homomorphisms with G a Λ-module and $\beta\alpha = 0$. Assume that the following is satisfied: whenever $\beta(y_1) = \beta(y_2)$, $y_1, y_2 \in Y$, one has $\alpha(g) + y_1 = y_2$, $g \in G$. Then, obviously, β is a normal Λ-homomorphism and $G \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is exact.

1.9. Lemma. *Suppose given a commutative diagram of Λ-semimodules and Λ-homomorphisms*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & Y \arrow[r, "\beta"] \arrow[d, "\varphi"'] & Z \arrow[d, "\psi"'] \\ X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \end{tikzcd}$$

such that f is surjective, φ is injective, and $\beta\alpha = 0$. Assume that the bottom row is exact and β' is normal. Then the top row is exact and β is normal.

Proof. Suppose that $\beta(y_1) = \beta(y_2)$, $y_1, y_2 \in Y$. Then $\beta'\varphi(y_1) = \beta'\varphi(y_2)$. Hence $\kappa_1 + \varphi(y_1) = \kappa_2 + \varphi(y_2)$, $\kappa_1, \kappa_2 \in \mathrm{Ker}(\beta')$. Since the bottom row is exact and f is onto, there exist $x_1, x_2 \in X$ such that $\kappa_1 = \alpha' f(x_1)$ and $\kappa_2 = \alpha' f(x_2)$. Then we get $\varphi\alpha(x_1)+\varphi(y_1) = \varphi\alpha(x_2)+\varphi(y_2)$. Whence, as φ is one-to-one, $\alpha(x_1)+y_1 = \alpha(x_2)+y_2$. Thus β is normal. Now assume that $\beta(y) = 0$, $y \in Y$. Then $\beta'\varphi(y) = 0$. Hence $\alpha' f(x) = \varphi(y)$ for some $x \in X$. This gives $\varphi\alpha(x) = \varphi(y)$. Whence $\alpha(x) = y$. $\square$

1.10. Definition ([4]). We say that a sequence of Λ-semimodules and Λ-homomorphisms

$$X : \cdots \rightrightarrows X_{n+1} \overset{\partial^+_{n+1}}{\underset{\partial^-_{n+1}}{\rightrightarrows}} X_n \overset{\partial^+_{n}}{\underset{\partial^-_{n}}{\rightrightarrows}} X_{n-1} \rightrightarrows \cdots, \quad n \in \mathbb{Z},$$

written $X = \{X_n, \partial^+_n, \partial^-_n\}$ for short, is a chain complex if

$$\partial^+_n\, \partial^+_{n+1} + \partial^-_n\, \partial^-_{n+1} = \partial^+_n\, \partial^-_{n+1} + \partial^-_n\, \partial^+_{n+1}$$

for each integer n. For every chain complex X we define the Λ-semimodule

$$Z_n(X) = \big\{x \in X_n | \partial^+_n(x) = \partial^-_n(x)\big\},$$

the n-cycles, and the n-th homology Λ-semimodule

$$H_n(X) = Z_n(X)/\rho_n(X),$$

where $\rho_n(X)$ is a congruence on $Z_n(X)$ defined as follows:

$$\begin{aligned} x\rho_n(X)y \;\; \Leftrightarrow \;\; & x + \partial^+_{n+1}(u) + \partial^-_{n+1}(v) = y + \partial^+_{n+1}(v) + \partial^-_{n+1}(u) \\ & \text{for some} \quad u, v \quad \text{in} \quad X_{n+1}. \end{aligned}$$

The Λ-homomorphisms $\partial^+_n, \partial^-_n$ are called differentials of the chain complex X.

A sequence $G = \{G_n, d^+_n, d^-_n\}$ of Λ-modules and Λ-homomorphisms is a chain complex if and only if

$$\cdots \longrightarrow G_n \xrightarrow{d^+_n - d^-_n} G_{n-1} \longrightarrow \cdots$$

is an ordinary chain complex of Λ-modules. Obviously, for any chain complex $G = \{G_n, d^+_n, d^-_n\}$ of Λ-modules, $H_*(G)$ coincides with the usual homology $H_*(\{G_n, d^+_n - d^-_n\})$.

1.11. One can think of an ordinary chain complex

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

of Λ-semimodules as a chain complex in the sense of Definition 1.10; namely, we identify $\{C_n, \partial_n\}$ with the chain complex

$$\cdots \rightrightarrows C_{n+1} \overset{\partial_{n+1}}{\underset{0}{\rightrightarrows}} C_n \overset{\partial_{n}}{\underset{0}{\rightrightarrows}} C_{n-1} \rightrightarrows \cdots.$$

Defining $H_k(\{C_n, \partial_n\})$ to be $H_k(\{C_n, \partial_n, 0\})$, one has $H_k(\{C_n, \partial_n\}) = \mathrm{Ker}(\partial_k)/\partial_{k+1}(C_{k+1})$.

1.12. Definition ([4]). Let $X = \{X_n, \partial_n^+, \partial_n^-\}$ and $X' = \{X'_n, \partial_n^{'+}, \partial_n^{'-}\}$ be chain complexes of Λ-semimodules. We say that a sequence $f = \{f_n\}$ of Λ-homomorphisms $f_n : X_n \longrightarrow X'_n$ is a $\pm$-morphism from X to X' if

$$f_{n-1}\partial_n^+ = \partial_n^{'+} f_n \quad \text{and} \quad f_{n-1}\partial_n^- = \partial_n^{'-} f_n \quad \text{for all} \quad n.$$

1.13. If $f = \{f_n\} : X \longrightarrow X'$ is a $\pm$-morphism of chain complexes, then $f_n(Z_n(X)) \subset Z_n(X')$, and the map

$$H_n(f) : H_n(X) \longrightarrow H_n(X'), \;\; H_n(f)(\mathrm{cl}(x)) = \mathrm{cl}(f_n(x)),$$

is a homomorphism of Λ-semimodules. Thus H_n is a covariant additive functor from the category of chain complexes and their $\pm$-morphisms to the category of Λ-semimodules.

