journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 2, Article 54, 15 pp.

CHEBYSHEV INEQUALITIES AND SELF-DUAL CONES

ZDZISŁAW OTACHEL

DEPARTMENT OF APPLIED MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF LIFE SCIENCES IN LUBLIN
AKADEMICKA 13, 20-950 LUBLIN, POLAND
zdzislaw.otachel@up.lublin.pl

Received 20 December, 2008; accepted 12 April, 2009
Communicated by S.S. Dragomir

ABSTRACT. The aim of this note is to give a general framework for Chebyshev inequalities and other classic inequalities. Some applications to Chebyshev inequalities are made. In addition, the relations of similar ordering, monotonicity in mean and synchronicity of vectors are discussed.

Key words and phrases: Chebyshev type inequality, Convex cone, Dual cone, Orthoprojector.

2000 *Mathematics Subject Classification.* 26D15, 26D20, 15A39.

1. INTRODUCTION AND SUMMARY

Let V be a real vector space provided with an inner product $\langle \cdot, \cdot \rangle$. For fixed $x \in V$ and $y, z \in V$ the inequality

$$\langle x, y\rangle \langle x, z\rangle \le \langle y, z\rangle \langle x, x\rangle \tag{1.1}$$

is called a *Chebyshev type inequality*.

A general method for finding vectors satisfying the above inequality is given by Niezgoda in [4]. The same author in [3] proved a projection inequality for the *Eaton system*, obtaining a *Chebyshev type inequality* as a particular case for orthoprojectors of rank one. Furthermore, the relation of synchronicity with respect to the *Eaton system* is introduced there. It generalizes commonly known relations of similarly ordered vectors (cf. for example, [6, chap. 7.1]).

This paper is organized as follows. Section 2 contains basic notions related to convex cones. In Section 3 a projection inequality in an abstract Hilbert space is studied. The framework covers the projection inequality for the Eaton system, Chebyshev sum and integral inequalities and others, see Examples 3.1 – 3.3. We modify and extend the applicability of the relation of synchronicity to vector spaces with infinite bases. The results are applied to the *Chebyshev sum inequality* in Section 4 and the *Chebyshev integral inequality* in Section 5.

001-09

2. PRELIMINARIES

In this note V is a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. A convex cone is a nonempty set $D \subset V$ such that $\alpha D + \beta D \subset D$ for all nonnegative scalars α and β. The closure of the convex cone of all nonnegative finite combinations in $H \subset V$ is denoted by $\operatorname{cone} H$. Similarly, $\operatorname{span} H$ denotes the closure of the subspace of all finite combinations in H. The dual cone of a subset $C \subset V$ is defined as follows

$$\operatorname{dual} C = \{v \in V : \langle v, C \rangle \geq 0\}.$$

It is known, that the dual cone of C is a closed convex cone and

$$\operatorname{dual} C = \operatorname{dual}(\operatorname{cone} C).$$

If for a subset $G \subset V$, a closed convex cone C is equal to $\operatorname{cone} G$, then we say that C is generated by G or G is a generator of C. The inclusion $A \subset B$ implies $\operatorname{dual} B \subset \operatorname{dual} A$. If C and D are convex cones, then

$$\operatorname{dual}(C + D) = \operatorname{dual} C \cap \operatorname{dual} D.$$

The dual cone of a subspace W is equal to its orthogonal complement $W^{\perp}$. If a set C is a closed convex cone, then

$$\operatorname{dual}\ \operatorname{dual} C = C,$$

(cf. [5, lemma 2.1]). The symbol $\operatorname{dual}_{V_1} C$ stands for $V_1 \cap \operatorname{dual} C$ and means the relative dual of C with respect to a closed subspace V_1 of V. If for a closed convex cone D the identity $\operatorname{dual}_{V_1} D = D$ holds, then D is called a self-dual cone w.r.t. V_1. For example, the convex cone generated by an orthogonal system of vectors is self-dual w.r.t. the subspace spanned by this system.

In other cases the standard mathematical notation is used.

3. PROJECTION INEQUALITY

From now on we make the following assumptions: P is an idempotent and symmetric operator (orthoprojector) defined on V, $V = V_1 + V_2$, where V_1 is the range of P and V_2 is its orthogonal complement, i.e. $V_1 = PV$ and $V_2 = (PV)^{\perp}$. The identity operator is denoted by id. All subspaces and convex cones of a real Hilbert space V are assumed to be closed.

For $y, z \in V$ we will consider a *projection inequality* (briefly (PI)) of the form

$$\langle y, Pz \rangle \geq 0.$$

If $y = z$, then (PI) holds for any orthoprojector P taking the form $\|Pz\|^2 \geq 0$. A general method of solution of (PI) is established by our following theorem (cf. [4, Theorem 3.1]).

Theorem 3.1. *For vectors $y, z \in V$ and a convex cone $C \subset V$ the following statements are mutually equivalent.*

- **i):** *(PI) holds for all $y \in C + V_2$*
- **ii):** $Pz \in \operatorname{dual} C$
- **iii):** $z \in \operatorname{dual} PC$.

Proof. Since i), the inequality (PI) holds for every $y \in C$. Thus

$$0 \leq \langle y, Pz \rangle = \langle Py, z \rangle .$$

Therefore $0 \leq \langle C, Pz \rangle = \langle PC, z \rangle$. Hence $Pz \in \operatorname{dual} C$ and $z \in \operatorname{dual} PC$. It proves that i) $\Rightarrow$ ii), iii).

