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SUPERQUADRACITY OF FUNCTIONS AND REARRANGEMENTS OF SETS

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ABSTRACT. In this paper we establish upper bounds of

$$\sum_{i=1}^{n}\left(f\left(\frac{x_i+x_{i+1}}{2}\right)+f\left(\frac{|x_i-x_{i+1}|}{2}\right)\right),\qquad x_{n+1}=x_1$$

when the function f is superquadratic and the set $(\mathbf{x})=(x_1,\ldots,x_n)$ is given except its arrangement.

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1. INTRODUCTION

We start with the definitions and results of [1] and [5] which we use in this paper.

Definition 1.1. The sets $(\mathbf{y}^-)=\left(y_1^-,\ldots,y_n^-\right)$ and $(^-\mathbf{y})=(^-y_1,\ldots,^-y_n)$ are symmetrically decreasing rearrangements of an ordered set $(\mathbf{y})=(y_1,\ldots,y_n)$ of n real numbers, if

$$y_1^- \le y_n^- \le y_2^- \le \cdots \le y^-_{\left[\frac{n+2}{2}\right]} \tag{1.1}$$

and

$$^-y_n \le {}^-y_1 \le {}^-y_{n-1} \le \cdots \le {}^-y_{\left[\frac{n+1}{2}\right]}. \tag{1.2}$$

A circular rearrangement of an ordered set $(\mathbf{y})=(y_1,\ldots,y_n)$ is a cyclic rearrangement of $(\mathbf{y})$ or a cyclic rearrangement followed by inversion.

Definition 1.2. An ordered set $(\mathbf{y})=(y_1,\ldots,y_n)$ of n real numbers is arranged in circular symmetric order if one of its circular rearrangements is symmetrically decreasing.

Theorem A ([1]). *Let $F(u,v)$ be a symmetric function defined for $\alpha\le u,v\le\beta$ for which* $\frac{\partial^2F(u,v)}{\partial u\partial v}\ge 0$.

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Let the set $(\mathbf{y}) = (y_1, \ldots, y_n)$, $\alpha \leq y_i \leq \beta$, $i = 1, \ldots, n$ *be given except its arrangement. Then*

$$\sum_{i=1}^{n} F(y_i, y_{i+1}), \qquad (y_{n+1} = y_1)$$

is maximal if $(\mathbf{y})$ *is arranged in circular symmetrical order.*

Definition 1.3 ([5])**.** A function f, defined on an interval $I = [0, L]$ or $[0, \infty)$ is superquadratic, if for each x in I, there exists a real number $C(x)$ such that

$$f(y) - f(x) \geq C(x)(y - x) + f(|y - x|)$$

for all $y \in I$.

A function is subquadratic if $-f$ is superquadratic.

Lemma A ([5])**.** *Let f be a superquadratic function with $C(x)$ as in Definition 1.3.*

(i) *Then* $f(0) \leq 0$.
(ii) *If* $f(0) = f'(0) = 0$, *then* $C(x) = f'(x)$ *whenever* f *is differentiable.*
(iii) *If* $f \geq 0$, *then* f *is convex and* $f(0) = f'(0) = 0$.

The following lemma presents a Jensen's type inequality for superquadratic functions.

Lemma B ([6, Lemma 2.3])**.** *Suppose that f is superquadratic. Let* $x_r \geq 0$, $1 \leq r \leq n$ *and let* $\overline{x} = \sum_{r=1}^{n} \lambda_r x_r$, *where* $\lambda_r \geq 0$, *and* $\sum_{r=1}^{n} \lambda_r = 1$. *Then*

$$\sum_{r=1}^{n} \lambda_r f(x_r) \geq f(\overline{x}) + \sum_{r=1}^{n} \lambda_r f(|x_r - \overline{x}|).$$

If $f(x)$ *is subquadratic, the reverse inequality holds.*

From Lemma B we get an immediate result which we state in the following lemma.