An important example of a $\pm$-morphism appears in a natural way: a map of presimplicial Λ-semimodules $f : S \longrightarrow S'$ induces a $\pm$-morphism $\underline{f} = f : \underline{S} \longrightarrow \underline{S}'$, where $\underline{S}$ and $\underline{S}'$ are the standard nonnegative chain complexes associated to S and S', respectively (see [4]).

1.14. Definition (cf. [5]). Let $X = \{X_n, \partial_n^+, \partial_n^-\}$ and $X' = \{X'_n, \partial_n^{'+}, \partial_n^{'-}\}$ be chain complexes of Λ-semimodules. We say that a sequence $f = \{f_n\}$ of Λ-homomorphisms $f_n : X_n \longrightarrow X'_n$ is a morphism from X to X' if

$$\partial_n^{'+} f_n + f_{n-1}\partial_n^- = \partial_n^{'-} f_n + f_{n-1}\partial_n^+ \quad \text{for all} \quad n.$$

Note that any $\pm$-morphism between chain complexes of Λ-semimodules is a morphism.

1.15. Definition. A sequence $E : A \overset{\varkappa}{\rightarrowtail} B \overset{\sigma}{\twoheadrightarrow} C$ of chain complexes and their morphisms is called a Schreier short exact sequence of chain complexes if each $E_n : A_n \overset{\varkappa_n}{\rightarrowtail} B_n \overset{\sigma_n}{\twoheadrightarrow} C_n$ is a Schreier extension of Λ-semimodules.

1.16. In general, a morphism $f = \{f_n\} : X \longrightarrow X'$ of chain complexes, unlike $\pm$-morphisms, does not induce a Λ-homomorphism from $H_n(X)$ to $H_n(X')$. However, if X' is a chain complex of cancellative Λ-semimodules, or X is an ordinary chain complex of Λ-semimodules, i.e., $\partial_n^- = 0$ for all n (see 1.11), then one can easily check that $f_n(Z_n(X)) \subset Z_n(X')$ and the map $H_n(f) : H_n(X) \longrightarrow H_n(X')$, $H_n(f)(\mathrm{cl}(x)) = \mathrm{cl}(f_n(x))$, is well-defined and is a Λ-homomorphism for all n. Besides, we have

Proposition. *Let $E : A \overset{\varkappa}{\rightarrowtail} B \overset{\sigma}{\twoheadrightarrow} C$ be a Schreier short exact sequence of chain complexes and their morphisms. If A is a chain complex of cancellative Λ-semimodules, then $\varkappa_n(Z_n(A)) \subset Z_n(B)$ and the map $H_n(\varkappa) : H_n(A) \longrightarrow H_n(B)$, $H_n(\varkappa)(\mathrm{cl}(a)) = \mathrm{cl}(\varkappa_n(a))$, is well-defined and is therefore a Λ-homomorphism for all n.*

Proof. Let d_n^+, d_n^- and $\partial_n^+, \partial_n^-$ denote the n-th differentials of A and B, respectively. Suppose $a \in Z_n(A)$, i.e., $d_n^+(a) = d_n^-(a)$. Since $\varkappa$ is a morphism, $\varkappa_{n-1}d_n^+(a) +$

$\partial_n^- \varkappa_n(a) = \varkappa_{n-1} d_n^-(a) + \partial_n^+ \varkappa_n(a)$. Whence,by 1.3, $\partial_n^- \varkappa_n(a) = \partial_n^+ \varkappa_n(a)$. That is, $\varkappa_n(a) \in Z_n(B)$. Now assume that $a_1, a_2 \in Z_n(A)$ and $a_1 \rho_n(A) a_2$. Hence

$$a_1 + d_{n+1}^+(p) + d_{n+1}^-(q) = a_2 + d_{n+1}^+(q) + d_{n+1}^-(p), \quad p, q \in A_{n+1}.$$

On the other hand,

$$\varkappa_n d_{n+1}^+(p) + \partial_{n+1}^- \varkappa_{n+1}(p) = \varkappa_n d_{n+1}^-(p) + \partial_{n+1}^+ \varkappa_{n+1}(p),$$
$$\varkappa_n d_{n+1}^+(q) + \partial_{n+1}^- \varkappa_{n+1}(q) = \varkappa_n d_{n+1}^-(q) + \partial_{n+1}^+ \varkappa_{n+1}(q).$$

These last three equalities imply

$$\varkappa_n(d_{n+1}^+(p) + d_{n+1}^-(q)) + \partial_{n+1}^+ \varkappa_{n+1}(p) + \partial_{n+1}^- \varkappa_{n+1}(q) + \varkappa_n(a_1) =$$
$$= \varkappa_n(d_{n+1}^+(p) + d_{n+1}^-(q)) + \partial_{n+1}^+ \varkappa_{n+1}(q) + \partial_{n+1}^- \varkappa_{n+1}(p) + \varkappa_n(a_2).$$

Whence, by 1.3, $\partial_{n+1}^+ \varkappa_{n+1}(p) + \partial_{n+1}^- \varkappa_{n+1}(q) + \varkappa_n(a_1) = \partial_{n+1}^+ \varkappa_{n+1}(q) + \partial_{n+1}^- \varkappa_{n+1}(p) + \varkappa_n(a_2)$. That is, $\varkappa_n(a_1) \rho_n(B) \varkappa_n(a_2)$. Thus $H_n(\varkappa)$ is well-defined. □

Definition 1.10 naturally leads us to new homology and cohomology monoids of monoids (in particular, groups) with coefficients in semimodules. The calculation of them for cyclic groups is an example of the effective use of morphisms which are not $\pm$-morphisms [6].

