Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 5, Article 101, 2003

# OSTROWSKI-GRÜSS TYPE INEQUALITIES IN TWO DIMENSIONS

NENAD UJEVIĆ

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SPLIT
TESLINA 12/III, 21000 SPLIT
CROATIA.
ujevic@pmfst.hr

*Received 08 January, 2003; accepted 07 August, 2003*
*Communicated by B.G. Pachpatte*

ABSTRACT. A general Ostrowski-Grüss type inequality in two dimensions is established. A particular inequality of the same type is also given.

*Key words and phrases:* Ostrowski's inequality, 2-dimensional generalization, Ostrowski-Grüss inequality.

2000 *Mathematics Subject Classification.* 26D10, 26D15.

## 1. INTRODUCTION

In 1938 A. Ostrowski proved the following integral inequality ([17] or [16, p. 468]).

**Theorem 1.1.** *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping differentiable in the interior $Int\ I$ of $I$, and let $a, b \in Int\ I$, $a < b$. If $|f'(t)| \leq M$, $\forall t \in [a, b]$, then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)M, \tag{1.1}$$

*for $x \in [a, b]$.*

The first (direct) generalization of Ostrowski's inequality was given by G.V. Milovanović and J. Pečarić in [14]. In recent years a number of authors have written about generalizations of Ostrowski's inequality. For example, this topic is considered in [2], [4], [6], [9] and [14]. In this way, some new types of inequalities have been formed, such as inequalities of Ostrowski-Grüss type, inequalities of Ostrowski-Chebyshev type, etc. The first inequality of Ostrowski-Grüss type was given by S.S. Dragomir and S. Wang in [6]. It was generalized and improved in [9]. X.L. Cheng gave a sharp version of the mentioned inequality in [4]. The first multivariate version of Ostrowski's inequality was given by G.V. Milovanović in [12] (see also [13] and [16, p. 468]). Multivariate versions of Ostrowski's inequality were also considered in [3], [7] and [11]. In this paper we give a general two-dimensional Ostrowski-Grüss inequality. For

ISSN (electronic): 1443-5756

003-03

that purpose, we introduce specially defined polynomials, which can be considered as harmonic or Appell-like polynomials in two dimensions. In Section 3 we use the mentioned general inequality to obtain a particular two-dimensional Ostrowski-Grüss type inequality.

## 2. A General Ostrowski-Grüss Inequality

Let $\Omega = [a,b] \times [a,b]$ and let $f : \Omega \to \mathbb{R}$ be a given function. Here we suppose that $f \in C^{2n}(\Omega)$. Let $P_k(s)$ and $Q_k(t)$ be harmonic or Appell-like polynomials, i.e.

$$P_k'(s) = P_{k-1}(s) \text{ and } Q_k'(t) = Q_{k-1}(t), \; k = 1,2,\dots,n+1, \tag{2.1}$$

with

$$P_0(s) = Q_0(t) = 1. \tag{2.2}$$

We also define

$$R_k(s,t) = P_k(s)Q_k(t), \; k = 0,1,2,\dots,n+1. \tag{2.3}$$

**Lemma 2.1.** *Let $R_k(s,t)$ be defined by (2.3). Then we have*

$$\frac{\partial^2 R_k(s,t)}{\partial s \partial t} = R_{k-1}(s,t) \tag{2.4}$$

*for $k = 1,2,\dots,n+1$.*

*Proof.* From (2.1) – (2.3) it follows that

$$\begin{aligned} \frac{\partial^2 R_k(s,t)}{\partial s \partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial R_k(s,t)}{\partial s}\right) \\ &= \frac{\partial}{\partial t}\left(P_k'(s)Q_k(t)\right) \\ &= P_{k-1}(s)Q_k'(t) \\ &= P_{k-1}(s)Q_{k-1}(t) = R_{k-1}(s,t). \end{aligned}$$

