APPROXIMATION BY MODIFIED SZÁSZ-MIRAKYAN OPERATORS

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Abstract: We introduce the modified Szász-Mirakyan operators $S_{n;r}$ related to the Borel methods B_r of summability of sequences. We give theorems on approximation properties of these operators in the polynomial weight spaces.

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1. Introduction

The approximation of functions by Szász-Mirakyan operators

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0,\ n \in \mathbb{N}, \tag{1.1}$$

$(\mathbb{R}_0 = [0,\infty), \mathbb{N} = \{1,2,...\})$ has been examined in many papers and monographs (e.g. [11], [1], [2], [4], [5]).

The above operators were modified by several authors (e.g. [3], [6], [9], [10], [12]) which showed that new operators have similar or better approximation properties than S_n. M. Becker in the paper [1] studied approximation problems for the operators S_n in the polynomial weight space C_p, $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, connected with the weight function w_p,

$$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1} \quad \text{if} \quad p \in \mathbb{N}, \tag{1.2}$$

for $x \in \mathbb{R}_0$. The C_p is the set of all functions $f : \mathbb{R}_0 \to \mathbb{R}$ $(\mathbb{R} = (-\infty,\infty))$ for which fw_p is uniformly continuous and bounded on $\mathbb{R}_0$ and the norm is defined by

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x\in\mathbb{R}_0} w_p(x)|f(x)|. \tag{1.3}$$

The space C_p^m, $m \in \mathbb{N}$, $p \in \mathbb{N}_0$, of m-times differentiable functions $f \in C_p$ with derivatives $f^{(k)} \in C_p$, $1 \le k \le m$, and the norm (1.3) was considered also in [1].

In [1] it was proved that S_n is a positive linear operator acting from the space C_p to C_p for every $p \in \mathbb{N}_0$. Moreover, for a fixed $p \in \mathbb{N}_0$ there exist $M_k(p) = const. > 0$, $k = 1,2$, depending only on p such that for every $f \in C_p$ there hold the inequalities:

$$\|S_n(f)\|_p \le M_1(p)\|f\|_p \quad \text{for} \quad n \in \mathbb{N}, \tag{1.4}$$

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and

$$(1.5)\qquad w_p(x)\,|S_n(f;x)-f(x)| \le M_2(p)\omega_2\left(f;C_p;\sqrt{\frac{x}{n}}\right),\quad x\in\mathbb{R}_0, n\in\mathbb{N},$$

where $\omega_2(f;C_p;\cdot)$ is the second modulus of continuity of f.

In this paper we introduce the following modified Szász-Mirakyan operators

$$(1.6)\qquad S_{n;r}(f;x) := \frac{1}{A_r(nx)}\sum_{k=0}^{\infty}\frac{(nx)^{rk}}{(rk)!}f\left(\frac{rk}{n}\right),\quad x\in\mathbb{R}_0, n\in\mathbb{N},$$

for $f\in C_p$ and every fixed $r\in\mathbb{N}$, where

$$(1.7)\qquad A_r(t) := \sum_{k=0}^{\infty}\frac{t^{rk}}{(rk)!}\quad \text{for}\quad t\in\mathbb{R}_0.$$

Clearly $A_1(t)=e^t$, $A_2(t)=\cosh t\equiv\frac{1}{2}(e^t+e^{-t})$ and $S_{n;1}(f;x)\equiv S_n(f;x)$ for $f\in C_p$, $x\in\mathbb{R}_0$ and $n\in\mathbb{N}$. (The operators $S_{n;2}$ were investigated in [9] for functions belonging to exponential weight spaces.)

We mention that the definition of $S_{n;r}$ is related to the Borel method of summability of sequences. It is well known ([7]) that a sequence $(a_n)_0^\infty$, $a_n\in\mathbb{R}$, is summable to g by the Borel method B_r, $r\in\mathbb{N}$, if the series $\sum_{k=0}^{\infty}\frac{x^{rk}}{(rk)!}a_k$ is convergent on $\mathbb{R}$ and

$$\lim_{x\to\infty} re^{-x}\sum_{k=0}^{\infty}\frac{x^{rk}}{(rk)!}a_k = g.$$

In Section 2 we shall give some elementary properties of $S_{n;r}$. The approximation theorems will be given in Section 3.

