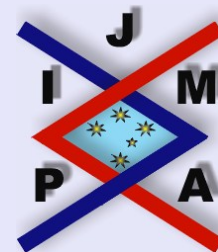


NENAD UJEVIĆ

Department of Mathematics
University of Split
Teslina 12/III, 21000 Split
CROATIA.
EMail: ujevic@pmfst.hr



volume 4, issue 5, article 101,
2003.

*Received 08 January, 2003;
accepted 07 August, 2003.*

Communicated by: B.G. Pachpatte

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

A general Ostrowski-Grüss type inequality in two dimensions is established. A particular inequality of the same type is also given.

2000 Mathematics Subject Classification: 26D10, 26D15.

Key words: Ostrowski's inequality, 2-dimensional generalization, Ostrowski-Grüss inequality.

Contents

1	Introduction	3
2	A General Ostrowski-Grüss Inequality	4
3	A Particular Inequality	18
References		



Ostrowski-Grüss type Inequalities in Two Dimensions

Nenad Ujević

Title Page

Contents



Go Back

Close

Quit

Page 2 of 26

1. Introduction

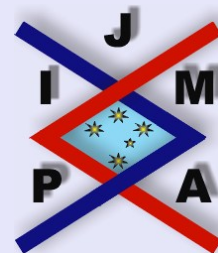
In 1938 A. Ostrowski proved the following integral inequality ([17] or [16, p. 468]).

Theorem 1.1. *Let $f : I \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping differentiable in the interior $\text{Int } I$ of I , and let $a, b \in \text{Int } I$, $a < b$. If $|f'(t)| \leq M$, $\forall t \in [a, b]$, then we have*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)M,$$

for $x \in [a, b]$.

The first (direct) generalization of Ostrowski's inequality was given by G.V. Milovanović and J. Pečarić in [14]. In recent years a number of authors have written about generalizations of Ostrowski's inequality. For example, this topic is considered in [2], [4], [6], [9] and [14]. In this way, some new types of inequalities have been formed, such as inequalities of Ostrowski-Grüss type, inequalities of Ostrowski-Chebyshev type, etc. The first inequality of Ostrowski-Grüss type was given by S.S. Dragomir and S. Wang in [6]. It was generalized and improved in [9]. X.L. Cheng gave a sharp version of the mentioned inequality in [4]. The first multivariate version of Ostrowski's inequality was given by G.V. Milovanović in [12] (see also [13] and [16, p. 468]). Multivariate versions of Ostrowski's inequality were also considered in [3], [7] and [11]. In this paper we give a general two-dimensional Ostrowski-Grüss inequality. For that purpose, we introduce specially defined polynomials, which can be considered as harmonic or Appell-like polynomials in two dimensions. In Section 3 we use the mentioned general inequality to obtain a particular two-dimensional Ostrowski-Grüss type inequality.



Ostrowski-Grüss type
Inequalities in Two Dimensions

Nenad Ujević

Title Page

Contents



Go Back

Close

Quit

Page 3 of 26

2. A General Ostrowski-Grüss Inequality

Let $\Omega = [a, b] \times [a, b]$ and let $f : \Omega \rightarrow \mathbb{R}$ be a given function. Here we suppose that $f \in C^{2n}(\Omega)$. Let $P_k(s)$ and $Q_k(t)$ be harmonic or Appell-like polynomials, i.e.

$$(2.1) \quad P'_k(s) = P_{k-1}(s) \text{ and } Q'_k(t) = Q_{k-1}(t), \quad k = 1, 2, \dots, n + 1,$$

with

$$(2.2) \quad P_0(s) = Q_0(t) = 1.$$

We also define

$$(2.3) \quad R_k(s, t) = P_k(s)Q_k(t), \quad k = 0, 1, 2, \dots, n + 1.$$

Lemma 2.1. *Let $R_k(s, t)$ be defined by (2.3). Then we have*

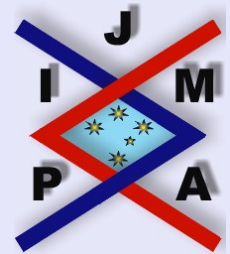
$$(2.4) \quad \frac{\partial^2 R_k(s, t)}{\partial s \partial t} = R_{k-1}(s, t)$$

for $k = 1, 2, \dots, n + 1$.

Proof. From (2.1) – (2.3) it follows that

$$\begin{aligned} \frac{\partial^2 R_k(s, t)}{\partial s \partial t} &= \frac{\partial}{\partial t} \left(\frac{\partial R_k(s, t)}{\partial s} \right) \\ &= \frac{\partial}{\partial t} (P'_k(s)Q_k(t)) \\ &= P_{k-1}(s)Q'_k(t) \\ &= P_{k-1}(s)Q_{k-1}(t) = R_{k-1}(s, t). \end{aligned}$$