□

We now define

$$J_k = \int_a^b \left[ R_k(b,t)\frac{\partial^{2k-1}f(b,t)}{\partial s^{k-1}\partial t^k} - R_k(a,t)\frac{\partial^{2k-1}f(a,t)}{\partial s^{k-1}\partial t^k}\right] dt, \tag{2.5}$$

$$u_{k-1}(t) = \frac{\partial^{k-1}f(b,t)}{\partial s^{k-1}}, \; v_{k-1}(t) = \frac{\partial^{k-1}f(a,t)}{\partial s^{k-1}}, \tag{2.6}$$

for $k = 1,2,\dots,n$. We also define

$$J_{k,1} = \int_a^b R_k(b,t)\frac{\partial^{2k-1}f(b,t)}{\partial s^{k-1}\partial t^k}dt = P_k(b)\int_a^b Q_k(t)u_{k-1}^{(k)}(t)dt \tag{2.7}$$

and

$$J_{k,2} = \int_a^b R_k(a,t)\frac{\partial^{2k-1}f(a,t)}{\partial s^{k-1}\partial t^k}dt = P_k(a)\int_a^b Q_k(t)v_{k-1}^{(k)}(t)dt \tag{2.8}$$

such that

$$J_k = J_{k,1} - J_{k,2}. \tag{2.9}$$

**Lemma 2.2.** *Let $J_{k,1}$ be defined by (2.7). Then we have*

$$J_{k,1} = P_k(b) \sum_{j=1}^{k} (-1)^{k-j} \left[ Q_j(b) u_{k-1}^{(j-1)}(b) - Q_j(a) u_{k-1}^{(j-1)}(a) \right] + (-1)^k P_k(b) \int_a^b u_{k-1}(t) dt, \tag{2.10}$$

*for $k = 1, 2, \ldots, n$.*

*Proof.* We introduce the notation

$$U_k(u_{k-1}) = \int_a^b Q_k(t) u_{k-1}^{(k)}(t) dt.$$

Then we have

$$\begin{aligned} (-1)^k U_k(u_{k-1}) &= (-1)^k \int_a^b Q_k(t) u_{k-1}^{(k)}(t) dt \\ &= (-1)^k \left[ Q_k(b) u_{k-1}^{(k-1)}(b) - Q_k(a) u_{k-1}^{(k-1)}(a) \right] \\ &\quad + (-1)^{k-1} \int_a^b Q_{k-1}(t) u_{k-1}^{(k-1)}(t) dt. \end{aligned}$$

We can write the above relation in the form

$$(-1)^k U_k(u_{k-1}) = (-1)^k \left[ Q_k(b) u_{k-1}^{(k-1)}(b) - Q_k(a) u_{k-1}^{(k-1)}(a) \right] + (-1)^{k-1} U_{k-1}(u_{k-1}).$$

In a similar way we get

$$\begin{aligned} (-1)^{k-1} U_{k-1}(u_{k-1}) &= (-1)^{k-1} \int_a^b Q_{k-1}(t) u_{k-1}^{(k-1)}(t) dt \\ &= (-1)^{k-1} \left[ Q_{k-1}(b) u_{k-1}^{(k-2)}(b) - Q_{k-1}(a) u_{k-1}^{(k-2)}(a) \right] \\ &\quad + (-1)^{k-2} \int_a^b Q_{k-2}(t) u_{k-1}^{(k-2)}(t) dt \end{aligned}$$

or

$$\begin{aligned} &(-1)^{k-1} U_{k-1}(u_{k-1}) \\ &\quad = (-1)^{k-1} \left[ Q_{k-1}(b) u_{k-1}^{(k-2)}(b) - Q_{k-1}(a) u_{k-1}^{(k-2)}(a) \right] + (-1)^{k-2} U_{k-2}(u_{k-1}). \end{aligned}$$

If we continue the above procedure then we obtain

$$\begin{aligned} &(-1)^k U_k(u_{k-1}) \\ &\quad = \sum_{j=1}^{k} (-1)^j \left[ Q_j(b) u_{k-1}^{(j-1)}(b) - Q_j(a) u_{k-1}^{(j-1)}(a) \right] + U_0(u_{k-1}) \\ &\quad = \sum_{j=1}^{k} (-1)^j \left[ Q_j(b) u_{k-1}^{(j-1)}(b) - Q_j(a) u_{k-1}^{(j-1)}(a) \right] + \int_a^b u_{k-1}(t) dt. \end{aligned}$$

Note now that

$$J_{k,1} = P_k(b) U_k(u_{k-1})$$

such that (2.10) holds. □

**Lemma 2.3.** *Let $J_{k,2}$ be defined by (2.8). Then we have*

$$J_{k,2} = P_k(a) \sum_{j=1}^{k} (-1)^{k-j} \left[ Q_j(b) v_{k-1}^{(j-1)}(b) - Q_j(a) v_{k-1}^{(j-1)}(a) \right] + (-1)^k P_k(a) \int_a^b v_{k-1}(t)dt, \tag{2.11}$$