1.17. If $X = \{X_n, \partial_n^+, \partial_n^-\}$ is a chain complex of Λ-semimodules, then $K(X) = \{K(X_n), K(\partial_n^+) - K(\partial_n^-)\}$ is an ordinary chain complex of $K(\Lambda)$-modules (i.e., Λ-modules). When each X_n is cancellative, then the converse is also true. Then, for any chain complex $X = \{X_n, \partial_n^+, \partial_n^-\}$ of Λ-semimodules, one has the Λ-homomorphisms $H_n(k_X) : H_n(X) \longrightarrow H_n(K(X))$, $H_n(k_X)(\mathrm{cl}(x)) = \mathrm{cl}(k_{X_n}(x)) = \mathrm{cl}[x, 0]$, induced by the canonical morphism $k_X = \{k_{X_n}\} : X \longrightarrow K(X)$ which is in fact a $\pm$-morphism from X to $\{K(X_n), K(\partial_n^+), K(\partial_n^-)\}$. When X is a chain complex of cancellative Λ-semimodules, then $H_n(k_X)$ is injective and therefore $H_n(X)$ is a cancellative Λ-semimodule. Further, if $f = \{f_n\} : X \longrightarrow X'$ is a morphism of chain complexes, then $K(f) = \{K(f_n) : K(X_n) \longrightarrow K(X'_n)\}$ is an usual chain map from $K(X)$ to $K(X')$. When X' is a chain complex of cancellative Λ-semimoduls, then the converse is also valid.

1.18. If $E : A \rightarrowtail^{\varkappa} B \overset{\sigma}{\twoheadrightarrow} C$ is a Schreier short exact sequence of chain complexes, then, by 1.5, $K(E) : K(A) \overset{K(\varkappa)}{\rightarrowtail} K(B) \overset{K(\sigma)}{\twoheadrightarrow} K(C)$ is a short exact sequence of ordinary chain complexes.

2. Main results

2.1. Proposition. *Suppose given a Schreier short exact sequence*

$$A \overset{\varkappa}{\rightarrowtail} B \overset{\sigma}{\twoheadrightarrow} C$$

of chain complexes and their morphisms such that each A_n is cancellative and each differential ∂_n^- of B preserves representatives. Assume that one of the following conditions holds:

(i) *σ is a $\pm$-morphism.*

(ii) *C is a chain complex of cancellative Λ-semimodules.*
Then there are Λ-homomorphisms $\partial_n(E): H_n(C) \longrightarrow H_{n-1}(A)$, called connecting homomorphisms, such that each diagram

$$\begin{tikzcd} H_n(C) \arrow[r, "\partial_n(E)"] \arrow[d, "H_n(k_C)"'] & H_{n-1}(A) \arrow[d, "H_{n-1}(k_A)"] \\ H_n(K(C)) \arrow[r, "\partial_n(K(E))"] & H_{n-1}(K(A)), \end{tikzcd}$$

where $\partial_n(K(E))$ is the usual connecting homomorphism induced by $K(E)$ (see 1.18), is commutative. Furthermore, $H_n(\varkappa)$ and $H_n(\sigma)$ are defined for all n, and the long sequence of homology semimodules

$$\cdots \to H_n(A) \xrightarrow{H_n(\varkappa)} H_n(B) \xrightarrow{H_n(\sigma)} H_n(C) \xrightarrow{\partial_n(E)} H_{n-1}(A) \xrightarrow{H_{n-1}(\varkappa)} H_{n-1}(B) \to \cdots$$

is an ordinary chain complex of Λ-semimodules.

Proof. Let d_n^+, d_n^- and δ_n^+, δ_n^- denote the n-th differentials of A and C, respectively. Take any $c \in Z_n(C)$. There is a representative u_c of $E_n : A_n \rightarrowtail^{\varkappa_n} B_n \xrightarrow{\sigma_n} C_n$ with $\sigma_n(u_c) = c$. When (i) holds, one can write $\sigma_{n-1}\partial_n^+(u_c) = \delta_n^+\sigma_n(u_c) = \delta_n^+(c) = \delta_n^-(c) = \delta_n^-\sigma_n(c) = \sigma_{n-1}\partial_n^-(u_c)$. If (ii) holds, then the equality $\sigma_{n-1}\partial_n^+(u_c) + \delta_n^-\sigma_n(u_c) = \sigma_{n-1}\partial_n^-(u_c) + \delta_n^+\sigma_n(u_c)$ implies $\sigma_{n-1}\partial_n^+(u_c) = \sigma_{n-1}\partial_n^-(u_c)$. Consequently, $\partial_n^+(u_c) = \varkappa_{n-1}(a) + \partial_n^-(u_c)$, $a \in A_{n-1}$, in both cases (see 1.1). Whence $[\partial_n^+(u_c), \partial_n^-(u_c)] = K(\varkappa_{n-1})([a, 0])$. On the other hand, $[\partial_n^+(u_c), \partial_n^-(u_c)] = (K(\partial_n^+) - K(\partial_n^-))([u_c, 0])$ and $K(\sigma_n)([u_c, 0]) = [c, 0] \in \operatorname{Ker}(K(\delta_n^+) - K(\delta_n^-))$. Therefore, by construction of $\partial_n(K(E))$, one concludes that $[a, 0] \in \operatorname{Ker}(K(d_{n-1}^+) - K(d_{n-1}^-))$ and $\partial_n(K(E))(\mathrm{cl}([c, 0])) = \mathrm{cl}([a, 0])$. As A_{n-2} is cancellative, the former gives $a \in Z_{n-1}(A)$. And we set

$$\partial_n(E)(\mathrm{cl}(c)) = \mathrm{cl}(a) \in H_{n-1}(A).$$

Clearly, since $\partial_n(K(E))H_n(k_C)(\mathrm{cl}(c)) = H_{n-1}(k_A)(\mathrm{cl}(a))$ and $H_{n-1}(k_A)$ is injective (see 1.17), we may write

$$\partial_n(E)(\mathrm{cl}(c)) = H_{n-1}(k_A)^{-1}\big(\partial_n(K(E))H_n(k_C)(\mathrm{cl}(c))\big).$$

Hence, $\partial_n(E)$ is well-defined and is a Λ-homomorphism, and $H_{n-1}(k_A)\partial_n(E) = \partial_n(K(E))H_n(k_C)$.