□



Ostrowski-Grüss type
Inequalities in Two Dimensions

Nenad Ujević

Title Page

Contents

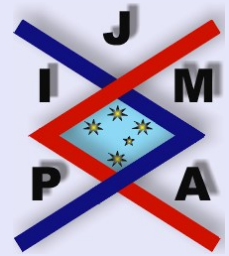


Go Back

Close

Quit

Page 4 of 26



Title Page

Contents



Go Back

Close

Quit

Page 5 of 26

We now define

$$(2.5) \quad J_k = \int_a^b \left[R_k(b, t) \frac{\partial^{2k-1} f(b, t)}{\partial s^{k-1} \partial t^k} - R_k(a, t) \frac{\partial^{2k-1} f(a, t)}{\partial s^{k-1} \partial t^k} \right] dt,$$

$$(2.6) \quad u_{k-1}(t) = \frac{\partial^{k-1} f(b, t)}{\partial s^{k-1}}, \quad v_{k-1}(t) = \frac{\partial^{k-1} f(a, t)}{\partial s^{k-1}},$$

for $k = 1, 2, \dots, n$. We also define

$$(2.7) \quad J_{k,1} = \int_a^b R_k(b, t) \frac{\partial^{2k-1} f(b, t)}{\partial s^{k-1} \partial t^k} dt = P_k(b) \int_a^b Q_k(t) u_{k-1}^{(k)}(t) dt$$

and

$$(2.8) \quad J_{k,2} = \int_a^b R_k(a, t) \frac{\partial^{2k-1} f(a, t)}{\partial s^{k-1} \partial t^k} dt = P_k(a) \int_a^b Q_k(t) v_{k-1}^{(k)}(t) dt$$

such that

$$(2.9) \quad J_k = J_{k,1} - J_{k,2}.$$

Lemma 2.2. Let $J_{k,1}$ be defined by (2.7). Then we have

$$(2.10) \quad J_{k,1} = P_k(b) \sum_{j=1}^k (-1)^{k-j} \left[Q_j(b) u_{k-1}^{(j-1)}(b) - Q_j(a) u_{k-1}^{(j-1)}(a) \right] \\ + (-1)^k P_k(b) \int_a^b u_{k-1}(t) dt,$$

for $k = 1, 2, \dots, n$.

Proof. We introduce the notation

$$U_k(u_{k-1}) = \int_a^b Q_k(t)u_{k-1}^{(k)}(t)dt.$$

Then we have

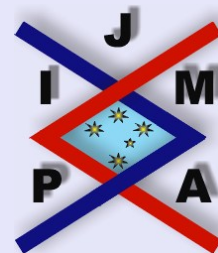
$$\begin{aligned} (-1)^k U_k(u_{k-1}) &= (-1)^k \int_a^b Q_k(t)u_{k-1}^{(k)}(t)dt \\ &= (-1)^k \left[Q_k(b)u_{k-1}^{(k-1)}(b) - Q_k(a)u_{k-1}^{(k-1)}(a) \right] \\ &\quad + (-1)^{k-1} \int_a^b Q_{k-1}(t)u_{k-1}^{(k-1)}(t)dt. \end{aligned}$$

We can write the above relation in the form

$$\begin{aligned} (-1)^k U_k(u_{k-1}) \\ = (-1)^k \left[Q_k(b)u_{k-1}^{(k-1)}(b) - Q_k(a)u_{k-1}^{(k-1)}(a) \right] + (-1)^{k-1} U_{k-1}(u_{k-1}). \end{aligned}$$

In a similar way we get

$$\begin{aligned} (-1)^{k-1} U_{k-1}(u_{k-1}) &= (-1)^{k-1} \int_a^b Q_{k-1}(t)u_{k-1}^{(k-1)}(t)dt \\ &= (-1)^{k-1} \left[Q_{k-1}(b)u_{k-1}^{(k-2)}(b) - Q_{k-1}(a)u_{k-1}^{(k-2)}(a) \right] \\ &\quad + (-1)^{k-2} \int_a^b Q_{k-2}(t)u_{k-1}^{(k-2)}(t)dt \end{aligned}$$



Title Page

Contents



Go Back

Close

Quit

Page 6 of 26

or

$$\begin{aligned} & (-1)^{k-1}U_{k-1}(u_{k-1}) \\ &= (-1)^{k-1} \left[Q_{k-1}(b)u_{k-1}^{(k-2)}(b) - Q_{k-1}(a)u_{k-1}^{(k-2)}(a) \right] \\ & \quad + (-1)^{k-2}U_{k-2}(u_{k-1}). \end{aligned}$$

If we continue the above procedure then we obtain

$$\begin{aligned} & (-1)^k U_k(u_{k-1}) \\ &= \sum_{j=1}^k (-1)^j \left[Q_j(b)u_{k-1}^{(j-1)}(b) - Q_j(a)u_{k-1}^{(j-1)}(a) \right] + U_0(u_{k-1}) \\ &= \sum_{j=1}^k (-1)^j \left[Q_j(b)u_{k-1}^{(j-1)}(b) - Q_j(a)u_{k-1}^{(j-1)}(a) \right] + \int_a^b u_{k-1}(t)dt. \end{aligned}$$

Note now that

$$J_{k,1} = P_k(b)U_k(u_{k-1})$$

such that (2.10) holds. □

Lemma 2.3. *Let $J_{k,2}$ be defined by (2.8). Then we have*

$$\begin{aligned} (2.11) \quad J_{k,2} &= P_k(a) \sum_{j=1}^k (-1)^{k-j} \left[Q_j(b)v_{k-1}^{(j-1)}(b) - Q_j(a)v_{k-1}^{(j-1)}(a) \right] \\ & \quad + (-1)^k P_k(a) \int_a^b v_{k-1}(t)dt, \end{aligned}$$



Title Page

Contents



Go Back

Close

Quit

Page 7 of 26

for $k = 1, 2, \dots, n$.