*for $k = 1, 2, \ldots, n$.*

*Proof.* The proof is almost identical to that of Lemma 2.2. □

We now define

$$K_k = \int_a^b \left[ \frac{\partial R_k(s,b)}{\partial s} \frac{\partial^{2k-2} f(s,b)}{\partial s^{k-1} \partial t^{k-1}} - \frac{\partial R_k(s,a)}{\partial s} \frac{\partial^{2k-2} f(s,a)}{\partial s^{k-1} \partial t^{k-1}} \right] ds, \tag{2.12}$$

for $k = 2, \ldots, n$,

$$x_{k-1}(s) = \frac{\partial^{k-1} f(s,b)}{\partial t^{k-1}}, \; y_{k-1}(s) = \frac{\partial^{k-1} f(s,a)}{\partial t^{k-1}} \tag{2.13}$$

and

$$K_1 = Q_1(b) \int_a^b x_0(s)ds - Q_1(a) \int_a^b y_0(s)ds. \tag{2.14}$$

We also define

$$K_{k,1} = \int_a^b \frac{\partial R_k(s,b)}{\partial s} \frac{\partial^{2k-2} f(s,b)}{\partial s^{k-1} \partial t^{k-1}} ds = Q_k(b) \int_a^b P_{k-1}(s) x_{k-1}^{(k-1)}(s)ds \tag{2.15}$$

and

$$K_{k,2} = \int_a^b \frac{\partial R_k(s,a)}{\partial s} \frac{\partial^{2k-2} f(s,a)}{\partial s^{k-1} \partial t^{k-1}} ds = Q_k(a) \int_a^b P_{k-1}(s) y_{k-1}^{(k-1)}(s)ds \tag{2.16}$$

such that

$$K_k = K_{k,1} - K_{k,2}, \; k = 1, 2, \ldots, n. \tag{2.17}$$

**Lemma 2.4.** *Let $K_{k,1}$ be defined by (2.15). Then we have*

$$K_{k,1} = Q_k(b) \sum_{j=2}^{k} (-1)^{k-j+1} \left[ P_{j-1}(b) x_{k-1}^{(j-2)}(b) - P_{j-1}(a) x_{k-1}^{(j-2)}(a) \right] + (-1)^{k-1} Q_k(b) \int_a^b x_{k-1}(s)ds, \tag{2.18}$$

*for $k = 2, \ldots, n$.*

*Proof.* We introduce the notation

$$U_{k-1}(x_{k-1}) = \int_a^b P_{k-1}(s) x_{k-1}^{(k-1)}(s)ds.$$

Then we have

$$(-1)^{k-1}U_{k-1}(x_{k-1}) = (-1)^{k-1}\int_a^b P_{k-1}(s)x_{k-1}^{(k-1)}(s)ds$$
$$= (-1)^{k-1}\left[P_{k-1}(b)x_{k-1}^{(k-2)}(b) - P_{k-1}(a)x_{k-1}^{(k-2)}(a)\right]$$
$$+ (-1)^{k-2}\int_a^b P_{k-2}(s)x_{k-1}^{(k-2)}(s)ds.$$

We can write the above relation in the form

$$(-1)^{k-1}U_{k-1}(x_{k-1})$$
$$= (-1)^{k-1}\left[P_{k-1}(b)x_{k-1}^{(k-2)}(b) - P_{k-1}(a)x_{k-1}^{(k-2)}(a)\right] + (-1)^{k-2}U_{k-2}(x_{k-1}).$$

In a similar way we get

$$(-1)^{k-2}U_{k-2}(x_{k-1}) = (-1)^{k-2}\int_a^b P_{k-2}(s)x_{k-1}^{(k-2)}(s)ds$$
$$= (-1)^{k-2}\left[P_{k-2}(b)x_{k-1}^{(k-3)}(b) - P_{k-2}(a)x_{k-1}^{(k-3)}(a)\right]$$
$$+ (-1)^{k-3}\int_a^b P_{k-3}(s)x_{k-1}^{(k-3)}(s)ds$$

or

$$(-1)^{k-2}U_{k-2}(x_{k-1})$$
$$= (-1)^{k-2}\left[P_{k-2}(b)x_{k-1}^{(k-3)}(b) - P_{k-2}(a)x_{k-1}^{(k-3)}(a)\right] + (-1)^{k-3}U_{k-3}(x_{k-1}).$$