It follows from 1.13 and 1.16 that $H_n(\sigma)$ and $H_n(\varkappa)$ are defined. Obviously $H_n(\sigma)H_n(\varkappa) = 0$. Using the usual long exact homology sequence for $K(E)$, one has

$$\begin{aligned} H_{n-1}(k_A)\partial_n(E)H_n(\sigma) &= \partial_n(K(E))H_n(k_C)H_n(\sigma) = \\ &= \partial_n(K(E))H_n(K(\sigma))H_n(k_B) = 0 \cdot H_n(k_B) = 0. \end{aligned}$$

Hence $\partial_n(E)H_n(\sigma) = 0$ since $H_{n-1}(k_A)$ is one-to-one. By definition of $\partial_n(E)$, $\partial_n(E)(\mathrm{cl}(c)) = \mathrm{cl}(a)$, $a \in Z_{n-1}(A)$, and a satisfies the equality $\partial_n^+(u_c) = \varkappa_{n-1}(a) +$

$\partial_n^-(u_c)$ for some representative u_c of E_n with $\sigma_n(u_c) = c$. Consequently, $H_{n-1}(\varkappa)\partial_n(E)(\mathrm{cl}(c)) = H_{n-1}(\varkappa)(\mathrm{cl}(a)) = \mathrm{cl}(\varkappa_{n-1}(a)) = 0$. Thus $H_{n-1}(\varkappa)\partial_n(E) = 0$. □

We see that for any Schreier short exact sequence of chain complexes $E : A \overset{\varkappa}{\rightarrowtail} B \overset{\sigma}{\twoheadrightarrow} C$ satisfying the hypotheses of Proposition 2.1, one has the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "H_n(\varkappa)"] \arrow[d, "H_n(k_A)"'] & H_n(B) \arrow[r, "H_n(\sigma)"] \arrow[d, "H_n(k_B)"'] & H_n(C) \arrow[r, "\partial_n(E)"] \arrow[d, "H_n(k_C)"'] & H_{n-1}(A) \arrow[r] \arrow[d, "H_{n-1}(k_A)"'] & \cdots \\
\cdots \arrow[r] & H_n(K(A)) \arrow[r, "H_n(K(\varkappa))"] & H_n(K(B)) \arrow[r, "H_n(K(\sigma))"] & H_n(K(C)) \arrow[r, "\partial_n(K(E))"] & H_{n-1}(K(A)) \arrow[r] & \cdots
\end{tikzcd}$$

induced by the canonical map $k_E = (k_A, k_B, k_C) : E \longrightarrow K(E)$. In fact $\partial_n(E)$ is natural in the following sense.

2.2. Proposition. *Suppose that*

$$\begin{tikzcd}
E: & A \arrow[r, tail, "\varkappa"] \arrow[d, "f"'] & B \arrow[r, two heads, "\sigma"] \arrow[d, "g"'] & C \arrow[d, "h"'] \\
E': & A' \arrow[r, tail, "\varkappa'"] & B' \arrow[r, two heads, "\sigma'"] & C'
\end{tikzcd}$$

is a commutative diagram of chain complexes and their morphisms such that E and E' are Schreier short exact sequences satisfying the hypotheses of Proposition 2.1*. Suppose further that $H_n(g)$ and $H_n(h)$ are defined for all n (see* 1.13 *and* 1.16*). Then the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "H_n(\varkappa)"] \arrow[d, "H_n(f)"'] & H_n(B) \arrow[r, "H_n(\sigma)"] \arrow[d, "H_n(g)"'] & H_n(C) \arrow[r, "\partial_n(E)"] \arrow[d, "H_n(h)"'] & H_{n-1}(A) \arrow[r] \arrow[d, "H_{n-1}(f)"'] & \cdots \\
\cdots \arrow[r] & H_n(A') \arrow[r, "H_n(\varkappa')"] & H_n(B') \arrow[r, "H_n(\sigma')"] & H_n(C') \arrow[r, "\partial_n(E')"] & H_{n-1}(A') \arrow[r] & \cdots
\end{tikzcd}$$

is commutative.

Proof. By Proposition 2.1 and the naturality of $\partial_n(K(E))$,

$$\begin{aligned}
&H_{n-1}(k_{A'})\partial_n(E')H_n(h) = \partial_n(K(E'))H_n(k_{C'})H_n(h) \\
&= \partial_n(K(E'))H_n(K(h))H_n(k_C) = H_{n-1}(K(f))\partial_n(K(E))H_n(k_C) \\
&= H_{n-1}(K(f))H_{n-1}(k_A)\partial_n(E) = H_{n-1}(k_{A'})H_{n-1}(f)\partial_n(E).
\end{aligned}$$

Therefore, by the injectivity of $H_{n-1}(k_{A'})$ (see 1.17), $\partial_n(E')H_n(h) = H_{n-1}(f)\partial_n(E)$. □

Before we state our main results, we note the following. Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be an exact sequence of Λ-semimodules and Λ-homomorphisms. If $\beta = 0$ then α is onto. But, unlike the situation for modules, one may have $\alpha = 0$ and yet not have β one-to-one. However, we have:

2.3. Suppose given an exact sequence of Λ-semimodules and Λ-homomorphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ with β a normal Λ-homomorphism (see 1.6). If $\alpha = 0$, then β is one-to-one.

This together with 1.7 motivates the following two theorems.

2.4. Theorem. *Let*

$$\begin{tikzcd}
 & A: \cdots \arrow[d, tail, "\varkappa"] \arrow[r, shift left] \arrow[r, shift right] & A_{n+1} \arrow[d, tail, "\varkappa_{n+1}"] \arrow[r, shift left, "d^+_{n+1}"] \arrow[r, shift right, "d^-_{n+1}"'] & A_n \arrow[d, tail, "\varkappa_n"] \arrow[r, shift left, "d^+_n"] \arrow[r, shift right, "d^-_n"'] & A_{n-1} \arrow[d, tail, "\varkappa_{n-1}"] \arrow[r, shift left] \arrow[r, shift right] & \cdots \\
E: & B: \cdots \arrow[d, two heads, "\sigma"] \arrow[r, shift left] \arrow[r, shift right] & B_{n+1} \arrow[d, two heads, "\sigma_{n+1}"] \arrow[r, shift left, "\partial^+_{n+1}"] \arrow[r, shift right, "\partial^-_{n+1}"'] & B_n \arrow[d, two heads, "\sigma_n"] \arrow[r, shift left, "\partial^+_n"] \arrow[r, shift right, "\partial^-_n"'] & B_{n-1} \arrow[d, two heads, "\sigma_{n-1}"] \arrow[r, shift left] \arrow[r, shift right] & \cdots \\
 & G: \cdots \arrow[r] & G_{n+1} \arrow[r, "\delta_{n+1}"] & G_n \arrow[r, "\delta_n"] & G_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