Proof. The proof is almost identical to that of Lemma 2.2. □

We now define

$$(2.12) \quad K_k = \int_a^b \left[\frac{\partial R_k(s, b)}{\partial s} \frac{\partial^{2k-2} f(s, b)}{\partial s^{k-1} \partial t^{k-1}} - \frac{\partial R_k(s, a)}{\partial s} \frac{\partial^{2k-2} f(s, a)}{\partial s^{k-1} \partial t^{k-1}} \right] ds,$$

for $k = 2, \dots, n$,

$$(2.13) \quad x_{k-1}(s) = \frac{\partial^{k-1} f(s, b)}{\partial t^{k-1}}, \quad y_{k-1}(s) = \frac{\partial^{k-1} f(s, a)}{\partial t^{k-1}}$$

and

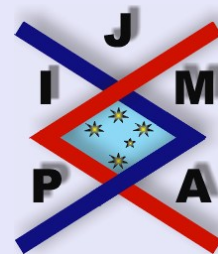
$$(2.14) \quad K_1 = Q_1(b) \int_a^b x_0(s) ds - Q_1(a) \int_a^b y_0(s) ds.$$

We also define

$$(2.15) \quad \begin{aligned} K_{k,1} &= \int_a^b \frac{\partial R_k(s, b)}{\partial s} \frac{\partial^{2k-2} f(s, b)}{\partial s^{k-1} \partial t^{k-1}} ds \\ &= Q_k(b) \int_a^b P_{k-1}(s) x_{k-1}^{(k-1)}(s) ds \end{aligned}$$

and

$$(2.16) \quad \begin{aligned} K_{k,2} &= \int_a^b \frac{\partial R_k(s, a)}{\partial s} \frac{\partial^{2k-2} f(s, a)}{\partial s^{k-1} \partial t^{k-1}} ds \\ &= Q_k(a) \int_a^b P_{k-1}(s) y_{k-1}^{(k-1)}(s) ds \end{aligned}$$



**Ostrowski-Grüss type
Inequalities in Two Dimensions**

Nenad Ujević

Title Page

Contents



Go Back

Close

Quit

Page 8 of 26

such that

$$(2.17) \quad K_k = K_{k,1} - K_{k,2}, \quad k = 1, 2, \dots, n.$$

Lemma 2.4. *Let $K_{k,1}$ be defined by (2.15). Then we have*

$$(2.18) \quad K_{k,1} = Q_k(b) \sum_{j=2}^k (-1)^{k-j+1} \left[P_{j-1}(b)x_{k-1}^{(j-2)}(b) - P_{j-1}(a)x_{k-1}^{(j-2)}(a) \right] \\ + (-1)^{k-1} Q_k(b) \int_a^b x_{k-1}(s) ds,$$

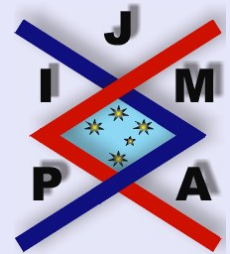
for $k = 2, \dots, n$.

Proof. We introduce the notation

$$U_{k-1}(x_{k-1}) = \int_a^b P_{k-1}(s)x_{k-1}^{(k-1)}(s) ds.$$

Then we have

$$(-1)^{k-1} U_{k-1}(x_{k-1}) = (-1)^{k-1} \int_a^b P_{k-1}(s)x_{k-1}^{(k-1)}(s) ds \\ = (-1)^{k-1} \left[P_{k-1}(b)x_{k-1}^{(k-2)}(b) - P_{k-1}(a)x_{k-1}^{(k-2)}(a) \right] \\ + (-1)^{k-2} \int_a^b P_{k-2}(s)x_{k-1}^{(k-2)}(s) ds.$$



Title Page

Contents



Go Back

Close

Quit

Page 9 of 26

We can write the above relation in the form

$$\begin{aligned} & (-1)^{k-1}U_{k-1}(x_{k-1}) \\ &= (-1)^{k-1} \left[P_{k-1}(b)x_{k-1}^{(k-2)}(b) - P_{k-1}(a)x_{k-1}^{(k-2)}(a) \right] \\ & \quad + (-1)^{k-2}U_{k-2}(x_{k-1}). \end{aligned}$$

In a similar way we get

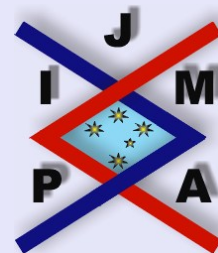
$$\begin{aligned} (-1)^{k-2}U_{k-2}(x_{k-1}) &= (-1)^{k-2} \int_a^b P_{k-2}(s)x_{k-1}^{(k-2)}(s)ds \\ &= (-1)^{k-2} \left[P_{k-2}(b)x_{k-1}^{(k-3)}(b) - P_{k-2}(a)x_{k-1}^{(k-3)}(a) \right] \\ & \quad + (-1)^{k-3} \int_a^b P_{k-3}(s)x_{k-1}^{(k-3)}(s)ds \end{aligned}$$

or

$$\begin{aligned} & (-1)^{k-2}U_{k-2}(x_{k-1}) \\ &= (-1)^{k-2} \left[P_{k-2}(b)x_{k-1}^{(k-3)}(b) - P_{k-2}(a)x_{k-1}^{(k-3)}(a) \right] \\ & \quad + (-1)^{k-3}U_{k-3}(x_{k-1}). \end{aligned}$$

