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SEVERAL INTEGRAL INEQUALITIES

FENG QI

Department of Mathematics
Jiaozuo Institute of Technology
Jiaozuo City, Henan 454000
THE PEOPLE'S REPUBLIC OF CHINA
EMail: qifeng@jzit.edu.cn
URL: http://rgmia.vu.edu.au/qi.html

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Abstract

In the article, some integral inequalities are presented by analytic approach and mathematical induction. An open problem is proposed.

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Contents

1. Several Integral Inequalities

In this article, we establish some integral inequalities by analytic method and induction.

Proposition 1.1. *Let* $f(x)$ *be differentiable on* (a,b) *and* $f(a)=0$. *If* $0\leqslant f'(x)\leqslant 1$, *then*

$$\int_a^b [f(x)]^3\,\mathrm{d}x \leqslant \left(\int_a^b f(x)\,\mathrm{d}x\right)^2. \tag{1.1}$$

If $f'(x)\geqslant 1$, *then inequality* (1.1) *reverses. The equality in* (1.1) *holds only if* $f(x)\equiv 0$ *or* $f(x)=x-a$.

Proof. For $a\leqslant t\leqslant b$, set

$$F(t)=\left(\int_a^t f(x)\,\mathrm{d}x\right)^2-\int_a^t [f(x)]^3\,\mathrm{d}x.$$

Simple computation yields

$$F'(t)=\left\{2\int_a^t f(x)\,\mathrm{d}x-[f(t)]^2\right\}f(t)\triangleq G(t)f(t),$$
$$G'(t)=2[1-f'(t)]f(t).$$

Since $f'(t)\geqslant 0$ and $f(a)=0$, thus $f(t)$ is increasing and $f(t)\geqslant 0$.

(1) When $0\leqslant f'(t)\leqslant 1$, we have $G'(t)\geqslant 0$, $G(t)$ increases and $G(t)\geqslant 0$ because of $G(a)=0$, hence $F'(t)=G(t)f(t)\geqslant 0$, $F(t)$ is increasing. Since $F(a)=0$, we have $F(t)\geqslant 0$, and $F(b)\geqslant 0$. Therefore, the inequality (1.1) holds.

(2) When $f'(t) \geqslant 1$, we have $G'(t) \leqslant 0$, $G(t)$ decreases, $G(t) \leqslant 0$, $F'(t) \leqslant 0$, and $F(t)$ is decreasing, then $F(t) \leqslant 0$, the inequality (1.1) reverses.

(3) Since the equality in (1.1) holds only if $f'(t) = 1$ or $f(t) = 0$, substitution of $f(t) = t + c$ into (1.1) and standard argument leads to $c = -a$.

The proof is completed. □

Corollary 1.2. *[3, p. 624] Let $f(x)$ be a continuous function on the closed interval $[0, 1]$ and $f(0) = 0$, its derivative of the first order is bounded by $0 \leqslant f'(x) \leqslant 1$ for $x \in (0, 1)$. Then*

$$\int_0^1 [f(x)]^3 \, dx \leqslant \left(\int_0^1 f(x) \, dx \right)^2. \tag{1.2}$$

Equality in (1.2) holds if and only if $f(x) = 0$ or $f(x) = x$.

Proposition 1.3. *Suppose $f(x)$ has continuous derivative of the n-th order on the interval $[a, b]$, $f^{(i)}(a) \geqslant 0$ and $f^{(n)}(x) \geqslant n!$, where $0 \leqslant i \leqslant n - 1$, then*

$$\int_a^b [f(x)]^{n+2} \, dx \geqslant \left(\int_a^b f(x) \, dx \right)^{n+1}. \tag{1.3}$$

Proof. Let

$$H(t) = \int_a^t [f(x)]^{n+2} \, dx - \left[\int_a^t f(x) \, dx \right]^{n+1}, \quad t \in [a, b]. \tag{1.4}$$

Direct calculation produces

$$H'(t) = \left\{ \left[f(x) \right]^{n+1} - (n+1) \left[\int_a^t f(x)\,\mathrm{d}x \right]^n \right\} f(t) \triangleq h_1(t) f(t),$$

$$h_1'(t) = (n+1) \left\{ \left[f(x) \right]^{n-1} f'(t) - n \left[\int_a^t f(x)\,\mathrm{d}x \right]^{n-1} \right\} f(t) \triangleq (n+1) h_2(t) f(t),$$

$$h_2'(t) = \left\{ \left[f(x) \right]^{n-2} f''(t) + (n-1) \left[f(t) \right]^{n-3} \left[f'(t) \right]^2 - n(n-1) \left[\int_a^t f(x)\,\mathrm{d}x \right]^{n-2} \right\} f(t) \triangleq h_3(t) f(t).$$