Conversely, if $Pz \in \operatorname{dual} C$ then for $y = c + x$, where $c \in C$ and $\langle x, V_1 \rangle = 0$ are arbitrary have $\langle y, Pz \rangle = \langle c, Pz \rangle + \langle x, Pz \rangle = \langle c, Pz \rangle \geq 0$. By a similar argument, if $z \in \operatorname{dual} PC$ then

$y \in C + V_2$ implies that $Py \in PC$. It leads to $\langle y, Pz\rangle = \langle Py, z\rangle \geq 0$. From this we conclude that ii),iii) $\Rightarrow$ i), which completes the proof. □

Example 3.1 (Bessel inequality)**.** For an orthoprojector P the inequality (PI) holds provided that $y = z$. Let $\{f_\nu\}$ be an orthogonal system in V. If P is the orthoprojector onto the subspace orthogonal to $\operatorname{span}\{f_\nu\}$, i.e. $P = \mathrm{id} - \sum_\nu \frac{\langle \cdot, f_\nu\rangle}{\|f_\nu\|^2} f_\nu$, then we obtain the classic Bessel inequality

$$\|z\|^2 \geq \sum_\nu \frac{\langle z, f_\nu\rangle^2}{\|f_\nu\|^2}.$$

Example 3.2 (Chebyshev type inequalities)**.** Let $x \in V$ be a fixed nonzero vector. Set $P = \mathrm{id} - \frac{\langle \cdot, x\rangle}{\|x\|^2} x$. It is clear that P is the orthoprojector onto the subspace orthogonal to x. In the case where the inequality (PI) becomes a *Chebyshev type inequality* (1.1):

$$\langle x, z\rangle \langle y, x\rangle \leq \langle y, z\rangle \|x\|^2.$$

In the space $V = \mathbb{R}^n$ under $x = (1, \dots, 1)$, inequality (1.1) transforms into the *Chebyshev sum inequality* (or (CHSI) for short):

$$\sum_{i=1}^n y_i \sum_{i=1}^n z_i \leq n \sum_{i=1}^n y_i z_i.$$

Consider the space $V = L^2$ of all 2-nd power integrable functions with respect to the Lebesgue measure μ on the unit interval $[0, 1]$. For $x \equiv 1$ inequality (1.1) takes the form of a *Chebeshev integral inequality* (or (CHII) for short):

$$\int y d\mu \int z d\mu \leq \int yz d\mu.$$

Example 3.3 (Projection inequality for Eaton systems)**.** Let G be a closed subgroup of the orthogonal group acting on V, $\dim V < \infty$, and $C \subset V$ be a closed convex cone. Let us assume:

- **i):** for each vector $a \in V$ there exist $g \in G$ and $b \in C$ satisfying $a = gb$,
- **ii):** $\langle a, gb\rangle \leq \langle a, b\rangle$ for all $a, b \in C$ and $g \in G$.

If P is the orthoprojector onto a subspace orthogonal to $\{a \in V : Ga = a\}$, then the inequality (PI) holds, provided that $y, z \in C$, (cf. [3, Theorem 2.1]).

The triplet (V, G, C) fulfiling the conditions i)-ii) is said to be an *Eaton system*, (see e.g. [3] and the references given therein). The main example of this structure is the permutation group acting on $\mathbb{R}^n$ and the cone of nonincreasing vectors.

Let $C \subset V$ be a convex cone. Every cone of the form $C + V_2$ has the representation:

$$C + V_2 = PC + V_2. \tag{3.1}$$

Therefore, on studying the projection inequality (PI), according to Theorem 3.1, it is sufficient to consider convex cones of the form $C = D + V_2$, where D is a convex cone in V_1. The following proposition is a simple consequence of Theorem 3.1.

Proposition 3.2. *Let $D \subset V_1$ be a convex cone. For $y, z \in V$ the following conditions are equivalent.*

- **i):** *(PI) holds for all $y \in D + V_2$*
- **ii):** $z \in \operatorname{dual} D$.

Let $D \subset V_1$ be a convex cone. Then $V_2 \subset \operatorname{dual} D$. This implies that $P \operatorname{dual} D = V_1 \cap \operatorname{dual} D$. Applying (3.1) to $\operatorname{dual} D + V_2 = \operatorname{dual} D$, we get

$$\operatorname{dual}_{V_1} D + V_2 = \operatorname{dual} D. \tag{3.2}$$

According to the above equation and the last proposition, we need to find for (PI) such cones D for which $D \cap \operatorname{dual}_{V_1} D$ are as wide as possible.

Proposition 3.3. *The inequality (PI) holds for $y, z \in D + V_2$, where D is an arbitrary self-dual cone w.r.t. V_1.*

Proof. By assumption, $D \subset V_1$, hence (3.2) gives $\operatorname{dual} D = D + V_2$. Proposition 3.2 implies that (PI) holds for $y, z \in (D + V_2) \cap \operatorname{dual} D = D + V_2$. □

If D is a self-dual cone w.r.t. V_1 then $D + V_2$ is a maximal cone for (PI) in the following sense.

Proposition 3.4. *Let D be a self-dual cone w.r.t. V_1 with $D + V_2 \subset C$, where $C \subset V$ is a convex cone.*

If (PI) holds for $y, z \in C$ then $C = D + V_2$.

Proof. Since $V_2 \subset C$, (3.1) yields $C = PC + V_2$. By Proposition 3.2, (PI) holds for $y, z \in (PC + V_2) \cap \operatorname{dual} PC$. The assumption that (PI) holds for $y, z \in C$ gives $PC + V_2 \subset \operatorname{dual} PC$. Since $D + V_2 \subset C$, $D = P(D + V_2) \subset PC$. From this we have $\operatorname{dual} PC \subset \operatorname{dual} D = D + V_2$, by (3.2), because $\operatorname{dual}_{V_1} D = D$. Combining these inclusions we can see that $C = PC + V_2 \subset D + V_2$.

The converse inclusion holds by the hypothesis, and thus the proof is complete. □

Let G_P denote the set of all unitary operators acting on V with $gV_2 = V_2$. Notice that G_P is a group of operators. The inequality (PI) is invariant with respect to G_P.

Theorem 3.5. *For fixed $g \in G_P$ the following statements are equivalent.*

i): *(PI) holds for y, z*

ii): *(PI) holds for gy, gz.*

Proof. Assume that g is a unitary operator satisfying $gV_2 = V_2$. This is equivalent to $g^*V_2 = V_2$, where g^* is the adjoint operator of g. We first show that $gV_1 \subset V_1$.