be a Schreier short exact sequence of chain complexes and their morphisms (*see* 1.11) *such that each* A_n *is a cancellative* Λ*-semimodule, every differential* ∂^-_n *preserves representatives, and each* G_n *is a* Λ*-module. Then the long homology sequence*

$$\cdots \longrightarrow H_{n+1}(G) \xrightarrow{\partial_{n+1}(E)} H_n(A) \xrightarrow{H_n(\varkappa)} H_n(B) \xrightarrow{H_n(\sigma)} \tag{2.4.1}$$
$$\xrightarrow{H_n(\sigma)} H_n(G) \xrightarrow{\partial_n(E)} H_{n-1}(A) \longrightarrow \cdots$$

is exact at $H_n(A)$ *and at* $H_n(B)$*,* $H_n(\sigma)(H_n(B)) \subset \operatorname{Ker}(\partial_n(E))$*, and* $H_n(\varkappa)$ *is normal. Furthermore,* (2.4.1) *is exact at* $H_n(G)$ *if and only if* $H_n(\sigma)(H_n(B)) = H_n(K(\sigma))(H_n(K(B)))$.

Proof. By Proposition 2.1, Sequence (2.4.1) is an ordinary chain complex. Then, the commutative diagram

$$\begin{tikzcd}
H_{n+1}(G) \arrow[r, "\partial_{n+1}(E)"] \arrow[d, equal] & H_n(A) \arrow[r, "H_n(\varkappa)"] \arrow[d, "H_n(k_A)"] & H_n(B) \arrow[d, "H_n(k_B)"] \\
H_{n+1}(G) \arrow[r, "\partial_{n+1}(K(E))"] & H_n(K(A)) \arrow[r, "H_n(K(\varkappa))"] & H_n(K(B))
\end{tikzcd}$$

satisfies the hypotheses of Lemma 1.9 (see 1.17). Hence (2.4.1) is exact at $H_n(A)$ and $H_n(\varkappa)$ is normal. We next show that $\mathrm{Ker}(H_n(\sigma)) \subset H_n(\varkappa)(H_n(A))$. Let $b \in Z_n(B)$, i.e., $\partial_n^+(b) = \partial_n^-(b)$. Assume that $H_n(\sigma)(\mathrm{cl}(b)) = 0$, i.e., $\mathrm{cl}(\sigma_n(b)) = 0$. Then $\sigma_n(b) = \delta_{n+1}(g)$ for some $g \in G_{n+1}$. Choose a representative $u = u_{-g}$ of

$$E_{n+1} : A_{n+1} \overset{\varkappa_{n+1}}{\rightarrowtail} B_{n+1} \overset{\sigma_{n+1}}{\twoheadrightarrow} G_{n+1}$$

with $\sigma_{n+1}(u) = -g$. Since σ is a morphism, $\sigma_n\partial_{n+1}^+(u) = \sigma_n\partial_{n+1}^-(u) + \delta_{n+1}\sigma_{n+1}(u)$. Whence $\sigma_n\partial_{n+1}^+(u) = \sigma_n\partial_{n+1}^-(u) - \sigma_n(b)$, i.e., $\sigma_n(\partial_{n+1}^+(u) + b) = \sigma_n\partial_{n+1}^-(u)$. Then, as ∂_{n+1}^- preserves representatives, if follows that

$$b + \partial_{n+1}^+(u) = \varkappa_n(a) + \partial_{n+1}^-(u), \quad a \in A_n. \tag{$*$}$$

This with the fact that $\varkappa$ is a morphism of chain complex gives

$$\begin{aligned} &\varkappa_{n-1}d_n^+(a) + \partial_n^-(b) + (\partial_n^-\partial_{n+1}^+ + \partial_n^+\partial_{n+1}^-)(u) = \\ &= \varkappa_{n-1}d_n^-(a) + \partial_n^+(b) + (\partial_n^+\partial_{n+1}^+ + \partial_n^-\partial_{n+1}^-)(u). \end{aligned}$$

But, by 1.4, B_{n-1} is cancellative. Therefore $\varkappa_{n-1}d_n^+(a) = \varkappa_{n-1}d_n^-(a)$ (see 1.10). Hence $d_n^+(a) = d_n^-(a)$, i.e., $a \in Z_n(A)$. Then, by $(*)$, one can write $H_n(\varkappa)(\mathrm{cl}(a)) = \mathrm{cl}(\varkappa_n(a)) = \mathrm{cl}(b)$. Thus (2.4.1) is exact at $H_n(B)$. Finally, the commutative diagram

$$\begin{tikzcd} H_n(B) \arrow[r, "H_n(\sigma)"] \arrow[d, "H_n(k_B)"'] & H_n(G) \arrow[r, "\partial_n(E)"] \arrow[d, equal] & H_{n-1}(A) \arrow[d, "H_{n-1}(k_A)"] \\ H_n(K(B)) \arrow[r, "H_n(K(B))"] & H_n(G) \arrow[r, "\partial_n(K(E))"] & H_{n-1}(K(A)) \end{tikzcd}$$

shows that if $H_n(\sigma)(H_n(B)) = H_n(K(\sigma))(H_n(K(B)))$, then (2.4.1) is exact at $H_n(G)$. The converse is also true since $H_{n-1}(k_A)$ is injective (see 1.17). $\square$