Lemma C. *Let* $f(x)$ *be superquadratic on* $[0, L]$ *and let* $x, y \in [0, L]$, $0 \leq \lambda \leq 1$, *then*

$$\begin{aligned} &\lambda f(x) + (1-\lambda) f(y) \\ &\qquad \geq f(\lambda x + (1-\lambda) y) + \lambda f((1-\lambda)|y-x|) + (1-\lambda) f(\lambda |y-x|) \\ &\qquad \geq f(\lambda x + (1-\lambda) y) + \sum_{k=0}^{t-1} \left(f\left(2\lambda(1-\lambda)|1-2\lambda|^k |x-y|\right)\right) \\ &\qquad\quad + \lambda f\left((1-\lambda)|1-2\lambda|^t |x-y|\right) + (1-\lambda) f\left(\lambda |1-2\lambda|^t |x-y|\right). \end{aligned}$$

If f is positive superquadratic we get that:

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y) + \sum_{k=0}^{t-1} \left(f\left(2\lambda(1-\lambda)|1-2\lambda|^k |x-y|\right)\right)$$

More results related to superquadracity were discussed in [2] *to* [6].

In this paper we refine the results in [7] by showing that for positive superquadratic functions we get better bounds than in [7].

Theorem B ([7, Thm. 1.2])**.** *If f is a convex function and* $x_1, x_2, \ldots, x_n$ *lie in its domain, then*

$$\begin{aligned} &\sum_{i=1}^{n} f(x_i) - f\left(\frac{x_1 + \cdots + x_n}{n}\right) \\ &\qquad \geq \frac{n-1}{n}\left[f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right) + f\left(\frac{x_n + x_1}{2}\right)\right]. \end{aligned}$$

Theorem C ([7, Thm. 1.4]). *If f is a convex function and $a_1, \dots, a_n$ lie in its domain, then*

$$(n-1)\left[f(b_1)+\cdots+f(b_n)\right] \le n\left[f(a_1)+\cdots+f(a_n)-f(a)\right],$$

where $a = \frac{a_1+\cdots+a_n}{n}$ and $b_i = \frac{na-a_i}{n-1},\ i=1,\dots,n$.

2. THE MAIN RESULTS

Theorem 2.1. *Let $f(x)$ be a superquadratic function on $[0,L]$. Then for $x_i \in [0,L],\ i = 1,\dots,n$, where $x_{n+1} = x_1$,*

$$\text{(2.1)}\quad \frac{n-1}{n}\sum_{i=1}^{n}\left(f\left(\sum_{i=1}^{n}\frac{x_i+x_{i+1}}{2}\right)+f\left(\sum_{i=1}^{n}\frac{|x_i-x_{i+1}|}{2}\right)\right) \le \left(\sum_{i=1}^{n}f(x_i)\right)-f\left(\sum_{i=1}^{n}\frac{x_i}{n}\right)-\frac{1}{n}\sum_{i=1}^{n}f\left(\left|x_i-\sum_{j=1}^{n}\frac{x_j}{n}\right|\right)$$

holds. If $f'''(x) \ge 0$ too, then

$$\text{(2.2)}\quad \frac{n-1}{n}\sum_{i=1}^{n}\left(f\left(\sum_{i=1}^{n}\frac{x_i+x_{i+1}}{2}\right)+f\left(\sum_{i=1}^{n}\frac{|x_i-x_{i+1}|}{2}\right)\right) \le \frac{n-1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+f\left(\frac{|\widehat{x}_i-\widehat{x}_{i+1}|}{2}\right)\right) \le \left(\sum_{i=1}^{n}f(x_i)\right)-f\left(\sum_{i=1}^{n}\frac{x_i}{n}\right)-\frac{1}{n}\sum_{i=1}^{n}f\left(\left|x_i-\sum_{j=1}^{n}\frac{x_j}{n}\right|\right),$$

where $(\widehat{\mathbf{x}}) = (\widehat{x}_1,\dots,\widehat{x}_n)$ is a circular symmetrical rearrangement of $(\mathbf{x}) = (x_1,\dots,x_n)$.