If we continue the above procedure then we get

$$\begin{aligned} & (-1)^{k-1}U_{k-1}(x_{k-1}) \\ &= \sum_{j=2}^k (-1)^{j-1} \left[P_{j-1}(b)x_{k-1}^{(j-2)}(b) - P_{j-1}(a)x_{k-1}^{(j-2)}(a) \right] + U_0(x_{k-1}) \end{aligned}$$



Ostrowski-Grüss type
Inequalities in Two Dimensions

Nenad Ujević

Title Page

Contents



Go Back

Close

Quit

Page 10 of 26

$$= \sum_{j=2}^k (-1)^{j-1} \left[P_{j-1}(b)x_{k-1}^{(j-2)}(b) - P_{j-1}(a)x_{k-1}^{(j-2)}(a) \right] + \int_a^b x_{k-1}(t)dt.$$

Note now that

$$K_{k,1} = Q_k(b)U_{k-1}(x_{k-1})$$

such that (2.18) holds. □

Lemma 2.5. *Let $K_{k,2}$ be defined by (2.16). Then we have*

$$(2.19) \quad K_{k,2} = Q_k(a) \sum_{j=2}^k (-1)^{k-j+1} \left[P_{j-1}(b)y_{k-1}^{(j-2)}(b) - P_{j-1}(a)y_{k-1}^{(j-2)}(a) \right] \\ + (-1)^{k-1} Q_k(a) \int_a^b y_{k-1}(s)ds,$$

for $k = 2, \dots, n$.

Proof. The proof is almost identical to that of Lemma 2.4. □

Let $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space and $e \in X$, $\|e\| = 1$. Let $\gamma, \varphi, \Gamma, \Phi$ be real numbers and $x, y \in X$ such that the conditions

$$(2.20) \quad \langle \Phi e - x, x - \varphi e \rangle \geq 0 \text{ and } \langle \Gamma e - y, y - \gamma e \rangle \geq 0$$

hold. In [5] we can find the inequality

$$(2.21) \quad |\langle x, y \rangle - \langle x, e \rangle \langle y, e \rangle| \leq \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma|.$$



**Ostrowski-Grüss type
Inequalities in Two Dimensions**

Nenad Ujević

Title Page

Contents



Go Back

Close

Quit

Page 11 of 26

We also have

$$(2.22) \quad |\langle x, y \rangle - \langle x, e \rangle \langle y, e \rangle| \leq (\|x\|^2 - \langle x, e \rangle^2)^{\frac{1}{2}} (\|y\|^2 - \langle y, e \rangle^2)^{\frac{1}{2}}.$$

Let $X = L_2(\Omega)$ and $e = 1/(b - a)$. If we define

$$(2.23) \quad T(f, g) = \frac{1}{(b - a)^2} \int_a^b \int_a^b f(t, s)g(t, s)dt ds \\ - \frac{1}{(b - a)^4} \int_a^b \int_a^b f(t, s)dt ds \int_a^b \int_a^b g(t, s)dt ds,$$

then from (2.20) and (2.21) we obtain the Grüss inequality in $L_2(\Omega)$,

$$(2.24) \quad |T(f, g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Phi - \varphi),$$

if

$$\gamma \leq f(x, y) \leq \Gamma, \quad \varphi \leq g(x, y) \leq \Phi, \quad (x, y) \in \Omega.$$

From (2.22), we have the pre-Grüss inequality

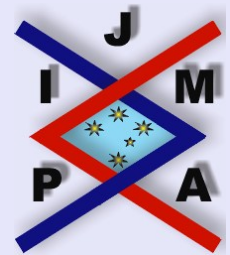
$$(2.25) \quad T(f, g)^2 \leq T(f, f)T(g, g).$$

We now define

$$(2.26) \quad I_n = \int_a^b \int_a^b R_n(s, t) \frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n} ds dt$$

and

$$(2.27) \quad S_n = \frac{1}{(b - a)^2} \int_a^b \int_a^b R_n(s, t) ds dt \int_a^b \int_a^b \frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n} ds dt.$$



Title Page

Contents



Go Back

Close

Quit

Page 12 of 26

Lemma 2.6. Let I_n and S_n be defined by (2.26) and (2.27), respectively. Then we have the inequality

$$(2.28) \quad |I_n - S_n| \leq \frac{M_{2n} - m_{2n}}{2} C(b - a)^2,$$

where

$$M_{2n} = \max_{(s,t) \in \Omega} \frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n}, \quad m_{2n} = \min_{(s,t) \in \Omega} \frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n}$$

and

$$(2.29) \quad C = \left\{ \frac{1}{(b-a)^2} \int_a^b P_n(s)^2 ds \int_a^b Q_n(t)^2 dt - \frac{1}{(b-a)^4} \left(\int_a^b P_n(s) ds \int_a^b Q_n(t) dt \right)^2 \right\}^{\frac{1}{2}}.$$

Proof. From (2.23), (2.26) and (2.27) we see that

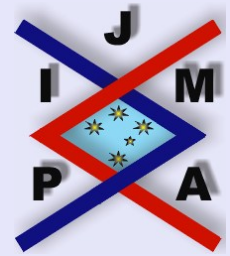
$$I_n - S_n = (b - a)^2 T \left(R_n(s, t), \frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n} \right).$$

Then from (2.25) we get

$$|I_n - S_n| \leq (b - a)^2 T(R_n(s, t), R_n(s, t))^{\frac{1}{2}} T \left(\frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n}, \frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n} \right)^{\frac{1}{2}}.$$