Suppose, contrary to our claim, that there exists a $u \in V_1$ with the property $gu = v_1 + v_2,\ v_i \in V_i,\ i = 1, 2,\ v_2 \neq 0$. We have:

$$\|u\|^2 = \|gu\|^2 = \|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2 \quad (g - \text{unitary},\ v_1 \perp v_2),$$

$$\begin{aligned}\|u - g^*v_2\|^2 &= \|g(u - g^*v_2)\|^2 \\ &= \|gu - v_2\|^2 \\ &= \|v_1\|^2 \quad (\text{since } g - \text{unitary},\ g^*g = \text{id}),\end{aligned}$$

$$\begin{aligned}\|u - g^*v_2\|^2 &= \|u\|^2 + \|g^*v_2\|^2 \\ &= \|u\|^2 + \|v_2\|^2 \quad (u \perp g^*v_2,\ g^* - \text{unitary}).\end{aligned}$$

Hence:

$$\begin{cases} \|u\|^2 = \|v_1\|^2 + \|v_2\|^2 \\ \\ \|v_1\|^2 = \|u\|^2 + \|v_2\|^2 \end{cases} \Rightarrow \|v_2\|^2 = 0 \Rightarrow v_2 = 0,$$

a contradiction. This completes the proof of $gV_1 \subset V_1$.

Note that $g^*V_1 \subset V_1$, too. This implies that $V_1 \subset gV_1$. Therefore

$$gV_1 = V_1. \tag{3.3}$$

Now, let $z \in V$ be arbitrary. We have $z = z_1 + z_2$, where $z_i \in V_i$, $i = 1, 2$. For an orthoprojector P onto V_1 we get:

$$gPz = gP(z_1 + z_2) = gz_1 = P(gz_1 + gz_2) = Pgz,$$

because $gz_1 \in V_1$ by (3.3) and $gz_2 \in V_2$ by assumption. Thus

$$Pg = gP. \tag{3.4}$$

By (3.4),

$$\langle gy, Pgz\rangle = \langle gy, gPz\rangle = \langle g^*gy, Pz\rangle = \langle y, Pz\rangle\,.$$

This proves required equivalence. □

A simple consequence of the above theorem is:

Remark 1. For a convex cone $C \subset V$ and $g_0 \in G_P$ the following statements are equivalent.

i): (PI) holds for $y, z \in C$
ii): (PI) holds for $y, z \in g_0C$.

In the remainder of this section we assume that V is a real separable Hilbert space.
Let $\{f_\nu\}$ be an orthogonal basis of V_1, i.e.

$$\langle f_\eta, f_\nu\rangle \begin{cases} > 0, & \eta = \nu \\ = 0, & \eta \neq \nu, \end{cases}$$

for integers η, ν.
Under the above assumption, the projection Pz takes the form:

$$Pz = \sum_\nu \frac{\langle z, f_\nu\rangle}{\|f_\nu\|^2} f_\nu. \tag{3.5}$$

From this, for $y, z \in V$ we have

$$\langle y, Pz\rangle = \sum_\nu \frac{\langle y, f_\nu\rangle\,\langle z, f_\nu\rangle}{\|f_\nu\|^2}.$$

Therefore the following remark is evident.

Remark 2. Let $\{f_\nu\}$ be an orthogonal basis of V_1.
For $y, z \in V$ the inequality (PI) holds if and only if

$$\sum_\nu \frac{\langle y, f_\nu\rangle\,\langle z, f_\nu\rangle}{\|f_\nu\|^2} \geq 0.$$

Set

$$D = \left\{x \in V : x = \sum_\nu \alpha_\nu f_\nu, \alpha_\nu \geq 0\right\}. \tag{3.6}$$

Clearly, D is a closed convex cone generated by the system $\{f_\nu\}$. The scalars $\alpha_\nu = \frac{\langle x, f_\nu\rangle}{\|f_\nu\|^2}$ are the Fourier coefficients of x w.r.t. the orthogonal system $\{f_\nu\}$. Moreover, D is a self-dual cone w.r.t. V_1. By Proposition 3.3 we get

Corollary 3.6. *If $\{f_\nu\}$ is an orthogonal basis of V_1, then (PI) holds for $y, z \in D + V_2$, where D is defined by (3.6).*

Let Ξ denote the set of all sequences $\xi = (\xi_1, \xi_2, \dots)$ with $\xi_\nu^2 = 1,\ \nu = 1, 2, \dots$. For given ξ, let us define the operator g_ξ on V as follows:

$$g_\xi x = x - Px + \sum_\nu \xi_\nu \frac{\langle x, f_\nu \rangle}{\|f_\nu\|^2} f_\nu.$$

This operator is an isometry, because

$$\begin{aligned} \|g_\xi x\|^2 &= \|x\|^2 - \|Px\|^2 + \sum_\nu \xi_\nu^2 \frac{\langle x, f_\nu \rangle^2}{\|f_\nu\|^2} \\ &= \|x\|^2 - \|Px\|^2 + \|Px\|^2 = \|x\|^2, \end{aligned}$$

by (3.5) and obvious orthogonality

$$x - Px \perp \sum_\nu \xi_\nu \frac{\langle x, f_\nu \rangle}{\|f_\nu\|^2} f_\nu.$$

If $x \in V_2$, then $\langle x, f_\nu \rangle = 0$ for all ν. Hence $Px = 0 = \sum_\nu \xi_\nu \frac{\langle x, f_\nu \rangle}{\|f_\nu\|^2} f_\nu$. For this reason

$$g_\xi x = x, \quad x \in V_2. \tag{3.7}$$

We write

$$G = \{g_\xi : \xi \in \Xi\}. \tag{3.8}$$

We will show that G is a group of operators. It is evident that:

$$g_\xi = \mathrm{id}, \quad \text{for } \xi = (1, 1, \dots). \tag{3.9}$$

Let $\zeta, \xi, \gamma \in \Xi$. We have:

$$g_\zeta f_\nu = f_\nu - Pf_\nu + \sum_\eta \zeta_\eta \frac{\langle f\nu, f_\eta \rangle}{\|f_\eta\|^2} f_\eta = \zeta_\nu f\nu,$$

because $Pf_\nu = f_\nu,\ \nu = 1, 2, \dots$. From this, by $x - Px \in V_2$ and (3.7) we get:

$$\begin{aligned} g_\zeta g_\xi x &= g_\zeta (x - Px + \sum_\nu \xi_\nu \frac{\langle x, f_\nu \rangle}{\|f_\nu\|^2} f_\nu) \\ &= g_\zeta (x - Px) + \sum_\nu \xi_\nu \frac{\langle x, f_\nu \rangle}{\|f_\nu\|^2} g_\zeta f_\nu \\ &= x - Px + \sum_\nu \zeta_\nu \xi_\nu \frac{\langle x, f_\nu \rangle}{\|f_\nu\|^2} f_\nu. \end{aligned}$$

