

Journal of Inequalities in Pure and Applied Mathematics

SEVERAL INTEGRAL INEQUALITIES

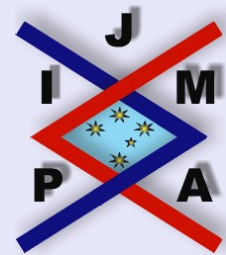
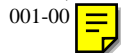
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Abstract

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Abstract

In the article, some integral inequalities are presented by analytic approach and mathematical induction. An open problem is proposed.

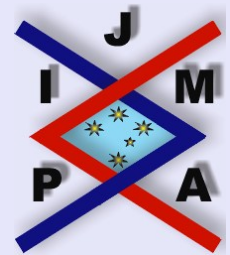
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1. Several Integral Inequalities

In this article, we establish some integral inequalities by analytic method and induction.

Proposition 1.1. *Let $f(x)$ be differentiable on (a, b) and $f(a) = 0$. If $0 \leq f'(x) \leq 1$, then*

$$(1.1) \quad \int_a^b [f(x)]^3 dx \leq \left(\int_a^b f(x) dx \right)^2.$$

If $f'(x) \geq 1$, then inequality (1.1) reverses. The equality in (1.1) holds only if $f(x) \equiv 0$ or $f(x) = x - a$.

Proof. For $a \leq t \leq b$, set

$$F(t) = \left(\int_a^t f(x) dx \right)^2 - \int_a^t [f(x)]^3 dx.$$

Simple computation yields

$$F'(t) = \left\{ 2 \int_a^t f(x) dx - [f(t)]^2 \right\} f(t) \triangleq G(t)f(t),$$
$$G'(t) = 2[1 - f'(t)]f(t).$$

Since $f'(t) \geq 0$ and $f(a) = 0$, thus $f(t)$ is increasing and $f(t) \geq 0$.

- (1) When $0 \leq f'(t) \leq 1$, we have $G'(t) \geq 0$, $G(t)$ increases and $G(t) \geq 0$ because of $G(a) = 0$, hence $F'(t) = G(t)f(t) \geq 0$, $F(t)$ is increasing. Since $F(a) = 0$, we have $F(t) \geq 0$, and $F(b) \geq 0$. Therefore, the inequality (1.1) holds.



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(2) When $f'(t) \geq 1$, we have $G'(t) \leq 0$, $G(t)$ decreases, $G(t) \leq 0$, $F'(t) \leq 0$, and $F(t)$ is decreasing, then $F(t) \leq 0$, the inequality (1.1) reverses.

(3) Since the equality in (1.1) holds only if $f'(t) = 1$ or $f(t) = 0$, substitution of $f(t) = t + c$ into (1.1) and standard argument leads to $c = -a$.

The proof is completed. \square

Corollary 1.2. [3, p. 624] Let $f(x)$ be a continuous function on the closed interval $[0, 1]$ and $f(0) = 0$, its derivative of the first order is bounded by $0 \leq f'(x) \leq 1$ for $x \in (0, 1)$. Then

$$(1.2) \quad \int_0^1 [f(x)]^3 dx \leq \left(\int_0^1 f(x) dx \right)^2.$$

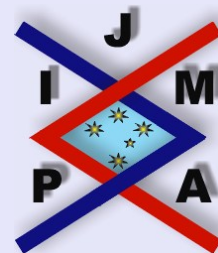
Equality in (1.2) holds if and only if $f(x) = 0$ or $f(x) = x$.

Proposition 1.3. Suppose $f(x)$ has continuous derivative of the n -th order on the interval $[a, b]$, $f^{(i)}(a) \geq 0$ and $f^{(n)}(x) \geq n!$, where $0 \leq i \leq n - 1$, then

$$(1.3) \quad \int_a^b [f(x)]^{n+2} dx \geq \left(\int_a^b f(x) dx \right)^{n+1}.$$

Proof. Let

$$(1.4) \quad H(t) = \int_a^t [f(x)]^{n+2} dx - \left[\int_a^t f(x) dx \right]^{n+1}, \quad t \in [a, b].$$