From (2.24) we have

$$T \left(\frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n}, \frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n} \right)^{\frac{1}{2}} \leq \frac{M_{2n} - m_{2n}}{2}.$$



Ostrowski-Grüss type
Inequalities in Two Dimensions

Nenad Ujević

Title Page

Contents



Go Back

Close

Quit

Page 13 of 26

We also have

$$T(R_n(s, t), R_n(s, t)) = \frac{1}{(b-a)^2} \int_a^b P_n(s)^2 ds \int_a^b Q_n(t)^2 dt - \frac{1}{(b-a)^4} \left(\int_a^b P_n(s) ds \int_a^b Q_n(t) dt \right)^2.$$

From the last three relations we see that (2.28) holds. □

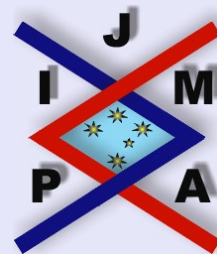
Theorem 2.7. Let $\Omega = [a, b] \times [a, b]$ and let $f : \Omega \rightarrow \mathbb{R}$ be a given function such that $f \in C^{2n}(\Omega)$. Let the conditions of Lemma 2.6 hold. If J_k, K_k are given by (2.9), (2.17), where $J_{k,1}, J_{k,2}, K_{k,1}, K_{k,2}$ are given by Lemmas 2.2 – 2.5, then we have the inequality

$$(2.30) \quad \left| \int_a^b \int_a^b f(s, t) ds dt + \sum_{k=1}^n J_k - \sum_{k=1}^n K_k - S_n \right| \leq \frac{M_{2n} - m_{2n}}{2} C(b-a)^2,$$

where

$$(2.31) \quad S_n = \frac{1}{(b-a)^2} [P_{n+1}(b) - P_{n+1}(a)] [Q_{n+1}(b) - Q_{n+1}(a)] \\ \times [v(b, b) - v(b, a) - v(a, b) + v(a, a)],$$

and $v(s, t) = \frac{\partial^{2n-2} t(s, t)}{\partial s^{n-1} \partial t^{n-1}}.$



Title Page

Contents



Go Back

Close

Quit

Page 14 of 26

Proof. We have

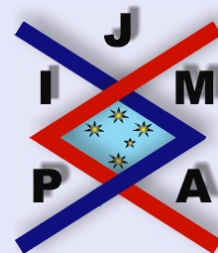
$$\begin{aligned}
 (2.32) \quad I_n &= \int_a^b \int_a^b R_n(s, t) \frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n} ds dt \\
 &= \int_a^b dt \int_a^b R_n(s, t) \frac{\partial}{\partial s} \left[\frac{\partial^{2n-1} f(s, t)}{\partial s^{n-1} \partial t^n} \right] ds \\
 &= \int_a^b \left[R_n(b, t) \frac{\partial^{2n-1} f(b, t)}{\partial s^{n-1} \partial t^n} - R_n(a, t) \frac{\partial^{2n-1} f(a, t)}{\partial s^{n-1} \partial t^n} \right] dt \\
 &\quad - \int_a^b \int_a^b \frac{\partial R_n(s, t)}{\partial s} \frac{\partial^{2n-1} f(s, t)}{\partial s^{n-1} \partial t^n} ds dt \\
 &= J_n - L_n,
 \end{aligned}$$

where

$$L_n = \int_a^b \int_a^b \frac{\partial R_n(s, t)}{\partial s} \frac{\partial^{2n-1} f(s, t)}{\partial s^{n-1} \partial t^n} ds dt.$$

We also have

$$\begin{aligned}
 L_n &= \int_a^b ds \int_a^b \frac{\partial R_n(s, t)}{\partial s} \frac{\partial}{\partial t} \left[\frac{\partial^{2n-2} f(s, t)}{\partial s^{n-1} \partial t^{n-1}} \right] dt \\
 &= \int_a^b \left[\frac{\partial R_n(s, b)}{\partial s} \frac{\partial^{2n-2} f(s, b)}{\partial s^{n-1} \partial t^{n-1}} - \frac{\partial R_n(s, a)}{\partial s} \frac{\partial^{2n-2} f(s, a)}{\partial s^{n-1} \partial t^{n-1}} \right] ds \\
 &\quad - \int_a^b \int_a^b R_{n-1}(s, t) \frac{\partial^{2n-2} f(s, t)}{\partial s^{n-1} \partial t^{n-1}} ds dt \\
 &= K_n - I_{n-1}.
 \end{aligned}$$



Ostrowski-Grüss type
Inequalities in Two Dimensions

Nenad Ujević

Title Page

Contents



Go Back

Close

Quit

Page 15 of 26

Hence, we have

$$I_n = J_n - K_n + I_{n-1}.$$

In a similar way we obtain

$$I_{n-1} = J_{n-1} - K_{n-1} + I_{n-2}.$$

If we continue this procedure then we get

$$(2.33) \quad I_n = \sum_{k=1}^n J_k - \sum_{k=1}^n K_k + I_0,$$

where

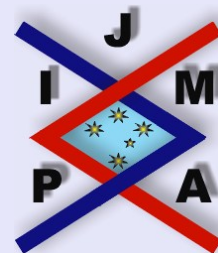
$$(2.34) \quad I_0 = \int_a^b \int_a^b f(s, t) ds dt.$$

We now consider the term

$$(2.35) \quad S_n = \frac{1}{(b-a)^2} \int_a^b \int_a^b R_n(s, t) ds dt \int_a^b \int_a^b \frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n} ds dt.$$