If we continue the above procedure then we get

$$(-1)^{k-1}U_{k-1}(x_{k-1})$$
$$= \sum_{j=2}^{k}(-1)^{j-1}\left[P_{j-1}(b)x_{k-1}^{(j-2)}(b) - P_{j-1}(a)x_{k-1}^{(j-2)}(a)\right] + U_0(x_{k-1})$$
$$= \sum_{j=2}^{k}(-1)^{j-1}\left[P_{j-1}(b)x_{k-1}^{(j-2)}(b) - P_{j-1}(a)x_{k-1}^{(j-2)}(a)\right] + \int_a^b x_{k-1}(t)dt.$$

Note now that

$$K_{k,1} = Q_k(b)U_{k-1}(x_{k-1})$$

such that (2.18) holds. □

**Lemma 2.5.** *Let $K_{k,2}$ be defined by (2.16). Then we have*

$$K_{k,2} = Q_k(a)\sum_{j=2}^{k}(-1)^{k-j+1}\left[P_{j-1}(b)y_{k-1}^{(j-2)}(b) - P_{j-1}(a)y_{k-1}^{(j-2)}(a)\right]$$
$$+ (-1)^{k-1}Q_k(a)\int_a^b y_{k-1}(s)ds, \tag{2.19}$$

*for $k = 2, \ldots, n$.*

*Proof.* The proof is almost identical to that of Lemma 2.4. □

Let $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space and $e \in X$, $\|e\| = 1$. Let $\gamma, \varphi, \Gamma, \Phi$ be real numbers and $x, y \in X$ such that the conditions

$$\langle \Phi e - x, x - \varphi e \rangle \geq 0 \text{ and } \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \tag{2.20}$$

hold. In [5] we can find the inequality

$$\left| \langle x, y \rangle - \langle x, e \rangle \langle y, e \rangle \right| \leq \frac{1}{4} \left| \Phi - \varphi \right| \left| \Gamma - \gamma \right|. \tag{2.21}$$

We also have

$$\left| \langle x, y \rangle - \langle x, e \rangle \langle y, e \rangle \right| \leq \left( \|x\|^2 - \langle x, e \rangle^2 \right)^{\frac{1}{2}} \left( \|y\|^2 - \langle e, y \rangle^2 \right)^{\frac{1}{2}}. \tag{2.22}$$

Let $X = L_2(\Omega)$ and $e = 1/(b-a)$. If we define

$$\begin{aligned} T(f,g) = &\frac{1}{(b-a)^2} \int_a^b \int_a^b f(t,s) g(t,s) dt ds \\ &- \frac{1}{(b-a)^4} \int_a^b \int_a^b f(t,s) dt ds \int_a^b \int_a^b g(t,s) dt ds, \end{aligned} \tag{2.23}$$

then from (2.20) and (2.21) we obtain the Grüss inequality in $L_2(\Omega)$,

$$|T(f,g)| \leq \frac{1}{4} (\Gamma - \gamma)(\Phi - \varphi), \tag{2.24}$$

if

$$\gamma \leq f(x,y) \leq \Gamma, \ \varphi \leq g(x,y) \leq \Phi, \ (x,y) \in \Omega.$$

From (2.22), we have the pre-Grüss inequality

$$T(f,g)^2 \leq T(f,f) T(g,g). \tag{2.25}$$

We now define

$$I_n = \int_a^b \int_a^b R_n(s,t) \frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n} ds dt \tag{2.26}$$

and

$$S_n = \frac{1}{(b-a)^2} \int_a^b \int_a^b R_n(s,t) ds dt \int_a^b \int_a^b \frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n} ds dt. \tag{2.27}$$

**Lemma 2.6.** *Let $I_n$ and $S_n$ be defined by (2.26) and (2.27), respectively. Then we have the inequality*

$$|I_n - S_n| \leq \frac{M_{2n} - m_{2n}}{2} C (b-a)^2, \tag{2.28}$$

*where*

$$M_{2n} = \max_{(s,t) \in \Omega} \frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n}, \ m_{2n} = \min_{(s,t) \in \Omega} \frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n}$$

*and*

$$\begin{aligned} C = \Bigg\{ &\frac{1}{(b-a)^2} \int_a^b P_n(s)^2 ds \int_a^b Q_n(t)^2 dt \\ &- \frac{1}{(b-a)^4} \left( \int_a^b P_n(s) ds \int_a^b Q_n(t) dt \right)^2 \Bigg\}^{\frac{1}{2}}. \end{aligned} \tag{2.29}$$

*Proof.* From (2.23), (2.26) and (2.27) we see that

$$I_n - S_n = (b-a)^2 T\left(R_n(s,t), \frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n}\right).$$

