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POLYNOMIALS AND CONVEX SEQUENCE INEQUALITIES

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ABSTRACT. In this note, motivated by an inequality of S. Haber, we consider how a polynomial, having certain properties, gives rise an inequality for a convex sequence.

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1. INTRODUCTION

The following inequality was proved in [1]. If a and b are positive then

$$\frac{1}{n+1}[a^n + a^{n-1}b + \cdots + b^n] \ \geq \left(\frac{a+b}{2}\right)^n \qquad (n = 0, 1, \dots).$$

If we replace $\frac{b}{a}$ by x this can equally well be written as

$$\sum_{0}^{n} \left[\frac{1}{n+1} - \frac{1}{2^n}\binom{n}{k}\right] x^k \geq 0 \tag{1.1}$$

in which we can take $x \geq 0$.

In [2] this inequality was generalized to the following: If the sequence $\{u_k\}$ is convex then

$$\sum_{0}^{n} \left[\frac{1}{n+1} - \frac{1}{2^n}\binom{n}{k}\right] u_k \geq 0. \tag{1.2}$$

The method used to prove the sequence result (1.2) was based largely on that used by Haber to obtain his polynomial result (1.1). Indeed, using the same technique again, another sequence result more general than (1.2) was obtained in [3].

The present author felt that it would be of interest to find those properties possessed by the polynomial on the left side of (1.1) which allow sequence results like those in (1.2) and [3] to

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be deduced directly from the polynomial itself. To this end, it is the purpose of the present note to prove the result of the next section.

2. From Polynomial to Sequence Inequality

Theorem 2.1. *Suppose that the polynomial*

$$\sum_0^n a_k x^k \tag{2.1}$$

has $x = 1$ as a double root and that when we write

$$b_k = \sum_{j=0}^{k} a_j \qquad (k = 0, \dots, n-1)$$

and

$$c_k = \sum_{j=0}^{k} b_j \qquad (k = 0, \dots, n-2)$$

we find that all the c_k are non-negative. Then we have

$$\sum_0^n a_k u_k \geq 0 \qquad \textit{if the sequence } \{u_k\} \textit{ is convex}. \tag{2.2}$$

Note. The coefficients c_k are simply the coefficients of

$$\frac{\sum_0^n a_k x^k}{(x-1)^2}$$

obtainable, in a simple case, by carrying out the actual division but it is convenient to have the above formulae for both b_k and c_k as it is often possible to discover the properties of these without any calculation.

Proof. Since the polynomial in (2.1) has 1 as a double root we can write

$$\sum_0^n a_k x^k = (1-x) \sum_0^{n-1} b_k x^k \tag{2.3}$$

and

$$\sum_0^{n-1} b_k x^k = (1-x) \sum_0^{n-2} c_k x^k. \tag{2.4}$$

Comparing coefficients and solving the resulting equations we get

$$b_k = \sum_{j=0}^{k} a_j \qquad (k = 0, \dots, n-1)$$

and

$$c_k = \sum_{j=0}^{k} b_j \qquad (k = 0, \dots, n-2).$$

We note, in passing, that if we extended each of the summations above to $k = n$ then, because the sums of all the a_k and all of the b_k are zero, we would have $b_n = 0$ and $c_n = c_{n-1} = 0$. These features are apparent in the second and third examples below.

Clearly, from (2.3) and (2.4) we have

$$\sum_0^n a_k x^k = (1-x)^2 \sum_0^{n-2} c_k x^k. \tag{2.5}$$

Now the identity (2.5) holds equally well in any commutative ring so we can replace the variable x by E where E is the numerical shift operator which acts on any sequence $\{u_k\}_{k=0}^n$ in $\mathbb{R}^{n+1}$ as follows:

$$E(u_k) = u_{k+1} \qquad (k = 0, 1, 2, \ldots, n-1).$$

So (2.5) yields the operator identity

$$\sum_0^n a_k E^k = \left[\sum_0^{n-2} c_k E^k\right](E-1)^2.$$

Given a convex sequence $\{u_k\}$, we allow each side of this to operate on u_0 when we get

$$\sum_0^n a_k u_k = \left[\sum_0^{n-2} c_k E^k\right](u_2 - 2u_1 + u_0).$$

That is

$$\sum_0^n a_k u_k = \sum_0^{n-2} c_k (u_{k+2} - 2u_{k+1} + u_k).$$

Since the sequence $\{u_k\}$ is convex and the c_k are all non-negative we arrive at the result

$$\sum_0^n a_k u_k \geq 0$$

and this completes the proof. □

3. EXAMPLES

The first example which presents itself is that discussed in the introduction. It is a simple matter to check that $x = 1$ is a root of the polynomial appearing on the left in (1.1) and of its derivative. That it is precisely a double root follows by Descartes' rule of signs. The coefficients in (1.1) are palindromic, taking signs (plus, minus, plus) as we proceed from left to right and by (2.3) their sum is zero. Hence the partial sums b_k take the signs (plus, minus) as we proceed from left to right and by (2.4) their sum is also zero. From this it follows that all the c_k, which are the partial sums of the b_k, are non-negative. Hence, by the theorem stated above, (1.2) follows from the properties of the polynomial in (1.1).

Note. We emphasize here that the theorem gives the sequence inequality, not from the polynomial inequality, but from the properties of the polynomial itself. The polynomial inequality is actually a special case of the sequence inequality. In the present example the inequality (1.1) is a consequence of (1.2).

As a second example we give a new proof of an inequality which appears in [4] (see the result 130 on p. 99 there). Changing the suffix notation there in an obvious way we consider the polynomial

$$\sum_0^{2n} a_k x^k,$$

where

$$a_k = \frac{1}{n+1}\ (k \text{ even}) \quad : \quad a_k = -\frac{1}{n}\ (k \text{ odd}).$$

Note. In this example k runs from 0 to $2n$ rather than from 0 to n. We find that

$$b_k = \frac{1}{n+1} - \frac{1}{n} + \frac{1}{n+1} - \frac{1}{n} + \cdots - \frac{1}{n} \quad (k+1 \text{ terms, if } k \text{ is odd})$$

and

$$b_k = \frac{1}{n+1} - \frac{1}{n} + \frac{1}{n+1} - \frac{1}{n} + \cdots + \frac{1}{n+1} \quad (k+1 \text{ terms, if } k \text{ is even}).$$

It is now a simple matter to calculate the c_k when we find that

$$c_k = \frac{(2n-k-1)(k+1)}{4n(n+1)} \quad (k \text{ odd}) \quad : \quad \frac{(2n-k)(k+2)}{4n(n+1)} \quad (k \text{ even}).$$

so that all the c_k are non-negative. So by the theorem of the last section we deduce that

$$\sum_{0}^{2n} a_k u_k \geq 0$$

when the u_k are convex. This is the result referred to in [4].

As a final example we consider the polynomial

$$2 + 3x - 7x^2 - 3x^3 + 5x^4.$$

This has 1 as a double root and the partial sums of the coefficients are $2, 5, -2, -5, 0$ and the partial sums of these are $2, 7, 5, 0, 0$ which are all non-negative and so if $\{u_k\}$ is a convex sequence then

$$2u_k + 3u_{k+1} + 5u_{k+4} \geq 7u_{k+2} + 3u_{k+3}.$$

Obviously this last example was contrived, starting from the result

$$5x^4 - 3x^3 - 7x^2 + 3x + 2 = (x-1)^2(5x^2 + 7x + 2),$$

but we included it because of its purely arithmetical nature.

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