We have

$$\begin{aligned} \int_a^b \int_a^b R_n(s, t) ds dt &= \int_a^b P_n(s) ds \int_a^b Q_n(t) dt \\ &= [P_{n+1}(b) - P_{n+1}(a)] [Q_{n+1}(b) - Q_{n+1}(a)] \end{aligned}$$



Title Page

Contents



Go Back

Close

Quit

Page 16 of 26

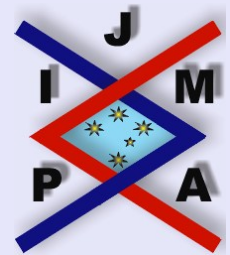
and

$$\begin{aligned}
 & \int_a^b \int_a^b \frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n} ds dt \\
 &= \int_a^b dt \int_a^b \frac{\partial}{\partial s} \left[\frac{\partial^{2n-1} f(s, t)}{\partial s^{n-1} \partial t^n} \right] ds \\
 &= \int_a^b \left[\frac{\partial^{2n-1} f(b, t)}{\partial s^{n-1} \partial t^n} - \frac{\partial^{2n-1} f(a, t)}{\partial s^{n-1} \partial t^n} \right] dt \\
 &= \frac{\partial^{2n-2} f(b, b)}{\partial s^{n-1} \partial t^{n-1}} - \frac{\partial^{2n-2} f(b, a)}{\partial s^{n-1} \partial t^{n-1}} - \frac{\partial^{2n-1} f(a, b)}{\partial s^{n-1} \partial t^{n-1}} + \frac{\partial^{2n-1} f(a, a)}{\partial s^{n-1} \partial t^{n-1}} \\
 &= [v(b, b) - v(b, a) - v(a, b) + v(a, a)],
 \end{aligned}$$

Thus (2.31) holds. From (2.33) – (2.35) we see that

$$I_n - S_n = \int_a^b \int_a^b f(s, t) ds dt + \sum_{k=1}^n J_k - \sum_{k=1}^n K_k - S_n.$$

Then from Lemma 2.6 we conclude that (2.30) holds. □



Ostrowski-Grüss type
Inequalities in Two Dimensions

Nenad Ujević

Title Page

Contents



Go Back

Close

Quit

Page 17 of 26

3. A Particular Inequality

Here we use the notations introduced in Section 2. In Theorem 2.7 we proved a general inequality of Ostrowski-Grüss type. Many particular inequalities can be obtained if we choose specific harmonic or Appell-like polynomials $P_k(s)$, $Q_k(t)$ in (2.30). For example, in [8] we can find the following harmonic polynomials

$$P_k(s) = \frac{1}{k!}(s-a)^k,$$

$$P_k(s) = \frac{1}{k!} \left(s - \frac{a+b}{2} \right)^k,$$

$$P_k(s) = \frac{(b-a)^k}{k!} B_k \left(\frac{s-a}{b-a} \right),$$

$$P_k(s) = \frac{(b-a)^k}{k!} E_k \left(\frac{s-a}{b-a} \right),$$

where $B_k(s)$ and $E_k(s)$ are Bernoulli and Euler polynomials, respectively. We shall not consider all possible combinations of these polynomials. Here we choose the following combination

$$(3.1) \quad P_k(s) = \frac{(b-a)^k}{k!} B_k \left(\frac{s-a}{b-a} \right),$$

$$Q_k(t) = \frac{(b-a)^k}{k!} B_k \left(\frac{t-a}{b-a} \right).$$

We now substitute the above polynomials in (2.10), (2.11), (2.18), (2.19) to obtain



Ostrowski-Grüss type
Inequalities in Two Dimensions

Nenad Ujević

Title Page

Contents



Go Back

Close

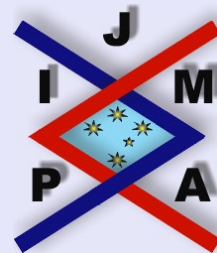
Quit

Page 18 of 26

$$\begin{aligned}
 (3.2) \quad J_{k,1} &= \bar{J}_{k,1} \\
 &= \frac{(b-a)^k}{k!} B_k(1) \sum_{j=1}^k (-1)^{k-j} \frac{(b-a)^j}{j!} \\
 &\quad \times \left[B_j(1) u_{k-1}^{(j-1)}(b) - B_j(0) u_{k-1}^{(j-1)}(a) \right] \\
 &\quad + (-1)^k B_k(1) \frac{(b-a)^k}{k!} \int_a^b u_{k-1}(t) dt,
 \end{aligned}$$

$$\begin{aligned}
 (3.3) \quad J_{k,2} &= \bar{J}_{k,2} \\
 &= \frac{(b-a)^k}{k!} B_k(0) \sum_{j=1}^k (-1)^{k-j} \frac{(b-a)^j}{j!} \\
 &\quad \times \left[B_j(1) v_{k-1}^{(j-1)}(b) - B_j(0) v_{k-1}^{(j-1)}(a) \right] \\
 &\quad + (-1)^k B_k(0) \frac{(b-a)^k}{k!} \int_a^b v_{k-1}(t) dt,
 \end{aligned}$$

$$\begin{aligned}
 (3.4) \quad K_{k,1} &= \bar{K}_{k,1} \\
 &= \frac{(b-a)^k}{k!} B_k(1) \sum_{j=2}^k (-1)^{k-j+1} \frac{(b-a)^{j-1}}{(j-1)!} \\
 &\quad \times \left[B_{j-1}(1) x_{k-1}^{(j-2)}(b) - B_{j-1}(0) x_{k-1}^{(j-2)}(a) \right] \\
 &\quad + (-1)^{k-1} \frac{(b-a)^k}{k!} B_k(1) \int_a^b x_{k-1}(s) ds,
 \end{aligned}$$