Then from (2.25) we get

$$|I_n - S_n| \le (b-a)^2 T\left(R_n(s,t), R_n(s,t)\right)^{\frac{1}{2}} T\left(\frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n}, \frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n}\right)^{\frac{1}{2}}.$$

From (2.24) we have

$$T\left(\frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n}, \frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n}\right)^{\frac{1}{2}} \le \frac{M_{2n} - m_{2n}}{2}.$$

We also have

$$\begin{aligned} &T\left(R_n(s,t), R_n(s,t)\right) \\ &\quad = \frac{1}{(b-a)^2} \int_a^b P_n(s)^2 ds \int_a^b Q_n(t)^2 dt - \frac{1}{(b-a)^4}\left(\int_a^b P_n(s) ds \int_a^b Q_n(t) dt\right)^2. \end{aligned}$$

From the last three relations we see that (2.28) holds. □

**Theorem 2.7.** *Let* $\Omega = [a,b] \times [a,b]$ *and let* $f : \Omega \to \mathbb{R}$ *be a given function such that* $f \in C^{2n}(\Omega)$*. Let the conditions of Lemma 2.6 hold. If* $J_k$*,* $K_k$ *are given by (2.9), (2.17), where* $J_{k,1}$*,* $J_{k,2}$*,* $K_{k,1}$*,* $K_{k,2}$ *are given by Lemmas 2.2 – 2.5, then we have the inequality*

$$\left| \int_a^b \int_a^b f(s,t) ds dt + \sum_{k=1}^n J_k - \sum_{k=1}^n K_k - S_n \right| \le \frac{M_{2n} - m_{2n}}{2} C (b-a)^2, \tag{2.30}$$

*where*

$$\begin{aligned} S_n = \frac{1}{(b-a)^2} &\left[P_{n+1}(b) - P_{n+1}(a)\right]\left[Q_{n+1}(b) - Q_{n+1}(a)\right] \\ &\times \left[v(b,b) - v(b,a) - v(a,b) + v(a,a)\right], \end{aligned} \tag{2.31}$$

*and* $v(s,t) = \frac{\partial^{2n-2} t(s,t)}{\partial s^{n-1} \partial t^{n-1}}$.

*Proof.* We have

$$\begin{aligned} I_n &= \int_a^b \int_a^b R_n(s,t) \frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n} ds dt \\ &= \int_a^b dt \int_a^b R_n(s,t) \frac{\partial}{\partial s}\left[\frac{\partial^{2n-1} f(s,t)}{\partial s^{n-1} \partial t^n}\right] ds \\ &= \int_a^b \left[R_n(b,t) \frac{\partial^{2n-1} f(b,t)}{\partial s^{n-1} \partial t^n} - R_n(a,t) \frac{\partial^{2n-1} f(a,t)}{\partial s^{n-1} \partial t^n}\right] dt \\ &\qquad - \int_a^b \int_a^b \frac{\partial R_n(s,t)}{\partial s} \frac{\partial^{2n-1} f(s,t)}{\partial s^{n-1} \partial t^n} ds dt \\ &= J_n - L_n, \end{aligned} \tag{2.32}$$

where

$$L_n = \int_a^b \int_a^b \frac{\partial R_n(s,t)}{\partial s} \frac{\partial^{2n-1} f(s,t)}{\partial s^{n-1} \partial t^n} ds dt.$$

We also have

$$\begin{aligned}L_n &= \int_a^b ds \int_a^b \frac{\partial R_n(s,t)}{\partial s}\frac{\partial}{\partial t}\left[\frac{\partial^{2n-2} f(s,t)}{\partial s^{n-1}\partial t^{n-1}}\right] dt \\ &= \int_a^b \left[\frac{\partial R_n(s,b)}{\partial s}\frac{\partial^{2n-2} f(s,b)}{\partial s^{n-1}\partial t^{n-1}} - \frac{\partial R_n(s,a)}{\partial s}\frac{\partial^{2n-2} f(s,a)}{\partial s^{n-1}\partial t^{n-1}}\right] ds \\ &\qquad - \int_a^b \int_a^b R_{n-1}(s,t)\frac{\partial^{2n-2} f(s,t)}{\partial s^{n-1}\partial t^{n-1}} dsdt \\ &= K_n - I_{n-1}.\end{aligned}$$