2.5. Theorem. *Suppose given a Schreier short exact sequence*

$$\begin{tikzcd} & G: \arrow[d, tail, "\varkappa"] & \cdots \arrow[r] & G_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, tail, "\varkappa_{n+1}"] & G_n \arrow[r, "d_n"] \arrow[d, tail, "\varkappa_n"] & G_{n-1} \arrow[r] \arrow[d, tail, "\varkappa_{n-1}"] & \cdots \\ E: & B: \arrow[d, two heads, "\sigma"] & \cdots \arrow[r, shift left] \arrow[r, shift right] & B_{n+1} \arrow[r, shift left, "\partial_{n+1}^+"] \arrow[r, shift right, "\partial_{n+1}^-"'] \arrow[d, two heads, "\sigma_{n+1}"] & B_n \arrow[r, shift left, "\partial_n^+"] \arrow[r, shift right, "\partial_n^-"'] \arrow[d, two heads, "\sigma_n"] & B_{n-1} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads, "\sigma_{n-1}"] & \cdots \\ & C: & \cdots \arrow[r, shift left] \arrow[r, shift right] & C_{n+1} \arrow[r, shift left, "\delta_{n+1}^+"] \arrow[r, shift right, "\delta_{n+1}^-"'] & C_n \arrow[r, shift left, "\delta_n^+"] \arrow[r, shift right, "\delta_n^-"'] & C_{n-1} \arrow[r, shift left] \arrow[r, shift right] & \cdots \end{tikzcd}$$

of chain complexes and their morphisms (*see* 1.11) *such that each* G_n *is a* Λ*-module* (*therefore each differential* ∂_n^- *obviously preserves representatives* (*see* 1.2)). *Assume that one of the following conditions holds:*

(i) σ *is a* $\pm$*-morphism.*

(ii) C *is a chain complex of cancellative* Λ*-semimodules.*

Then the long homology sequence

$$\cdots \to H_n(G) \xrightarrow{H_n(\varkappa)} H_n(B) \xrightarrow{H_n(\sigma)} H_n(C) \xrightarrow{\partial_n(E)} \tag{2.5.1}$$
$$\xrightarrow{\partial_n(E)} H_{n-1}(G) \xrightarrow{H_{n-1}(\varkappa)} H_{n-1}(B) \longrightarrow \cdots$$

(*here* $\partial_n(E)$ *evidently coincides with* $\partial_n(K(E))H_n(k_C)$) *is exact at* $H_n(B)$ *and at* $H_n(C)$, $\partial_n(E)(H_n(C)) \subset \mathrm{Ker}(H_{n-1}(\varkappa))$, *and* $H_n(\sigma)$ *is a normal* Λ*-homomorphism. Furthermore, if* $\partial_n(K(E))H_n(k_C)(H_n(C)) = \partial_n(K(E))(H_n(K(C)))$ *then* (2.5.1) *is exact at* $H_{n-1}(G)$. *When* (ii) *holds, the converse is also valid.*

Proof. By Proposition 2.1, Sequence (2.5.1) is an ordinary chain complex. Assume that (ii) holds. Then, by 1.4, the commutative diagram

$$\begin{tikzcd}
H_n(G) \arrow[r, "H_n(\varkappa)"] \arrow[d, equal] & H_n(B) \arrow[r, "H_n(\sigma)"] \arrow[d, "H_n(k_B)"] & H_n(C) \arrow[d, "H_n(k_C)"] \\
H_n(G) \arrow[r, "H_n(K(\varkappa))"] & H_n(K(B)) \arrow[r, "H_n(K(\sigma))"] & H_n(K(C))
\end{tikzcd}$$

satisfies the hypotheses of Lemma 1.9 (see 1.17). Therefore (2.5.1) is exact at $H_n(B)$ and $H_n(\sigma)$ is normal. When (i) holds, we prove the same as follows. Let $b_1, b_2 \in Z_n(B)$, i.e., $\partial_n^+(b_1) = \partial_n^-(b_1)$ and $\partial_n^+(b_2) = \partial_n^-(b_2)$, and let $H_n(\sigma)(\mathrm{cl}(b_1)) = H_n(\sigma)(\mathrm{cl}(b_2))$, i.e., $\mathrm{cl}(\sigma_n(b_1)) = \mathrm{cl}(\sigma_n(b_2))$. Then $\sigma_n(b_1) + \delta_{n+1}^+(p) + \delta_{n+1}^-(q) = \sigma_n(b_2) + \delta_{n+1}^+(q) + \delta_{n+1}^-(p)$ for some $p, q \in C_{n+1}$. Take $x, y \in B_{n+1}$ with $\sigma_{n+1}(x) = p$ and $\sigma_{n+1}(y) = q$, and write $\sigma_n(b_1) + \delta_{n+1}^+\sigma_{n+1}(x) + \delta_{n+1}^-\sigma_{n+1}(y) = \sigma_n(b_2) + \delta_{n+1}^+\sigma_{n+1}(y) + \delta_{n+1}^-\sigma_{n+1}(x)$. This, as σ is a $\pm$-morphism, implies

$$\sigma_n\big(b_1 + \partial_{n+1}^+(x) + \partial_{n+1}^-(y)\big) = \sigma_n\big(b_2 + \partial_{n+1}^+(y) + \partial_{n+1}^-(x)\big).$$

Therefore, by 1.2,

$$b_1 + \partial_{n+1}^+(x) + \partial_{n+1}^-(y) = \varkappa_n(g) + b_2 + \partial_{n+1}^+(y) + \partial_{n+1}^-(x), \quad g \in G_n. \tag{$**$}$$

Whence

$$\partial_n^+(b_1) + \partial_n^+\partial_{n+1}^+(x) + \partial_n^+\partial_{n+1}^-(y) =$$
$$= \partial_n^+\varkappa_n(g) + \partial_n^+ b_2 + \partial_n^+\partial_{n+1}^+(y) + \partial_n^+\partial_{n+1}^-(x)$$

and

$$\begin{aligned}&\partial_n^-(b_1)+\partial_n^-\partial_{n+1}^+(x)+\partial_n^-\partial_{n+1}^-(y)=\\&=\partial_n^-\varkappa_n(g)+\partial_n^-(b_2)+\partial_n^-\partial_{n+1}^+(y)+\partial_n^-\partial_{n+1}^-(x).\end{aligned}$$

The last two equalities give

$$\begin{aligned}&\partial_n^+\varkappa_n(g)+\partial_n^+(b_2)+\partial_n^+\partial_{n+1}^+(y)+\partial_n^+\partial_{n+1}^-(x)+\\&\quad+\partial_n^-(b_1)+\partial_n^-\partial_n^+(x)+\partial_n^-\partial_{n+1}^-(y)=\\&=\partial_n^-\varkappa_n(g)+\partial_n^-(b_2)+\partial_n^-\partial_{n+1}^+(y)+\partial_n^-\partial_{n+1}^-(x)+\\&\quad+\partial_n^+(b_1)+\partial_n^+\partial_n^+(x)+\partial_n^+\partial_{n+1}^-(y).\end{aligned}$$