Title Page

Contents



Go Back

Close

Quit

Page 19 of 26

and

$$\begin{aligned}
 (3.5) \quad K_{k,2} &= \bar{K}_{k,2} \\
 &= \frac{(b-a)^k}{k!} B_k(0) \sum_{j=2}^k (-1)^{k-j+1} \frac{(b-a)^{j-1}}{(j-1)!} \\
 &\quad \times \left[B_{j-1}(1) y_{k-1}^{(j-2)}(b) - B_{j-1}(0) y_{k-1}^{(j-2)}(a) \right] \\
 &\quad + (-1)^{k-1} \frac{(b-a)^k}{k!} B_k(0) \int_a^b y_{k-1}(s) ds.
 \end{aligned}$$

We have

$$(3.6) \quad J_k = \bar{J}_k = \bar{J}_{k,1} - \bar{J}_{k,2}, \quad k = 1, 2, \dots, n,$$

$$(3.7) \quad K_k = \bar{K}_k = \bar{K}_{k,1} - \bar{K}_{k,2}, \quad k = 2, \dots, n$$

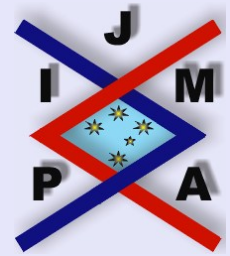
and

$$(3.8) \quad \bar{K}_1 = \frac{b-a}{2} \left[\int_a^b x_0(s) ds + \int_a^b y_0(s) ds \right],$$

where $\bar{J}_{k,1}$, $\bar{J}_{k,2}$, $\bar{K}_{k,1}$, $\bar{K}_{k,2}$ are defined by (3.2) – (3.5), respectively.

Basic properties of Bernoulli polynomials can be found in [1]. Here we emphasize the following properties:

$$(3.9) \quad \int_0^1 B_k(s) ds = 0, \quad k = 1, 2, \dots$$



Title Page

Contents



Go Back

Close

Quit

Page 20 of 26

and

$$(3.10) \quad \int_0^1 B_k(s)B_j(s)ds = (-1)^{k-1} \frac{k!j!}{(k+j)!} B_{k+j}, \quad k, j = 1, 2, \dots,$$

where

$$(3.11) \quad B_k = B_k(0), \quad k = 0, 1, 2, \dots$$

are Bernoulli numbers. We also have

$$(3.12) \quad B_{2i+1} = 0, \quad i = 1, 2, \dots,$$

$$(3.13) \quad B_k(0) = B_k(1) = B_k, \quad k = 0, 2, 3, 4, \dots,$$

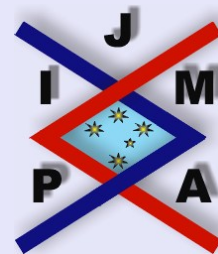
and, in particular,

$$(3.14) \quad B_1(0) = -\frac{1}{2}, \quad B_1(1) = \frac{1}{2}.$$

From (3.2) – (3.8) and (3.12) we see that

$$(3.15) \quad \bar{J}_{2i+1} = \bar{K}_{2i+1} = 0, \quad i = 1, 2, \dots, n.$$

Note also that sums in (3.2) – (3.5) have only even-indexed terms and the term for $j = 1$ ($j = 2$) is non-zero.



**Ostrowski-Grüss type
Inequalities in Two Dimensions**

Nenad Ujević

Title Page

Contents



Go Back

Close

Quit

Page 21 of 26

Theorem 3.1. Under the assumptions of Theorem 2.7 we have

$$(3.16) \quad \left| \int_a^b \int_a^b f(s, t) ds dt + \sum_{k=1}^n \bar{J}_k - \sum_{k=1}^n \bar{K}_k \right| \leq \frac{M_{2n} - m_{2n}}{2} \cdot \frac{|B_{2n}|}{(2n)!} (b-a)^{2n+2},$$

where B_k are Bernoulli numbers and \bar{J}_k, \bar{K}_k are given by (3.6), (3.7), respectively.

Proof. The proof follows from the proof of Theorem 2.7, since the following is valid. Let P_n and Q_n be defined by (3.1), for $k = n$.

Firstly, we have

$$S_n = \frac{1}{(b-a)^2} \int_a^b \int_a^b R_n(s, t) ds dt \int_a^b \int_a^b \frac{\partial^{2n} f(s, t)}{\partial s^n \partial t^n} ds dt = 0,$$

since

$$\begin{aligned} \int_a^b \int_a^b (s, t) ds dt &= \int_a^b P_n(s) ds \int_a^b Q_n(t) dt \\ &= \left(\int_a^b P_n(s) ds \right)^2 \\ &= \left[\frac{(b-a)^{n+1}}{n!} \int_0^1 B_n(s) ds \right]^2 = 0, \end{aligned}$$



Ostrowski-Grüss type
Inequalities in Two Dimensions

Nenad Ujević

Title Page

Contents



Go Back

Close

Quit

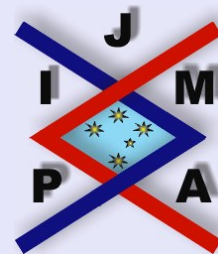
Page 22 of 26

because of (3.9).