Hence, we have

$$I_n = J_n - K_n + I_{n-1}.$$

In a similar way we obtain

$$I_{n-1} = J_{n-1} - K_{n-1} + I_{n-2}.$$

If we continue this procedure then we get

$$I_n = \sum_{k=1}^{n} J_k - \sum_{k=1}^{n} K_k + I_0, \tag{2.33}$$

where

$$I_0 = \int_a^b \int_a^b f(s,t) dsdt. \tag{2.34}$$

We now consider the term

$$S_n = \frac{1}{(b-a)^2}\int_a^b \int_a^b R_n(s,t)dsdt \int_a^b \int_a^b \frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n} dsdt. \tag{2.35}$$

We have

$$\begin{aligned}\int_a^b \int_a^b R_n(s,t)dsdt &= \int_a^b P_n(s)ds \int_a^b Q_n(t)dt \\ &= \left[P_{n+1}(b) - P_{n+1}(a)\right]\left[Q_{n+1}(b) - Q_{n+1}(a)\right]\end{aligned}$$

and

$$\begin{aligned}&\int_a^b \int_a^b \frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n} dsdt \\ &\qquad = \int_a^b dt \int_a^b \frac{\partial}{\partial s}\left[\frac{\partial^{2n-1} f(s,t)}{\partial s^{n-1}\partial t^{n}}\right] ds \\ &\qquad = \int_a^b \left[\frac{\partial^{2n-1} f(b,t)}{\partial s^{n-1}\partial t^{n}} - \frac{\partial^{2n-1} f(a,t)}{\partial s^{n-1}\partial t^{n}}\right] dt \\ &\qquad = \frac{\partial^{2n-2} f(b,b)}{\partial s^{n-1}\partial t^{n-1}} - \frac{\partial^{2n-2} f(b,a)}{\partial s^{n-1}\partial t^{n-1}} - \frac{\partial^{2n-1} f(a,b)}{\partial s^{n-1}\partial t^{n-1}} + \frac{\partial^{2n-1} f(a,a)}{\partial s^{n-1}\partial t^{n-1}} \\ &\qquad = \left[v(b,b) - v(b,a) - v(a,b) + v(a,a)\right],\end{aligned}$$

Thus (2.31) holds. From (2.33) – (2.35) we see that

$$I_n - S_n = \int_a^b \int_a^b f(s,t)dsdt + \sum_{k=1}^{n} J_k - \sum_{k=1}^{n} K_k - S_n.$$

Then from Lemma 2.6 we conclude that (2.30) holds. □

## 3. A PARTICULAR INEQUALITY

Here we use the notations introduced in Section 2. In Theorem 2.7 we proved a general inequality of Ostrowski-Grüss type. Many particular inequalities can be obtained if we choose specific harmonic or Appell-like polynomials $P_k(s)$, $Q_k(t)$ in (2.30). For example, in [8] we can find the following harmonic polynomials

$$\begin{aligned} P_k(s) &= \frac{1}{k!}(s-a)^k, \\ P_k(s) &= \frac{1}{k!}\left(s - \frac{a+b}{2}\right)^k, \\ P_k(s) &= \frac{(b-a)^k}{k!} B_k\left(\frac{s-a}{b-a}\right), \\ P_k(s) &= \frac{(b-a)^k}{k!} E_k\left(\frac{s-a}{b-a}\right), \end{aligned}$$

where $B_k(s)$ and $E_k(s)$ are Bernoulli and Euler polynomials, respectively. We shall not consider all possible combinations of these polynomials. Here we choose the following combination

$$P_k(s) = \frac{(b-a)^k}{k!} B_k\left(\frac{s-a}{b-a}\right), \qquad Q_k(t) = \frac{(b-a)^k}{k!} B_k\left(\frac{t-a}{b-a}\right). \tag{3.1}$$

We now substitute the above polynomials in (2.10), (2.11), (2.18), (2.19) to obtain

$$\begin{aligned} J_{k,1} &= \bar{J}_{k,1} \\ &= \frac{(b-a)^k}{k!} B_k(1) \sum_{j=1}^{k} (-1)^{k-j} \frac{(b-a)^j}{j!} \\ &\quad \times \left[ B_j(1) u_{k-1}^{(j-1)}(b) - B_j(0) u_{k-1}^{(j-1)}(a) \right] + (-1)^k B_k(1) \frac{(b-a)^k}{k!} \int_a^b u_{k-1}(t)dt, \end{aligned} \tag{3.2}$$