Thus

$$g_\zeta g_\xi = g_{\zeta \cdot \xi} = g_\xi g_\zeta, \tag{3.10}$$

where $\zeta \cdot \xi = (\zeta_1 \xi_1, \zeta_2 \xi_2, \dots)$. This clearly gives:

$$g_\zeta (g_\xi g_\gamma) = g_{\zeta \cdot \xi \cdot \gamma} = (g_\xi g_\zeta) g_\gamma \tag{3.11}$$

and

$$g_\xi g_\xi = g_{\xi \cdot \xi} = \mathrm{id},$$

which is equivalent to

$$(g_\xi)^{-1} = g_\xi. \tag{3.12}$$

Since g_ξ is an isometry and invertible,

$$g_\xi - \text{unitary}, \quad \forall_{\xi \in \Xi}. \tag{3.13}$$

By (3.13), (3.7), (3.9) – (3.12) we can assert that G is an Abelian group of unitary operators that are identities on V_2. As a consequence, $G \subset G_P$.

Given any $x \in V$, we define $\xi_x = (\xi_{x,1}, \xi_{x,2}, \dots)$ by

$$\xi_{x,\nu} = \begin{cases} 1, & \langle x, f_\nu \rangle \geq 0 \\ -1, & \langle x, f_\nu \rangle < 0. \end{cases} \tag{3.14}$$

It is clear that $\xi_{x,\nu} \langle x, f_\nu \rangle = |\langle x, f_\nu \rangle|$. Hence

$$g_{\xi_x} x = x - Px + \sum_\nu \frac{|\langle x, f_\nu \rangle|}{\|f_\nu\|^2} f_\nu,$$

where $x - Px \in V_2$ and $\sum\limits_\nu \frac{|\langle x, f_\nu \rangle|}{\|f_\nu\|^2} f_\nu \in D$. Therefore

$$g_{\xi_x} x \in D + V_2. \tag{3.15}$$

Assertion (3.15) is simply the statement that

$$(Gx) \cap C \neq \emptyset, \qquad \forall_{x \in V} \tag{3.16}$$

with $C = D + V_2$. This condition ensures that the sum of the cones gC, where g runs over G, covers the whole space V. Now, we show that (3.16) holds for G_P and for every cone $C = PC + V_2,\ PC \neq \{0\}$.

Fix $v \in V$. Clearly, $v = v_1 + v_2,\ v_i \in V_i,\ i = 1, 2$. If $v_1 = 0$ then $v \in G_P v \subset V_2 \subset C$, i.e. (3.16) holds. Assume that $0 \neq v_1$ and note that there exists $0 \neq u_1 \in PC$. Let us construct the two orthogonal bases $\{e_\nu\}$ and $\{f_\nu\}$ of V_1 with $e_1 = v_1$ and $f_1 = u_1$. Set $u = \|v_1\| \frac{u_1}{\|u_1\|} + v_2$ and

$$g = \mathrm{id} - P + \sum_\nu \frac{\langle \cdot, e_\nu \rangle}{\|e_\nu\| \|f_\nu\|} f_\nu. \tag{3.17}$$

Observe that $u \in C$, $gv = u$ and g is the identity operator on V_2. Now, we prove that g is unitary. Firstly, we note that for any $x \in V$

$$\|gx\|^2 = \|x\|^2 - \|Px\|^2 + \sum_\nu \frac{\langle x, e_\nu \rangle^2}{\|e_\nu\|^2} = \|x\|^2,$$

because $\|Px\|^2 = \sum\limits_\nu \frac{\langle x, e_\nu \rangle^2}{\|e_\nu\|^2}$. Our next goal is to show that $gV = V$. To do this, fix $y \in V$. We have

$$y = y - Py + \sum_\nu \frac{\langle y, f_\nu \rangle}{\|f_\nu\|} \frac{f_\nu}{\|f_\nu\|}.$$

Set

$$x = y - Py + \sum_\nu \frac{\langle y, f_\nu \rangle}{\|f_\nu\|} \frac{e_\nu}{\|e_\nu\|}.$$

It is easily seen that $gx = y$. So, g is unitary.

Finally, g is a unitary operator on V with $gV_2 = V_2$ and $gv = u$. It gives $u \in G_P v \cap C$, as desired.

We are now in a position to introduce a notion of synchronicity of vectors for (PI). For an orthoprojector P let C be a convex cone which admits the representation

$$C = PC + V_2,$$

where PC is nontrivial. Let G be a subgroup of G_P with the property (3.16).

The two vectors $y, z \in V$ are said to be G-synchronous (with respect to C) if there exists a $g \in G$ such that $gy, gz \in C$. If $G = G_P$, then we simply say that y and z are synchronous.

The definition is motivated by [3, sec. 2]. It generalizes the notion of synchronicity with respect to Eaton systems. Obviously, G-synchronicity forces synchronicity under fixed C. In the sequel, for special cones we show that synchronicity is equivalent to (PI) but G-synchronicity is a sufficient condition for (PI).

According to Theorem 3.5, by the notion of synchronicity, it is possible to extend (PI) beyond a cone C if only (PI) holds for vectors in C.

Proposition 3.7. *Let $C \subset V$ be a convex cone with $C = PC + V_2,\ PC \neq \{0\}$ and let G be a subgroup of G_P with property (3.16).*

The following statements are equivalent.

i): *(PI) holds for $y, z \in C$*

ii): *(PI) holds for the vectors y and z which are G-synchronous w.r.t. C.*

Proof. i) $\Rightarrow$ ii). Assume y and z are G-synchronous w.r.t. C. There exists $g \in G$ with $gy, gz \in C$. Since i), (PI) holds for gy, gz. By Theorem 3.5 we conclude that (PI) holds for y and z.

The converse implication is evident because $y, z \in C$ are of course G-synchronous. $\square$

Now, we are able to give an equivalent condition for G-synchronicity. Simultaneously, the condition is sufficient for synchronicity w.r.t. $D + V_2$.