Secondly, we have

$$\begin{aligned}
 C &= \left\{ \frac{1}{(b-a)^2} \int_a^b P_n(s)^2 ds \int_a^b Q_n(t)^2 dt \right. \\
 &\quad \left. - \frac{1}{(b-a)^4} \left(\int_a^b P_n(s) ds \int_a^b Q_n(t) dt \right)^2 \right\}^{\frac{1}{2}} \\
 &= \left[\frac{1}{(b-a)^2} \int_a^b P_n(s)^2 ds \int_a^b Q_n(t)^2 dt \right]^{\frac{1}{2}} \\
 &= \frac{1}{b-a} \int_a^b P_n(s)^2 ds \\
 &= \frac{1}{b-a} \cdot \frac{(b-a)^{2n+1}}{(n!)^2} \int_0^1 B_n(s)^2 ds \\
 &= \frac{(b-a)^{2n}}{(n!)^2} \frac{(n!)^2}{(2n)!} |B_{2n}| = \frac{|B_{2n}|}{(2n)!} (b-a)^{2n},
 \end{aligned}$$

since (3.10) holds. □



**Ostrowski-Grüss type
Inequalities in Two Dimensions**

Nenad Ujević

Title Page

Contents



Go Back

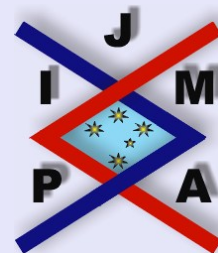
Close

Quit

Page 23 of 26

References

- [1] M. ABRAMOWITZ AND I.A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series 55, 4th printing, Washington, 1965
- [2] G.A. ANASTASSIOU, Multivariate Ostrowski type inequalities, *Acta Math. Hungar.*, **76** (1997), 267–278.
- [3] G.A. ANASTASSIOU, Ostrowski type inequalities, *Proc. Amer. Math. Soc.*, **123**(12) (1995), 3775–3781.
- [4] X.L. CHENG, Improvement of some Ostrowski-Grüss type inequalities, *Comput. Math. Appl.*, **42** (2001), 109–114.
- [5] S.S. DRAGOMIR, A generalization of Grüss inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74–82.
- [6] S.S. DRAGOMIR AND S. WANG, An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Comput. Math. Appl.*, **33**(11) (1997), 16–20.
- [7] G. HANNA, P. CERONE AND J. ROUMELIOTIS, An Ostrowski type inequality in two dimensions using the three point rule, *ANZIAM J.*, **42**(E) (2000), 671–689.



Ostrowski-Grüss type
Inequalities in Two Dimensions

Nenad Ujević

Title Page

Contents



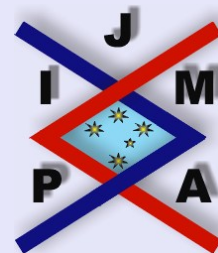
Go Back

Close

Quit

Page 24 of 26

- [8] M. MATIĆ, J. PEČARIĆ AND N. UJEVIĆ, On new estimation of the remainder in generalized Taylor's formula, *Math. Inequal. Appl.*, **2**(3), (1999), 343–361.
- [9] M. MATIĆ, J. PEČARIĆ AND N. UJEVIĆ, Improvement and further generalization of some inequalities of Ostrowski-Grüss type, *Comput. Math. Appl.*, **39** (2000), 161–175.
- [10] M. MATIĆ, J. PEČARIĆ AND N. UJEVIĆ, Generalizations of weighted version of Ostrowski's inequality and some related results, *J. Inequal. Appl.*, **5** (2000), 639–666.
- [11] M. MATIĆ, J. PEČARIĆ AND N. UJEVIĆ, Weighted version of multivariate Ostrowski type inequalities, *Rocky Mountain J. Math*, **31**(2) (2001), 511–538.
- [12] G.V. MILOVANOVIĆ, On some integral inequalities, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No 498-541, (1975), 119–124.
- [13] G.V. MILOVANOVIĆ, O nekim funkcionalnim nejednakostima, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No 599, (1977), 1–59.
- [14] G.V. MILOVANOVIĆ AND J.E. PEČARIĆ, On generalization of the inequality of A. Ostrowski and some related applications, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No 544-576, (1976), 155-158.
- [15] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.



Ostrowski-Grüss type
Inequalities in Two Dimensions

Nenad Ujević

Title Page

Contents



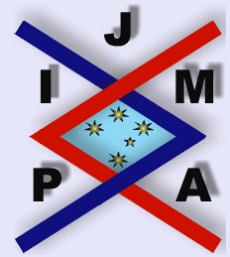
Go Back

Close

Quit

Page 25 of 26

- [16] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Acad. Publ., Dordrecht, Boston/London, 1991.
- [17] A. OSTROWSKI, Über die Absolutabweichung einer Differentiebaren Funktion von ihren Integralmittelwert, *Comment. Math. Helv.*, **10** (1938), 226–227.
- [18] B.G. PACHPATTE, On multidimensional Grüss type inequalities, *J. Inequal. Pure Appl. Math.*, **3**(2) (2002), Article 27. [ONLINE http://jipam.vu.edu.au/v3n2/063_01.html].



Ostrowski-Grüss type
Inequalities in Two Dimensions

Nenad Ujević

Title Page

Contents



Go Back

Close

Quit

Page 26 of 26