$$\begin{aligned} J_{k,2} &= \bar{J}_{k,2} \\ &= \frac{(b-a)^k}{k!} B_k(0) \sum_{j=1}^{k} (-1)^{k-j} \frac{(b-a)^j}{j!} \left[ B_j(1) v_{k-1}^{(j-1)}(b) - B_j(0) v_{k-1}^{(j-1)}(a) \right] \\ &\quad + (-1)^k B_k(0) \frac{(b-a)^k}{k!} \int_a^b v_{k-1}(t)dt, \end{aligned} \tag{3.3}$$

$$\begin{aligned} K_{k,1} &= \bar{K}_{k,1} \\ &= \frac{(b-a)^k}{k!} B_k(1) \sum_{j=2}^{k} (-1)^{k-j+1} \frac{(b-a)^{j-1}}{(j-1)!} \\ &\quad \times \left[ B_{j-1}(1) x_{k-1}^{(j-2)}(b) - B_{j-1}(0) x_{k-1}^{(j-2)}(a) \right] \\ &\quad + (-1)^{k-1} \frac{(b-a)^k}{k!} B_k(1) \int_a^b x_{k-1}(s)ds, \end{aligned} \tag{3.4}$$

and

$$K_{k,2} = \bar{K}_{k,2} \tag{3.5}$$
$$= \frac{(b-a)^k}{k!} B_k(0) \sum_{j=2}^{k} (-1)^{k-j+1} \frac{(b-a)^{j-1}}{(j-1)!} \times \left[ B_{j-1}(1) y_{k-1}^{(j-2)}(b) - B_{j-1}(0) y_{k-1}^{(j-2)}(a) \right] + (-1)^{k-1} \frac{(b-a)^k}{k!} B_k(0) \int_a^b y_{k-1}(s)ds.$$

We have

$$J_k = \bar{J}_k = \bar{J}_{k,1} - \bar{J}_{k,2},\ k = 1, 2, \dots, n, \tag{3.6}$$

$$K_k = \bar{K}_k = \bar{K}_{k,1} - \bar{K}_{k,2},\ \ k = 2, \dots, n \tag{3.7}$$

and

$$\bar{K}_1 = \frac{b-a}{2} \left[ \int_a^b x_0(s)ds + \int_a^b y_0(s)ds \right], \tag{3.8}$$

where $\bar{J}_{k,1}, \bar{J}_{k,2}, \bar{K}_{k,1}, \bar{K}_{k,2}$ are defined by (3.2) – (3.5), respectively.

Basic properties of Bernoulli polynomials can be found in [1]. Here we emphasize the following properties:

$$\int_0^1 B_k(s)ds = 0,\ \ k = 1, 2, \dots \tag{3.9}$$

and

$$\int_0^1 B_k(s) B_j(s) ds = (-1)^{k-1} \frac{k!j!}{(k+j)!} B_{k+j},\ \ k, j = 1, 2, \dots, \tag{3.10}$$

where

$$B_k = B_k(0),\ k = 0, 1, 2, \dots \tag{3.11}$$

are Bernoulli numbers. We also have

$$B_{2i+1} = 0,\ i = 1, 2, \dots, \tag{3.12}$$

$$B_k(0) = B_k(1) = B_k,\ k = 0, 2, 3, 4, \dots, \tag{3.13}$$

and, in particular,

$$B_1(0) = -\frac{1}{2},\ \ B_1(1) = \frac{1}{2}. \tag{3.14}$$

From (3.2) – (3.8) and (3.12) we see that

$$\bar{J}_{2i+1} = \bar{K}_{2i+1} = 0,\ \ i = 1, 2, \dots, n. \tag{3.15}$$

Note also that sums in (3.2) – (3.5) have only even-indexed terms and the term for $j = 1$ $(j = 2)$ is non-zero.

**Theorem 3.1.** *Under the assumptions of Theorem 2.7 we have*

$$\left| \int_a^b \int_a^b f(s,t)dsdt + \sum_{k=1}^n \bar{J}_k - \sum_{k=1}^n \bar{K}_k \right| \le \frac{M_{2n} - m_{2n}}{2} \cdot \frac{|B_{2n}|}{(2n)!}(b-a)^{2n+2}, \tag{3.16}$$

*where $B_k$ are Bernoulli numbers and $\bar{J}_k$, $\bar{K}_k$ are given by (3.6), (3.7), respectively.*

*Proof.* The proof follows from the proof of Theorem 2.7, since the following is valid. Let $P_n$ and $Q_n$ be defined by (3.1), for $k = n$.