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2. Auxiliary Results

It is known ([1]) that for $e_k(x) = x^k$, $k = 0, 1, 2$, there holds: $S_n(e_0; x) = 1$, $S_n(e_1; x) = x$ and $S_n(e_2; x) = x^2 + \frac{x}{n}$, which imply that

$$S_n\left((e_1(t) - e_1(x))^2; x\right) = \frac{x}{n} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}. \tag{2.1}$$

Moreover, for every fixed $q \in \mathbb{N}$, there exists a polynomial $\mathcal{P}_q(x) = \sum_{k=0}^{q} a_k x^k$ with real coefficients a_k, $a_q \neq 0$, depending only on q such that

$$S_n\left((e_1(t) - e_1(x))^{2q}; x\right) \leq \mathcal{P}_q(x) n^{-q} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}. \tag{2.2}$$

From (1.1) – (1.4), (1.6) and (1.7) we deduce that $S_{n;r}$ is a positive linear operator well defined on every space C_p, $p \in \mathbb{N}_0$, and

$$S_{n;r}(e_0; x) = 1, \tag{2.3}$$

$$S_{n;r}(e_1; x) = \frac{x}{n} \frac{A_r'(nx)}{A_r(nx)}, \tag{2.4}$$

$$S_{n;r}(e_2; x) = \frac{x^2}{n^2} \frac{A_r''(nx)}{A_r(nx)} + \frac{x}{n^2} \frac{A_r'(nx)}{A_r(nx)}, \tag{2.5}$$

for $x \in \mathbb{R}_0$ and $n, r \in \mathbb{N}$, and

$$S_{n;r}(f; 0) = f(0) \quad \text{for} \quad f \in C_p, \; n, r \in \mathbb{N}. \tag{2.6}$$

Here we derive a simpler formula for A_r.

Lemma 2.1. *Let $r \in \mathbb{N}$ be a fixed number. Then A_r defined by (1.7) can be rewritten in the form: $A_1(t) = e^t$, $A_2(t) = \cosh t$,*

$$A_{2m}(t) = \frac{1}{m}\left[\cosh t + \sum_{k=1}^{m-1} \exp\left(t \cos\frac{k\pi}{m}\right) \cos\left(t \sin\frac{k\pi}{m}\right)\right], \tag{2.7}$$

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for $2 \le m \in \mathbb{N}$, *and*

$$A_{2m+1}(t) = \frac{1}{2m+1}\left[e^t + 2\sum_{k=1}^{m} \exp\left(t\cos\frac{2k\pi}{2m+1}\right)\cos\left(t\sin\frac{2k\pi}{2m+1}\right)\right], \tag{2.8}$$

for $m \in \mathbb{N}$ *and* $t \in \mathbb{R}_0$.

Proof. The formulas for A_1 and A_2 are obvious by (1.7). For $r \ge 3$ and $t \in \mathbb{R}_0$ we have

$$e^t = \sum_{k=0}^{\infty} \frac{t^{rk}}{(rk)!} + \sum_{k=0}^{\infty} \frac{t^{rk+1}}{(rk+1)!} + \cdots + \sum_{k=0}^{\infty} \frac{t^{rk+r-1}}{(rk+r-1)!}$$

which by (1.7) can be written in the form

$$e^t = A_r(t) + \int_0^t A_r(u)\,du + \int_0^t \int_0^{v_1} A_r(u)\,du\,dv_1 + \cdots + \int_0^t \int_0^{v_1} \cdots \int_0^{v_{r-2}} A_r(u)\,du\,dv_{r-2}\ldots dv_1.$$

By $(r-1)$-times differentiation we get the equality

$$e^t = A_r^{(r-1)}(t) + A_r^{(r-2)}(t) + \cdots + A_r'(t) + A_r(t) \quad \text{for} \quad t \in \mathbb{R}_0,$$

which shows that $y = A_r(t)$ is the solution of the differential equation

$$y^{(r-1)} + y^{(r-2)} + \cdots + y' + y = e^t \tag{2.9}$$

satisfying the initial conditions

$$y(0) = 1, \quad y'(0) = y''(0) = \cdots = y^{(r-2)}(0) = 0. \tag{2.10}$$

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Using now the Laplace transformation

$$\mathcal{L}[y(t)] = Y(s) := \int_0^\infty y(t)e^{-st}dt, \quad s = x + iy,$$

we have by (2.10)

$$\mathcal{L}\left[y^{(k)}(t)\right] = s^k Y(s) - s^{k-1} \quad \text{for} \quad k = 1, \dots, r-1,$$

and consequently we get from (2.7)

$$\left(s^{r-1} + s^{r-2} + \cdots + s + 1\right) Y(s) = \frac{1}{s-1} + s^{r-2} + s^{r-3} + \cdots + s + 1,$$

and

$$Y(s) = \frac{s^{r-1}}{s^r - 1}. \tag{2.11}$$

By the inverse Laplace transformation we get

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^{r-1}}{s^r - 1}\right] \quad \text{for} \quad t \in \mathbb{R}_0, \tag{2.12}$$

and this $\mathcal{L}^{-1}$ transform can be calculated by the residues of Y.

It is known that the inverse transform of a rational function $\frac{P(s)}{Q(s)}$ with the simple poles s_k can be written as follows

$$\mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right] = \sum_{s_k}{}^{*} \frac{P(s_k)e^{s_k t}}{Q'(s_k)} + 2re \sum_{s_k}{}^{**} \frac{P(s_k)e^{s_k t}}{Q'(s_k)}, \tag{2.13}$$

where $\sum^*$ denotes the sum for all real s_k and $\sum^{**}$ denotes the sum for all complex $s_k = x_k + iy_k$ with a positive y_k.