Proposition 3.8. *Let G be the group defined by (3.8) and let D be the cone defined by (3.6).*

The vectors $y, z \in V$ are G-synchronous w.r.t. $D + V_2$ if and only if

$$\langle y, f_\nu \rangle \langle z, f_\nu \rangle \geq 0, \quad \forall_\nu.$$

Proof. If y, z are G-synchronous, then there exists a ξ such that $g_\xi y, g_\xi z \in D + V_2$. Hence $\xi_\nu \langle y, f_\nu \rangle \geq 0$ and $\xi_\nu \langle z, f_\nu \rangle \geq 0$ for all ν. Multiplying the above inequalities side by side we obtain $0 \leq \xi_\nu^2 \langle y, f_\nu \rangle \langle z, f_\nu \rangle = \langle y, f_\nu \rangle \langle z, f_\nu \rangle$ for every ν.

Conversely, suppose that $\langle y, f_\nu \rangle \langle z, f_\nu \rangle \geq 0$ for every ν. In this situation, the sequences defined for y and z by (3.14) are equal. Hence y and z are G-synchronous by (3.15). $\square$

Summarizing the above considerations we give sufficient and necessary conditions for (PI) to hold.

Theorem 3.9. *Let $\{f_\nu\}$ be an orthogonal basis of V_1. Set $C = D + V_2$, where D is defined by (3.6). The following statements are equivalent.*

i): *y and z are synchronous w.r.t. C*

ii): *(PI) holds for y and z.*

In particular, if

$$\langle y, g_0 f_\nu \rangle \langle z, g_0 f_\nu \rangle \geq 0, \quad \forall \nu, \tag{3.18}$$

then (PI) holds, where $g_0 \in G_P$ is fixed.

Proof. The first part, i)$\Rightarrow$ii). It is a consequence of Corollary 3.6 and Proposition 3.7.

Conversely, if ii), then $\langle Py, Pz \rangle \geq 0$. Firstly, suppose that $Pz = \alpha Py$. Clearly, $\alpha \geq 0$. By (3.16), which holds for G_P and C, there exists a $g \in G_P$ such that $gy \in C$. Hence $Pgy \in PC = D$. By (3.4), $gPy \in D$. Since $\alpha \geq 0$, $\alpha g P y \in D$. Since $z - Pz \in V_2$, $g(z - Pz) \in V_2$, because $gV_2 = V_2$. Hence

$$gz = gPz + g(z - Pz) = \alpha g P y + g(z - Pz) \in C.$$

Therefore $gy, gz \in C$, i.e. y and z are synchronous.

Next, assume that Py and Pz are linearly independent. Let us construct an orthogonal basis $\{e_\nu\}$ of V_1 with

$$e_1 = Py, \quad e_2 = Pz - \frac{\langle Pz, Py\rangle}{\|Py\|^2} Py$$

and let $g \in G_P$ be defined by (3.17). There is no difficulty to showing that

$$gy = \underbrace{y - Py}_{\in V_2} + \underbrace{\frac{\|Py\|}{\|f_1\|} f_1}_{\in D} \in C,$$

$$gz = \underbrace{z - Pz}_{\in V_2} + \underbrace{\underbrace{\tfrac{\langle z,Py\rangle}{\|e_1\|\|f_1\|}}_{\geq 0,\ \text{by (PI)}} f_1 + \underbrace{\tfrac{\|Pz\|^2\|Py\|^2 - \langle Pz,Py\rangle^2}{\|Py\|^2\|e_2\|\|f_2\|}}_{\geq 0,\ \text{by Cauchy}-\text{Schwarz ineq.}} f_2}_{\in D} \in C.$$

Therefore, y and z are synchronous as required.

Now, let us note that (3.18) is equivalent to

$$\langle g_0^* y, f_\nu\rangle \langle g_0^* z, f_\nu\rangle \geq 0, \quad \forall \nu.$$

By Proposition 3.8, $g_0^* y$ and $g_0^* z$ are G-synchronous w.r.t. C. Hence there exists a $g \in G$ such that $gg_0^* y, gg_0^* \in C$. Since $gg_0^* \in G_P$, y and z are synchronous w.r.t. C. For this reason (PI) holds, by the first part of this proposition. The proof is complete. □

4. APPLICATIONS TO THE CHEBYSHEV SUM INEQUALITY

Throughout this section, $V = \mathbb{R}^n$ with the standard inner product $\langle \cdot, \cdot \rangle$. Let $\{s_i\}$ be the basis of $\mathbb{R}^n$, where $s_i = (\underbrace{1, \dots, 1}_{i}, 0, \dots, 0)$, $i = 1, \dots, n$. The symbols V_1 and V_2 stand for the subspace orthogonal to s_n and its orthogonal complement, respectively, i.e.

$$V_1 = \left\{ (x_1, \dots, x_n) : \sum_i x_i = 0 \right\}, \quad V_2 = \operatorname{span}\{s_n\}.$$

Let P be the orthoprojector onto V_1, i.e. $P = \mathrm{id} - \frac{\langle \cdot, s_n\rangle}{n} s_n$. In this situation, by Example 3.2, (PI) becomes the *Chebyshev sum inequality* (CHSI).

It is known that the convex cone of nonincreasing vectors

$$C = \{x = (x_1, \dots, x_n) : x_1 \geq x_2 \geq \dots \geq x_n\}$$

is generated by $\{s_1, \dots, s_n, -s_n\}$. On the other side,

$$\{(1, -1, 0, \dots, 0),\ (0, 1, -1, 0, \dots, 0),\ (0, \dots, 0, 1, -1)\}$$

is a generator of

$$\operatorname{dual} C = \left\{ x = (x_1, \dots, x_n) : \sum_{i=1}^n x_i = 0,\ \sum_{i=1}^k x_i \geq 0,\ k = 1, \dots, n-1 \right\}.$$

Set $e_i = nPs_i,\ i = 1, \dots, n-1$. Clearly,

$$e_i = ns_i - is_n = (\underbrace{n-i, \dots, n-i}_{i}, \underbrace{-i, \dots, -i}_{n-i}), \quad i = 1, \dots, n-1. \tag{4.1}$$

Write

$$D = \operatorname{cone}\{e_i\}.$$

Clearly, $PC = D$ and $V_2 \subset C$. Hence by (3.1),

$$C = D + V_2.$$

Applying Proposition 3.2, we conclude that (PI) holds for $y, z \in (D + V_2) \cap \operatorname{dual} D = C \cap \operatorname{dual} D$. With the aid of generators we can check that $D \subset \operatorname{dual} C$. Hence $C = \operatorname{dual}\ \operatorname{dual} C \subset \operatorname{dual} D$.