Firstly, we have

$$S_n = \frac{1}{(b-a)^2} \int_a^b \int_a^b R_n(s,t)dsdt \int_a^b \int_a^b \frac{\partial^{2n} f(s,t)}{\partial s^n \partial t^n} dsdt = 0,$$

since

$$\begin{aligned} \int_a^b \int_{an}^b (s,t)dsdt &= \int_a^b P_n(s)ds \int_a^b Q_n(t)dt \\ &= \left( \int_a^b P_n(s)ds \right)^2 \\ &= \left[ \frac{(b-a)^{n+1}}{n!} \int_0^1 B_n(s)ds \right]^2 = 0, \end{aligned}$$

because of (3.9).

Secondly, we have

$$\begin{aligned} C &= \left\{ \frac{1}{(b-a)^2} \int_a^b P_n(s)^2 ds \int_a^b Q_n(t)^2 dt - \frac{1}{(b-a)^4} \left( \int_a^b P_n(s)ds \int_a^b Q_n(t)dt \right)^2 \right\}^{\frac{1}{2}} \\ &= \left[ \frac{1}{(b-a)^2} \int_a^b P_n(s)^2 ds \int_a^b Q_n(t)^2 dt \right]^{\frac{1}{2}} \\ &= \frac{1}{b-a} \int_a^b P_n(s)^2 ds \\ &= \frac{1}{b-a} \cdot \frac{(b-a)^{2n+1}}{(n!)^2} \int_0^1 B_n(s)^2 ds \\ &= \frac{(b-a)^{2n}}{(n!)^2} \frac{(n!)^2}{(2n)!} |B_{2n}| = \frac{|B_{2n}|}{(2n)!}(b-a)^{2n}, \end{aligned}$$

since (3.10) holds. □

## REFERENCES


[1] M. ABRAMOWITZ AND I.A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series 55, 4th printing, Washington, 1965

[2] G.A. ANASTASSIOU, Multivariate Ostrowski type inequalities, *Acta Math. Hungar.*, **76** (1997), 267–278.

[3] G.A. ANASTASSIOU, Ostrowski type inequalities, *Proc. Amer. Math. Soc.*, **123**(12) (1995), 3775–3781.

[4] X.L. CHENG, Improvement of some Ostrowski-Grüss type inequalities, *Comput. Math. Appl.*, **42** (2001), 109–114.

[5] S.S. DRAGOMIR, A generalization of Grüss inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74–82.

[6] S.S. DRAGOMIR AND S. WANG, An inequality of Ostrowski–Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Comput. Math. Appl.*, **33**(11) (1997), 16–20.

[7] G. HANNA, P. CERONE AND J. ROUMELIOTIS, An Ostrowski type inequality in two dimensions using the three point rule, *ANZIAM J.*, **42**(E) (2000), 671–689.

[8] M. MATIĆ, J. PEČARIĆ AND N. UJEVIĆ, On new estimation of the remainder in generalized Taylor's formula, *Math. Inequal. Appl.*, **2**(3), (1999), 343–361.

[9] M. MATIĆ, J. PEČARIĆ AND N. UJEVIĆ, Improvement and further generalization of some inequalities of Ostrowski-Grüss type, *Comput. Math. Appl.*, **39** (2000), 161–175.

[10] M. MATIĆ, J. PEČARIĆ AND N. UJEVIĆ, Generalizations of weighted version of Ostrowski's inequality and some related results, *J. Inequal. Appl.*, **5** (2000), 639–666.

[11] M. MATIĆ, J. PEČARIĆ AND N. UJEVIĆ, Weighted version of multivariate Ostrowski type inequalities, *Rocky Mountain J. Math*, **31**(2) (2001), 511–538.

[12] G.V. MILOVANOVIĆ, On some integral inequalities, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No 498-541, (1975), 119–124.

[13] G.V. MILOVANOVIĆ, O nekim funkcionalnim nejednakostima, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No 599, (1977), 1–59.

[14] G.V. MILOVANOVIĆ AND J.E. PEČARIĆ, On generalization of the inequality of A. Ostrowski and some related applications, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No 544-576, (1976), 155-158.

[15] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[16] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Acad. Publ., Dordrecht, Boston/London, 1991.

[17] A. OSTROWSKI, Über die Absolutabweichung einer Differentiebaren Funktion von ihren Integralmittelwert, *Comment. Math. Helv.*, **10** (1938), 226–227.

[18] B.G. PACHPATTE, On multidimensional Grüss type inequalities, *J. Inequal. Pure Appl. Math.*, **3**(2) (2002), Article 27. [ONLINE `http://jipam.vu.edu.au/v3n2/063_01.html`].