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Direct calculation produces

$$H'(t) = \left\{ [f(x)]^{n+1} - (n+1) \left[\int_a^t f(x) dx \right]^n \right\} f(t) \triangleq h_1(t)f(t),$$

$$h'_1(t) = (n+1) \left\{ [f(x)]^{n-1} f'(t) - n \left[\int_a^t f(x) dx \right]^{n-1} \right\} f(t) \triangleq (n+1)h_2(t)f(t),$$

$$h'_2(t) = \left\{ [f(x)]^{n-2} f''(t) + (n-1) [f(t)]^{n-3} [f'(t)]^2 - n(n-1) \left[\int_a^t f(x) dx \right]^{n-2} \right\} f(t) \triangleq h_3(t)f(t).$$

By induction, we obtain

$$(1.5) \quad h'_i(t) = \left\{ f^{(i)}(t) [f(t)]^{n-i} + p_i(t) - \frac{n!}{(n-i)!} \left[\int_a^t f(x) dx \right]^{n-i} \right\} f(t) \triangleq h_{i+1}(t)f(t),$$

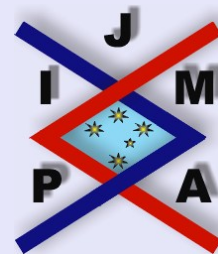
where $2 \leq i \leq n$ and

$$(1.6) \quad \begin{aligned} p_2(t) &= (n-1) [f(t)]^{n-3} [f'(t)]^2, \\ p_{i+1}(t)f(t) &= p'_i(t) + (n-i)f^{(i)}(t) [f(t)]^{n-i-1} f'(t). \end{aligned}$$

From $f^{(n)}(t) \geq n!$ and $f^{(i)}(a) \geq 0$ for $0 \leq i \leq n-1$, it follows that $f^{(i)}(t) \geq 0$ and are increasing for $0 \leq i \leq n-1$.

Using mathematical induction, it is easy to see that

$$p_i(t) = \sum_{j_0 + \sum_{k=1}^{i-1} k \cdot j_k = n-1} C(j_0, j_1, \dots, j_{i-1}) \prod_{k=0}^{i-1} [f^{(k)}(t)]^{j_k},$$



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where j_k and $C(j_0, j_1, \dots, j_{i-1})$ are nonnegative integers, $0 \leq k \leq i - 1$.

Therefore, we obtain $p'_k(t) \geq 0$ and $p_{k+1}(t) \geq 0$, then $p'_{k-1}(t)$ and $p_k(t)$ are increasing for $2 \leq k \leq n$. Straightforward computation yields

$$h_{n+1}(t) = f^{(n)}(t) + p_n(t) - n!$$

Considering $f^{(n)}(t) \geq n!$, we get $h_{n+1}(t) \geq 0$, and $h'_n(t) \geq 0$, then $h_n(t)$ increases.

By our definitions of $h_i(t)$, we have, for $1 \leq i \leq n - 1$,

$$h_{i+1}(a) = f^{(i)}(a) [f(a)]^{n-i} + p_i(a) \geq 0.$$

Therefore, using induction on i , we obtain $h'_i(t) \geq 0$, $h_i(t) \geq 0$, and $h_i(t)$ are increasing for $1 \leq i \leq n$. Then $H'(t) \geq 0$ and increases, and $H(t) \geq 0$. The inequality (1.3) follows from $H(b) \geq 0$. Thus, Proposition 1.3 is proved. \square

Corollary 1.4. *Let $f(x)$ be n -times differentiable on $[a, b]$, $f^{(i)}(a) \geq 0$ and $f^{(n)}(x) \geq n!$ for $0 \leq i \leq n - 1$. Then the functions $H(t)$, $h_j(t)$ and $p_k(t)$ defined by the formulae (1.4), (1.5) and (1.6) are increasing and convex, where $1 \leq j \leq n - 1$ and $2 \leq k \leq n - 2$.*

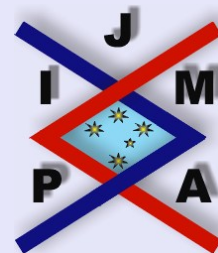
Remark 1.1. The inequality (1.3) is not found in [1, 2, 4, 5]. So maybe it is a new inequality.

Lastly, we propose the following open problem:

Theorem 1.5 (Open Problem). *Under what conditions does the inequality*

$$(1.7) \quad \int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x) dx \right)^{t-1}$$

hold for $t > 1$?



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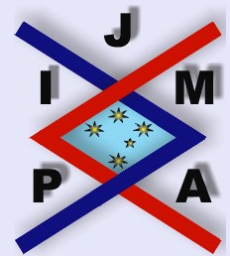
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