Example 2.1. *The functions*

$$f(x) = x^n, \qquad n \ge 2, \qquad x \ge 0,$$

and the function

$$f(x) = \begin{cases} x^2 \log x, & x > 0, \\ 0, & x = 0 \end{cases}$$

are superquadratic with an increasing second derivative and therefore (2.2) holds for these functions.

Proof. Let f be a superquadratic function on $[0,L]$. Then by Lemma B we get for $0 \le \alpha \le 1,\quad 1 \le k \le n$ and $x_i \in [0,L]$, $x_{n+1} = x_1$,

$$\text{(2.3)}\quad \sum_{i=1}^{n}f(x_i) = \frac{n-k}{n}\sum_{i=1}^{n}f(x_i)+\frac{k}{n}\sum_{i=1}^{n}f(x_i) = \frac{n-k}{n}\sum_{i=1}^{n}\left(\alpha f(x_i)+(1-\alpha)f(x_{i+1})\right)+\frac{k}{n}\sum_{i=1}^{n}f(x_i)$$

$$\geq \frac{n-k}{n}\sum_{i=1}^{n} f\left(\alpha x_i + (1-\alpha)\, x_{i+1}\right)$$
$$+\frac{n-k}{n}\sum_{i=1}^{n}\left(\alpha f\left((1-\alpha)\left|x_{i+1}-x_i\right|\right)+(1-\alpha)\, f\left(\alpha\left|x_{i+1}-x_i\right|\right)\right)$$
$$+k\left(f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)+\sum_{i=1}^{n}\frac{1}{n} f\left(\left|x_i-\frac{\sum_{i=1}^{n} x_i}{n}\right|\right)\right).$$

For $k=1$ and $\alpha=\frac{1}{2}$ we get that (2.1) holds.

If $f'''(x)\geq 0$, then $\frac{\partial^2 F(u,v)}{\partial u \partial v}\geq 0$, where

$$F(u,v)=f(u+v)+f(|u-v|)\,, \qquad u,v\in[0,L]\,.$$

Therefore according to Theorem A, the sum

$$\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}}{2}\right)+f\left(\frac{|x_i+x_{i+1}|}{2}\right), \qquad x_{n+1}=x_1,$$

is maximal for $(\widehat{\mathbf{x}})=(\widehat{x}_1,\ldots,\widehat{x}_n)$, which is the circular symmetric rearrangement of $(\mathbf{x})$. Therefore in this case (2.2) holds as well. □

Remark 2.2. For a positive superquadratic function f, which according to Lemma A is also a convex function, (2.1) is a refinement of Theorem B.

If $f'''(x)\geq 0$, (2.2) is a refinement of Theorem B as well.

Remark 2.3. Theorem B is refined by

$$\sum_{i=1}^{n} f(x_i)-f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\geq\frac{n-1}{n}\left(\sum_{i=1}^{n} f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)\right)$$
$$\geq\frac{n-1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}}{2}\right),$$

because a convex function f satisfies the conditions of Theorem A for $F(u,v)=f(u+v)$.

The following inequality is a refinement of Theorem C for a positive superquadratic function f, which is therefore also convex. The inequality results easily from Lemma B and the identity

$$\sum_{i=1}^{n} f(a_i)=\sum_{i=1}^{n}\left(\frac{1}{n-1}\sum_{j=1}^{n} f(a_j)(1-\delta_{ij})\right)$$

(where $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$), therefore the proof is omitted.

Theorem 2.4. *Let f be a superquadratic function on $[0,L]$, and let $x_i\in[0,L]$, $i=1,\ldots,n$. Then*

$$\frac{n}{n-1}\left(\left(\sum_{i=1}^{n} f(x_i)\right)-f(\overline{x})\right)-\sum_{i=1}^{n} f(y_i)$$
$$\geq\frac{1}{n-1}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} f(|y_i-x_j|)(1-\delta_{ij})\right)+\frac{1}{n-1}\sum_{i=1}^{n} f(|\overline{x}-x_i|)\,,$$

where $\overline{x}=\sum_{i=1}^{n}\frac{x_i}{n}$, $y_i=\left(\frac{n\overline{x}-x_i}{n-1}\right)$, $i=1,\ldots,n$.

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