By the above considerations, for arbitrary $y, z \in C$, the inequality (CHSI) holds. This is a classic Chebyshev result.

The system $\{e_i,\ i = 1, \dots, n-1\}$ constitutes a basis of V_1. Observe that

$$\langle e_i, e_j \rangle = i(n-j)n, \quad i \le j,\ i, j = 1, \dots, n-1.$$

Hence, easy computations lead to

$$\left\langle e_{k+1} - \frac{n-k-1}{n-k} e_k, e_i \right\rangle = 0, \quad i = 1, \dots, k;\ k = 1, \dots, n-2.$$

From this, the Gram-Schmidt orthogonalization gives the orthogonal system $\{q_i\}$ for the basis $\{e_i\}$ as follows:

$$\begin{cases} q_1 = e_1, \\ q_{k+1} = \frac{n-k}{n}\left(e_{k+1} - \frac{n-k-1}{n-k} e_k\right), \quad k = 1, \dots, n-2. \end{cases} \tag{4.2}$$

According to (4.1) and (4.2) we obtain the explicit form of the orthogonal basis $\{q_i\}$

$$q_k = (\underbrace{0, \dots, 0}_{k-1}, n-k, \underbrace{-1, \dots, -1}_{n-k}), \quad k = 1, \dots, n-1. \tag{4.3}$$

Let us denote

$$K = \widetilde{D} + V_2,$$

where $\widetilde{D}$ stands for the $\operatorname{cone}\{q_k\}$. The convex cone $\widetilde{D}$ is self-dual w.r.t. V_1.

According to Proposition 3.3 we can assert that (CHSI) holds for $y, z \in K$.

Let $g_0(x_1, \dots, x_n) = (-x_n, \dots, -x_1)$. Clearly, $g_0 \in G_P$. By Remark 1, (CHSI) holds for $y, z \in g_0 K$. Have:

$$g_0 K = g_0(\widetilde{D} + V_2) = g_0 \widetilde{D} + V_2 = \operatorname{cone}\{g_0 q_k\} + V_2.$$

Define $f_k = g_0 q_{n-k},\ k = 1, \dots, n-1$. Since $g_0 \in G_P$, g_0 is unitary and $g_0 V_1 = V_1$, by (3.3). Hence, $\{f_k\}$ is an orthogonal basis of V_1. Observe

$$f_k = (\underbrace{1, \dots, 1}_{k}, -k, 0, \dots, 0), \quad k = 1, \dots, n-1. \tag{4.4}$$

Write

$$M = \operatorname{cone}\{f_k\} + V_2.$$

By Remark 1, it is evident that (CHSI) holds for $y, z \in M$.

Proposition 4.1. *For* $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$$x \in K \iff \textit{the sequence } \left\{ \frac{1}{n-k+1} \sum_{i=k}^{n} x_i \right\}_{k=1}^{n} \textit{ is nonincreasing,}$$

$$x \in M \iff \textit{the sequence } \left\{ \frac{1}{k} \sum_{i=1}^{k} x_i \right\}_{k=1}^{n} \textit{ is nonincreasing.}$$

Proof. We prove only the first equivalence. The second one uses a similar procedure.

By (3.2), $K = \operatorname{dual} \widetilde{D}$, because $\widetilde{D}$ is self-dual w.r.t. V_1. Hence by (4.3) we can assert that $x \in K$ is equivalent to

$$(n-k)x_k \geq \sum_{i=k+1}^{n} x_i, \qquad k = 1, \dots, n-1.$$

Adding to both of sides $(n-k)\sum_{i=k+1}^{n} x_i$ and dividing by $(n-k)(n-k+1)$, we obtain

$$\frac{1}{n-k+1}\left(\sum_{i=k}^{n} x_i\right) \geq \frac{1}{n-k}\left(\sum_{i=k+1}^{n} x_i\right), \qquad k = 1, \dots, n-1.$$

This is equivalent to our claim. □

By the above proposition, we can see that $C \subset K$ and $C \subset M$. The cone M is said to be a cone of vectors nonincreasing in mean. It is easily seen that (CHSI) holds for $y, z \in -K$ and for $y, z \in -M$ (for e.g., by taking $C = K, M$ and substituting $-\operatorname{id}$ into g_0 in Remark 1). The statement that (CHSI) holds for vectors monotonic in mean is due to Biernacki, see [1].

The remainder of this section will be devoted to (CHSI) for synchronous vectors. We will consider relations between synchronicity and similar ordering.

Here G_P is the group of all orthogonal matrices such that the sum of the entries of each row and column is equal to 1 or -1. The group of all $n \times n$ permutation matrices is a subgroup of G_P, which together with the cone C fulfil (3.16). The permutation group synchronicity w.r.t. C is simply the relation "to be similarly ordered". It implies synchronicity w.r.t. every cone which contains C, e.g. M or K.

The two vectors $x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n$ are said to be similarly ordered if

$$(x_i - x_j)(y_i - y_j) \geq 0, \qquad \forall_{i,j}. \tag{4.5}$$

The assertion that (CHSI) holds for similarly ordered vectors is a consequence of Proposition 3.7.

Theorem 3.9 states that (CHSI) is equivalent to synchronicity w.r.t. $\operatorname{cone}\{f_k\} + V_2$ where $\{f_k\}$ is an arbitrarily chosen orthogonal basis of V_1. Moreover, G-synchronicity gives (CHSI), where G is the group (3.8) acting on $\mathbb{R}^n$. For this reason, the specification of Theorem 3.9 can be as follows.

Let $\{f_k\}$ be defined by (4.4) and G by (3.8) in compliance with the basis.

Corollary 4.2. *(CHSI) holds for $y,\ z$ if and only if y and z are synchronous w.r.t. M.*

In particular, (CHSI) is satisfied by y and z such that

$$\langle y, Uf_k\rangle \langle z, Uf_k\rangle \geq 0, \qquad k = 1, \dots, n-1,$$

where U is a fixed unitary operation with $Us_n = s_n$ or $Us_n = -s_n$, i.e. U is represented by an orthogonal matrix whose rows and columns sum up to 1 or to -1.