But $\partial_n^+(b_1)=\partial_n^-(b_1)$, $\partial_n^+(b_2)=\partial_n^-(b_2)$, $\partial_n^+\varkappa_n(g)=\varkappa_{n-1}d_n(g)+\partial_n^-\varkappa_n(g)$ and $\partial_n^+\partial_{n+1}^+ + \partial_n^-\partial_{n+1}^- = \partial_n^+\partial_{n+1}^- + \partial_n^-\partial_{n+1}^+$. Consequently, we have

$$\varkappa_{n-1}d_n(g)+w=w,\quad w\in B_{n-1}.$$

Whence, by 1.2, $d_n(g)=0$. That is, $g\in\operatorname{Ker}(d_n)$. Then, by $(**)$, one can write

$$\operatorname{cl}(b_1)=H_n(\varkappa)(\operatorname{cl}(g))+\operatorname{cl}(b_2).$$

Thus, by 1.8, we conclude that (2.5.1) is exact at $H_n(B)$ and $H_n(\sigma)$ is normal.

We next show that $\operatorname{Ker}(\partial_n(E))\subset H_n(\sigma)(H_n(B))$. Let $c\in Z_n(C)$. Take any $b\in B_n$ with $\sigma_n(b)=c$. By definition of $\partial_n(E)$, $\partial_n(E)(\operatorname{cl}(c))=\operatorname{cl}(g)$, $g\in Z_{n-1}(G)$, and g satisfies the equality $\partial_n^+(b)=\varkappa_{n-1}(g)+\partial_n^-(b)$ (see 1.2). Assume that $\partial_n(E)(\operatorname{cl}(c))=0$, i.e., $\operatorname{cl}(g)=0$. Then $g=d_n(h)$, $h\in G_n$. As $\varkappa$ is a morphism, we can write $\partial_n^+(b-\varkappa_n(h))=\partial_n^+(b)-\partial_n^+\varkappa_n(h)=\varkappa_{n-1}(g)+\partial_n^-(b)-(\varkappa_{n-1}d_n(h)+\partial_n^-\varkappa_n(h))=\partial_n^-(b)-\partial_n^-\varkappa_n(h)=\partial_n^-(b-\varkappa_n(h))$. Hence $b-\varkappa_n(h)\in Z_n(B)$. Clearly, $H_n(\sigma)(\operatorname{cl}(b-\varkappa_n(h)))=\operatorname{cl}(c)$. Thus (2.5.1) is exact at $H_n(C)$.

Finally, the commutative diagram

$$\begin{tikzcd} H_n(C) \arrow[r, "\partial_n(E)"] \arrow[d, "H_n(k_C)"'] & H_{n-1}(G) \arrow[r, "H_{n-1}(\varkappa)"] \arrow[d, equal] & H_{n-1}(B) \arrow[d, "H_{n-1}(k_B)"] \\ H_n(K(C)) \arrow[r, "\partial_n(K(E))"] & H_{n-1}(G) \arrow[r, "H_{n-1}(K(\varkappa))"] & H_{n-1}(K(B)) \end{tikzcd}$$

shows that if $\partial_n(K(E))H_n(k_C)(H_n(C))=\partial_n(K(E))(H_n(K(C)))$, then (2.5.1) is exact at $H_{n-1}(G)$. When (ii) holds, the converse is also true since $H_{n-1}(k_B)$ is injective (see 1.4 and 1.17). □

2.6. Remark. For a Schreier short exact sequence $E:\ G\overset{\varkappa}{\rightarrowtail}B\overset{\sigma}{\twoheadrightarrow}C$ of chain complexes, where G is an ordinary chain complex of Λ-modules, one always has the connecting homomorphism $\partial_n(K(E))H_n(k_C)$. But in general $H_{n-1}(\varkappa)\partial_n(K(E))H_n(k_C)\neq 0$. Moreover, if neither (i) nor (ii) holds, Theorem 2.5 need not hold even in the case when $H_{n-1}(\varkappa)\partial_n(K(E))H_n(k_C)=0$ and $H_n(\sigma)$ is defined for all n (see 1.16). Indeed, consider the following diagram

$$\begin{tikzcd}
G: \arrow[d, tail, "\varkappa"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, tail] & 0 \arrow[r] \arrow[d, tail] & 0 \arrow[r] \arrow[d, tail] & 0 \arrow[r] \arrow[d, tail] & 0 \arrow[r] \arrow[d, tail] & \cdots \\
B: \arrow[d, tail, "\sigma"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, tail] & M \arrow[r, "1"] \arrow[d, tail, "1"'] & M \arrow[r, "0"] \arrow[d, tail, "1"'] & M \arrow[r] \arrow[d, tail, "1"'] & 0 \arrow[r] \arrow[d, tail] & \cdots \\
C: & \cdots \arrow[r, shift left] \arrow[r, shift right] & 0 \arrow[r, shift left] \arrow[r, shift right] & M \arrow[r, shift left, "1+1"] \arrow[r, shift right, "1"'] & M \arrow[r, shift left, "1"] \arrow[r, shift right, "1"'] & M \arrow[r, shift left] \arrow[r, shift right] & 0 \arrow[r, shift left] \arrow[r, shift right] & \cdots
\end{tikzcd}$$

in which M is an abelian monoid and $1 = 1_M$. Clearly, $\sigma = (\dots, 0, 1, 1, 1, 0, \dots)$ is a morphism of chain complexes (see 1.14 and 1.11). One can easily see that the long homology sequence associated to this diagram coincides with the sequence

$$\cdots \longrightarrow 0 \longrightarrow E(M) \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow$$

$$\longrightarrow 0 \longrightarrow M \xrightarrow{\ k\ } M' \longrightarrow 0 \longrightarrow \cdots$$

where $E(M)$ is the monoid of all idempotents of M, M' denotes the largest cancellative homomorphic image of M, and k is the canonical homomorphism. ($M' = M/\!\sim$, $m_1 \sim m_2$, $m_1, m_2 \in M \Leftrightarrow m_1 + m = m_2 + m$, $m \in M$. $\mathrm{cl}_\sim(m_1) + \mathrm{cl}_\sim(m_2) = \mathrm{cl}_\sim(m_1 + m_2)$, $k(m) = \mathrm{cl}_\sim(m)$.) Let $\delta_1^+ = 1 + 1$. Hence $H_1(C) = E(M)$, $H_{-1}(B) = M$, $H_{-1}(C) = M'$ and $H_{-1}(\sigma) = k$. If $E(M) \neq 0$, then this sequence is not exact at $H_1(C)$ as well as at $H_{-1}(B)$.