By induction, we obtain

(1.5)
$$h_i'(t) = \left\{ f^{(i)}(t) \left[f(t) \right]^{n-i} + p_i(t) - \frac{n!}{(n-i)!} \left[\int_a^t f(x)\,\mathrm{d}x \right]^{n-i} \right\} f(t) \triangleq h_{i+1}(t) f(t),$$

where $2 \leqslant i \leqslant n$ and

(1.6)
$$p_2(t) = (n-1) \left[f(t) \right]^{n-3} \left[f'(t) \right]^2,$$
$$p_{i+1}(t) f(t) = p_i'(t) + (n-i) f^{(i)}(t) \left[f(t) \right]^{n-i-1} f'(t).$$

From $f^{(n)}(t) \geqslant n!$ and $f^{(i)}(a) \geqslant 0$ for $0 \leqslant i \leqslant n-1$, it follows that $f^{(i)}(t) \geqslant 0$ and are increasing for $0 \leqslant i \leqslant n-1$.

Using mathematical induction, it is easy to see that

$$p_i(t) = \sum_{j_0 + \sum_{k=1}^{i-1} k \cdot j_k = n-1} C(j_0, j_1, \dots, j_{i-1}) \prod_{k=0}^{i-1} \left[f^{(k)}(t) \right]^{j_k},$$

where j_k and $C(j_0, j_1, \dots, j_{i-1})$ are nonnegative integers, $0 \leqslant k \leqslant i-1$.

Therefore, we obtain $p'_k(t) \geqslant 0$ and $p_{k+1}(t) \geqslant 0$, then $p'_{k-1}(t)$ and $p_k(t)$ are increasing for $2 \leqslant k \leqslant n$. Straightforward computation yields

$$h_{n+1}(t) = f^{(n)}(t) + p_n(t) - n!.$$

Considering $f^{(n)}(t) \geqslant n!$, we get $h_{n+1}(t) \geqslant 0$, and $h'_n(t) \geqslant 0$, then $h_n(t)$ increases.

By our definitions of $h_i(t)$, we have, for $1 \leqslant i \leqslant n-1$,

$$h_{i+1}(a) = f^{(i)}(a)\big[f(a)\big]^{n-i} + p_i(a) \geqslant 0.$$

Therefore, using induction on i, we obtain $h'_i(t) \geqslant 0$, $h_i(t) \geqslant 0$, and $h_i(t)$ are increasing for $1 \leqslant i \leqslant n$. Then $H'(t) \geqslant 0$ and increases, and $H(t) \geqslant 0$. The inequality (1.3) follows from $H(b) \geqslant 0$. Thus, Proposition 1.3 is proved. □

Corollary 1.4. *Let $f(x)$ be n-times differentiable on $[a,b]$, $f^{(i)}(a) \geqslant 0$ and $f^{(n)}(x) \geqslant n!$ for $0 \leqslant i \leqslant n-1$. Then the functions $H(t)$, $h_j(t)$ and $p_k(t)$ defined by the formulae* (1.4), (1.5) *and* (1.6) *are increasing and convex, where $1 \leqslant j \leqslant n-1$ and $2 \leqslant k \leqslant n-2$.*

Remark 1.1. The inequality (1.3) is not found in [1, 2, 4, 5]. So maybe it is a new inequality.

Lastly, we propose the following open problem:

Theorem 1.5 (Open Problem). *Under what conditions does the inequality*

$$\int_a^b \big[f(x)\big]^t \,\mathrm{d}x \geqslant \left(\int_a^b f(x)\,\mathrm{d}x\right)^{t-1} \tag{1.7}$$

hold for $t > 1$?

References

[1] E. F. BECKENBACH AND R. BELLMAN, *Inequalities*, Springer, Berlin, 1983.

[2] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, 2nd edition, Cambridge University Press, Cambridge, 1952.

[3] JI-CHANG KUANG, *Applied Inequalities*, 2nd edition, Hunan Education Press, Changsha, China, 1993. (Chinese)

[4] D.S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[5] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.