By Proposition 3.8 we have:

Remark 3. The vectors $y = (y_1, \dots, y_n)$ and $z = (z_1, \dots, z_n)$ are G-synchronous w.r.t. M if and only if

$$\left[\sum_{i=1}^{k} y_i - k y_{k+1}\right]\left[\sum_{i=1}^{k} z_i - k z_{k+1}\right] \geq 0, \qquad k = 1, \dots, n-1.$$

Relations of similar ordering and G-synchronicity w.r.t. M are not comparable, i.e. there exist similarly ordered vectors which are not synchronous and there exist synchronous vectors that are not similarly ordered. On the other hand, both relations imply synchronicity w.r.t. M and as a consequence, (CHSI) holds.

Example 4.1. Consider $\mathbb{R}^n,\ n > 3$.

For $0 < \alpha < 1 < \beta < n-1$ set $y = (0, \dots, 0, 1-n, -\alpha),\ z = (0, \dots, 0, 1-n, -\beta)$. According to (4.5) and Remark 3 the vectors y and z are similarly ordered and are not G-synchronous, but they are synchronous w.r.t. M, so (CHSI) holds.

Now, set $y' = f_1 + f_2$, $z' = f_2 + f_3$, where f_i are defined by (4.4). The vectors y' and z' are G-synchronous w.r.t. M, because $y', z' \in M$, so (CHSI) holds.

On the other hand $y' = (2, 0, -2, 0, \dots, 0),\ z' = (2, 2, -1, -3, 0, \dots, 0)$ are not similarly ordered by (4.5), because $(y_3' - y_4')(z_3' - z_4') = -2(-1+3) < 0$.

5. Applications to the Chebyshev Integral Inequality

Set $V = L^2$ as in Example 3.2. The characteristic function of the measurable set $A \subset [0,1]$ is denoted by I_A. Additionally we will write $e_s = I_{[0,s]},\ 0 \le s \le 1$. The symbol V_1 stands for the subspace orthogonal to $V_2 = \operatorname{span}\{e_1\}$, i.e. $V_1 = \left\{x \in L^2 : \int x d\mu = 0\right\}$. By Example 3.2, it is known that for the orthoprojector P onto V_1 (PI) transforms into the *Chebyshev integral inequality* (CHII). Let $C \subset L^2$ be the closed convex cone of all nonincreasing $\mu\ a.e.$ functions. It is known (see [5, Theorem 3.1 and 3.3]) that:

$$\begin{aligned} C &= \operatorname{cone}\left(\{e_s : 0 \le s \le 1\} \cup \{-e_1\}\right), \\ \operatorname{dual} C &= \operatorname{cone}\{I_\Pi - I_{\Pi+\varepsilon} : \varepsilon > 0, \Pi, \Pi+\varepsilon \subset [0,1]\}, \end{aligned} \tag{5.1}$$

where Π stands for an interval.

The Haar system:

$$\begin{aligned} &\chi_0^0 = e_1 \\ &\chi_n^k(t) = \begin{cases} 2^{n/2}, & \frac{2k-2}{2^{n+1}} \le t < \frac{2k-1}{2^{n+1}} \\ -2^{n/2}, & \frac{2k-1}{2^{n+1}} \le t < \frac{2k}{2^{n+1}} \\ 0, & \text{otherwise} \end{cases} \\ &n = 0, 1, \dots, \qquad k = 1, 2, \dots, 2^n \end{aligned} \tag{5.2}$$

forms an orthonormal basis of L^2. In particular, $\mathcal{H} = \{\chi_n^k : n = 0, 1, \dots,\ k = 1, \dots, 2^n\}$ is an orthonormal basis of V_1.

Let

$$D = \operatorname{cone} \mathcal{H}.$$

The cone D is self-dual w.r.t. V_1, so by (3.2) we have:

$$\operatorname{dual} D = D + V_2. \tag{5.3}$$

By (5.1), observe that $\mathcal{H} \subset \operatorname{dual} C$, hence $C = \operatorname{dual}\ \operatorname{dual} C \subset \operatorname{dual} \mathcal{H} = \operatorname{dual} D$. Combining this with (5.3), we obtain

$$C \subset D + V_2. \tag{5.4}$$

From (5.4) and Corollary 3.6 it follows that

Corollary 5.1. *(CHII) holds for $y, z \in D + V_2$.*

The cone $D + V_2$ contains the cone of all nonincreasing $\mu\ a.e.$ functions in L^2.

It is easily seen that the cone $D+V_2$ contains functions which are not nonincreasing μ $a.e.$

Let G be the group (3.8) acting on L^2 with the Haar system. Employing the G-synchronicity relation w.r.t. $D+V_2$, by Theorem 3.9 we get:

Corollary 5.2. *(CHII) holds for $y,z\in L^2$ if only*

$$\langle y,\chi\rangle\langle z,\chi\rangle\geq 0,\qquad \forall_{\chi\in\mathcal{H}}. \tag{5.5}$$

We next discuss the relation between the condition (5.5) and the known sufficient conditions for (CHII). One of these is the condition that y and z are similarly ordered, i.e.

$$[y(s)-y(t)]\,[z(s)-z(t)]]\geq 0,\qquad \text{for all } 0\leq s,t\leq 1 \tag{5.6}$$

(see e.g. [6, pp. 198-199]). Now, we show by an example that the G-synchronicity condition (5.5) is not stronger than the condition of similar ordering (5.6) in L^2.

Example 5.1. In L^2 let $y=\chi_2^1+\chi_2^2$, $z=\chi_2^2+\chi_2^3$, where χ_i^j are defined by (5.2). The vectors y and z are G-synchronous w.r.t. $D+V_2$, because they are in D.

On the other hand

$$\begin{array}{ll} y(s)=2, & 0\leq s\leq \frac{1}{8}\\ y(t)=0, & \frac{4}{8}\leq t\leq\frac{5}{8}\end{array},\qquad \begin{array}{ll} z(s)=0, & 0\leq s\leq \frac{1}{8}\\ z(t)=2, & \frac{4}{8}\leq t\leq\frac{5}{8}\end{array}.$$

From this, $[y(s)-y(t)]\,[z(s)-z(t)]]=[2-0][0-2]\leq 0$ for any $0\leq s\leq\frac{1}{8}$ and $\frac{4}{8}\leq t\leq\frac{5}{8}$. Thus y and z are not similarly ordered.

Now, we recall that a function $y\in L^2$ is nonincreasing (nondecreasing, monotone) in mean if the function $s\mapsto\frac{1}{s}\int_0^s y d\mu,$ is nonincreasing (nondecreasing, monotone).