2.7. Example. Let $f = \{f_n\} : X = \{X_n, \partial_n^+, \partial_n^-\} \longrightarrow X' = \{X'_n, {\partial'}_n^+, {\partial'}_n^-\}$ be a morphism of chain complexes. The *mapping cone* of f is the chain complex

$$C_f = \big\{(C_f)_n, d_n^+, d_n^-\big\}, \quad (C_f)_n = X_{n-1} \oplus X'_n,$$

$$d_n^+(x, x') = \big(\partial_{n-1}^-(x), {\partial'}_n^+(x') + f_{n-1}(x)\big), \quad d_n^-(x, x') = \big(\partial_{n-1}^+(x), {\partial'}_n^-(x')\big).$$

There is a Schreier short exact sequence of chain complexes and their $\pm$-morphisms

$$\begin{tikzcd}
& X': \cdots \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail, "i_f"] & X'_{n+1} \arrow[r, shift left, "{\partial'}_{n+1}^+"] \arrow[r, shift right, "{\partial'}_{n+1}^-"'] \arrow[d, tail, "(i_f)_{n+1}"] & X'_n \arrow[r, shift left, "{\partial'}_n^+"] \arrow[r, shift right, "{\partial'}_n^-"'] \arrow[d, tail, "(i_f)_n"] & X'_{n-1} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail, "(i_f)_{n-1}"] & \cdots \\
E_f: & C_f: \cdots \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads, "p_f"] & X_n \oplus X'_{n+1} \arrow[r, shift left, "d_{n+1}^+"] \arrow[r, shift right, "d_{n+1}^-"'] \arrow[d, two heads, "(p_f)_{n+1}"] & X_{n-1} \oplus X'_n \arrow[r, shift left, "d_n^+"] \arrow[r, shift right, "d_n^-"'] \arrow[d, two heads, "(p_f)_n"] & X_{n-2} \oplus X'_{n-1} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads, "(p_f)_{n-1}"] & \cdots \\
& X[-1]: \cdots \arrow[r, shift left] \arrow[r, shift right] & X_n \arrow[r, shift left, "\partial_n^-"] \arrow[r, shift right, "\partial_n^+"'] & X_{n-1} \arrow[r, shift left, "\partial_{n-1}^-"] \arrow[r, shift right, "\partial_{n-1}^+"'] & X_{n-2} \arrow[r, shift left] \arrow[r, shift right] & \cdots
\end{tikzcd}$$

where $(i_f)_n$ sends x' to $(0, x')$, and $(p_f)_n$ sends (x, x') to x. An element (x, x') of

$X_{n-1} \oplus X'_n$ is a representative of $(E_f)_n$ if and only if $x' \in U(X'_n)$, where $U(X'_n)$ denotes the maximal Λ-submodule of X'_n. Therefore each d_n^- obviously preserves representatives. Assume that X' is a chain complex of cancellative Λ-semimodules. Then, by Proposition 2.1, we have the long homology sequence

$$H(E_f): \quad \cdots \to H_n(X') \xrightarrow{H_n(i_f)} H_n(C_f) \xrightarrow{H_n(p_f)} H_n(X[-1]) \xrightarrow{\partial_n(E_f)}$$

$$\xrightarrow{\partial_n(E_f)} H_{n-1}(X') \xrightarrow{\quad H_{n-1}(i_f) \quad} H_{n-1}(C_f) \longrightarrow \cdots$$

associated to E_f. One can easily see that in fact $H_n(X[-1]) = H_{n-1}(X)$, and $\partial_n(E_f) = H_{n-1}(f)$. Furthermore, Theorems 2.4 and 2.5 together with 1.7 imply the following

Corollary. *Let $f = \{f_n\} : X = \{X_n, \partial_n^+, \partial_n^-\} \longrightarrow X' = \{X'_n, \partial_n'^+, \partial_n'^-\}$ be a morphism of chain complexes and suppose that one of the following holds:*

(i) For each n, X'_n is a cancellative Λ-semimodule, X_n a Λ-module, and $H_n(p_f)(H_n(C_f)) = H_n(p_{K(f)})(H_n(C_{K(f)}))$.

(ii) For each n, X'_n is a Λ-module and $H_n(f)(H_n(X)) = H_n(K(f))(H_n(K(X)))$.

Then $H(E_f)$ is exact everywhere, and $H_n(i_f)$, $H_n(p_f)$ and $\partial_n(E_f)(= H_{n-1}(f))$ are normal Λ-homomorphisms for all n (see 2.3).

Proof. Suppose (i) holds. Since K commutes with mapping cones, it follows easily that $H_n(p_f)(H_n(C_f)) = H_n(K(p_f))(H_n(K(C_f)))$ for all n. Therefore, thinking of $p_f = \{(p_f)_n\}$ as a morphism from C_f to $\{X_{n-1}, \partial_{n-1}^- - \partial_{n-1}^+\}$, we conclude, by 2.4, that $H(E_f)$ is exact everywhere and $H_n(i_f)$ is normal. By 1.7, $H_n(p_f)$ and $\partial_n(E_f)$ are also normal. When (ii) holds, the assertion is clear since $H_{n-1}(f) = \partial_n(E_f) = \partial_n(K(E_f))H_n(k_{X[-1]})$ and $H_{n-1}(K(f)) = \partial_n(E_{K(f)}) = \partial_n(K(E_f))$ (see 2.5 and 1.7). □

In subsequent papers we shall give applications of 2.4, 2.5 and 2.7.

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