Differentiating $\frac{1}{s}\int_0^s y d\mu$ we can easy obtain that y is nonincreasing in mean if and only if $\frac{1}{s}\int_0^s y d\mu\geq y(s),$ μ $a.e.$

It is known that (CHII) holds for y and z which are monotone in mean in the same direction (see [1], cf. also [6, pp. 198-199]). Johnson in [2] gave a more general condition. Namely, if

$$\left[\frac{1}{s}\int_0^s y d\mu-y(s)\right]\left[\frac{1}{s}\int_0^s z d\mu-z(s)\right]\geq 0,\qquad \forall_{0<s<1} \tag{5.7}$$

then (CHII) holds for y and z.

Remark 4.

(1) There exist functions in $\operatorname{cone}\mathcal{H}$ which are not nonincreasing in mean.
(2) There exist functions nonincreasing in mean which are not in $\operatorname{cone}\mathcal{H}$.
(3) There exist functions in $\operatorname{cone}\mathcal{H}$ for which (5.7) does not hold, i.e. the condition (5.5) is not stronger than (5.7).

Proof. An easy verification shows that:

Ad. 1) $\chi_n^k\in\mathcal{H},$ $k>1$ are not nonincreasing in mean.

Ad. 2) Set $f=I_{[0,1/2)}-2I_{[1/2,3/4)}$. f is nonincreasing in mean and is not in $\operatorname{cone}\mathcal{H}$ because $\langle f,\chi_1^2\rangle<0$.

Ad. 3) Set $y=\chi_1^2,$ $z=\chi_2^3$. For $\frac{5}{8}<s<\frac{6}{8}$ have:

$$\left[\frac{1}{s}\int_0^s y d\mu-y(s)\right]\left[\frac{1}{s}\int_0^s z d\mu-z(s)\right]=-\frac{\sqrt{2}/2}{s}\cdot\frac{3/2}{s}<0.$$

□

The set of all L^2-functions nonincreasing in mean constitutes a convex cone. It will be denoted by M. Let $\mathcal{M}_0$ be the class of all step functions of the form

$$g_{s,t} = I_{[0,s)} - \frac{s}{t-s} I_{[s,t]}, \qquad 0 < s < t < 1.$$

Proposition 5.3.

$$\begin{aligned} M &= \operatorname{dual} \mathcal{M}_0, \\ PM &= \operatorname{dual}_{V_1} \mathcal{M}_0 = \operatorname{cone} \mathcal{M}_0, \\ M &= \operatorname{cone} \mathcal{M}_0 + V_2. \end{aligned}$$

Proof. By definition, $f \in M$ if and only if

$$\frac{1}{s} \int_0^s f d\mu \geq \frac{1}{t} \int_0^t f d\mu \quad \text{for all } 0 < s < t < 1.$$

After equivalent transformations we obtain

$$\int_0^s f d\mu \geq \frac{s}{t-s} \int_s^t f d\mu \quad \text{for all } 0 < s < t < 1.$$

This is simply $f \in \operatorname{dual} \mathcal{M}_0$, so the first equation holds.

To show the second equation, note that $\mathcal{M}_0 \subset M \cap V_1$. Hence

$$\operatorname{dual}(M \cap V_1) \subset \operatorname{dual} \mathcal{M}_0 = M,$$

by the first equation. It follows that $V_1 \cap \operatorname{dual}(M \cap V_1) \subset M \cap V_1$, i.e.

$$\operatorname{dual}_{V_1}(M \cap V_1) \subset M \cap V_1. \tag{5.8}$$

Fix $f \in M \cap V_1$ and let $g \in M \cap V_1$ be arbitrary. For such f and g (CHII) holds and takes the form:

$$\int_0^1 fg d\mu \geq \int_0^1 f d\mu \cdot \int_0^1 g d\mu = 0 \cdot 0 = 0,$$

i.e. $f \in \operatorname{dual}_{V_1}(M \cap V_1)$. Therefore

$$M \cap V_1 \subset \operatorname{dual}_{V_1}(M \cap V_1). \tag{5.9}$$

Since $M = \operatorname{dual} \mathcal{M}_0$, $\operatorname{dual} M = \operatorname{cone} \mathcal{M}_0$. Now, observe that $V_2 \subset M$. This implies by (3.1) that $M = PM + V_2$. Furthermore, in this situation $PM = M \cap V_1$. The above gives

$$\begin{aligned} \operatorname{dual} M &= \operatorname{dual}(M \cap V_1 + V_2) \\ &= V_1 \cap \operatorname{dual}(M \cap V_1) = \operatorname{dual}_{V_1}(M \cap V_1). \end{aligned}$$

Hence

$$\operatorname{dual}_{V_1}(M \cap V_1) = \operatorname{cone} \mathcal{M}_0. \tag{5.10}$$

Combining (5.8), (5.9) and (5.10) we obtain the required equations.

The third equation is a consequence of the second one. The proof is complete. □

The second equation of the above propositions immediately gives:

Remark 5. The convex cone of all L^2-functions nonincreasing in mean with integral equal to 0 is self-dual w.r.t. V_1.

Taking $C = M$ in Theorem 3.1, by Proposition 5.3 we easily obtain:

Corollary 5.4. *If $\int f d\mu \int g d\mu \leq \int fg d\mu$ holds for all functions $f \in L^2$ monotone in mean, then $g \in L^2$ is also monotone in mean.*

REFERENCES

[1] M. BIERNACKI, Sur une inégalité entre les intégrales due á Tchébyscheff, *Ann. Univ. Mariae Curie-Skłodowska*, **A5** (1951), 23–29.

[2] R. JOHNSON, Chebyshev's inequality for functions whose averages are monotone, *J. Math. Anal. Appl.*, **172** (1993), 221–232.

[3] M. NIEZGODA, On the Chebyshev functional, *Math. Inequal. Appl.*, **10**(3) (2007), 535–546.

[4] M. NIEZGODA, Bifractional inequalities and convex cones, *Discrete Math.*, **306**(2) (2006), 231–243.

[5] Z. OTACHEL, Spectral orders and isotone functionals, *Linear Algebra Appl.*, **252** (1997), 159–172.

[6] J.E. PEČARIĆ, F. PROSCHAN AND Y.L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Mathematics in Science and Engineering, Vol. 187, Academic Press, Inc. (1992)