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The function Y defined by (2.11) has the simple poles $s_k = \sqrt[r]{1} = e^{2k\pi i/r}$ for $k = 0, 1, \dots, r-1$. From this and (2.12) and (2.13) for $r = 2m$, $2 \le m \in \mathbb{N}$, we get

$$y(t) = \frac{1}{2m}\left(\sum_{s_k}{}^{*} e^{s_k t} + 2re\sum_{s_k}{}^{**} e^{s_k t}\right)$$
$$= \frac{1}{m}\left[\cosh t + \sum_{k=1}^{m-1} \exp\left(t\cos\frac{k\pi}{m}\right)\cos\left(t\sin\frac{k\pi}{m}\right)\right].$$

This shows that the formula (2.7) is proved.

Analogously by (2.12) and (2.13) we obtain (2.8). □

From (2.7) and (2.8) we have that

$$A_3(t) = \frac{1}{3}\left(e^t + 2e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)\right),$$
$$A_4(t) = \frac{1}{2}(\cosh t + \cos t),$$
$$A_6(t) = \frac{1}{3}\left(\cosh t + 2\cosh\frac{t}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)\right), \quad \text{for} \quad t \in \mathbb{R}_0.$$

Applying the formula (1.7) and Lemma 2.1, we immediately obtain the following:

Lemma 2.2. *For every fixed $r \in \mathbb{N}$ there exists a positive constant $M_3(r)$ depending only on r such that*

$$1 \le \frac{e^{nx}}{A_r(nx)} \le M_3(r) \quad \textit{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}. \tag{2.14}$$

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Lemma 2.3. *Let $r \in \mathbb{N}$. Then for $e_1(x) = x$ there holds*

$$\lim_{n\to\infty} nS_{n;r}\left(e_1(t) - e_1(x); x\right) = 0 \tag{2.15}$$

and

$$\lim_{n\to\infty} nS_{n;r}\left(\left(e_1(t) - e_1(x)\right)^2; x\right) = x,$$

at every $x \in \mathbb{R}_0$. Moreover, we have

$$S_{n;r}\left(\left(e_1(t) - e_1(x)\right)^{2q}; x\right) \leq M_3(r) S_n\left(\left(e_1(t) - e_1(x)\right)^{2q}; x\right) \tag{2.16}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and every fixed $q \in \mathbb{N}$.

Proof. The inequality (2.16) is obvious by (1.1), (1.6) and (2.14).

We shall prove only (2.15) for $r = 2m$, $m \in \mathbb{N}$.

If $r = 2$ then $A_2(t) = \cosh t$ and by (2.4) we have

$$\begin{aligned} S_{n;2}\left(e_1(t) - e_1(x); x\right) &= x\left(\frac{\sinh nx}{\cosh nx} - 1\right) \\ &= \frac{-2x}{e^{2nx} + 1} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}, \end{aligned}$$

which implies (2.15).

If $r = 2m$ with $2 \leq m \in \mathbb{N}$, then by (2.4), (2.7) and (2.14) we get

$$\left|S_{n;2m}\left(e_1(t) - e_1(x); x\right)\right| = \frac{x}{A_{2m}(nx)}\left|\frac{1}{n} A'_{2m}(nx) - A_{2m}(nx)\right|$$

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$$
\begin{aligned}
&= \frac{x}{mA_{2m}(nx)} \Bigg| \sinh nx - \cosh nx \\
&\qquad + \sum_{k=1}^{m-1} \exp\left(nx\cos\frac{k\pi}{m}\right)\left[\cos\frac{k\pi}{m}\cos\left(nx\sin\frac{k\pi}{m}\right)\right. \\
&\qquad \left. - \sin\frac{k\pi}{m}\sin\left(nx\sin\frac{k\pi}{m}\right) - \cos\left(nx\sin\frac{k\pi}{m}\right)\right] \Bigg| \\
&\le M_3(2m)\frac{x}{m}\left[e^{-2nx} + 3\sum_{k=1}^{m-1}\exp\left(-2nx\sin^2\frac{k\pi}{m}\right)\right]
\end{aligned}
$$

and from this we immediately obtain (2.15). □

From (1.6), (1.1) – (1.4) and (2.14) the following lemma results.

Lemma 2.4. *The operator* $S_{n;r}$, $n, r \in \mathbb{N}$, *is linear and positive, and acts from the space* C_p *to* C_p *for every* $p \in \mathbb{N}_0$. *For* $f \in C_p$

$$
\begin{aligned}
\|S_{n;r}(f)\|_p &\le \|f\|_p \|S_{n;r}(1/w_p)\|_p \\
&\le M_3(r)\|f\|_p \cdot \|S_n(1/w_p)\|_p \le M_4(p,r)\|f\|_p \quad for \quad n, r, \in \mathbb{N},
\end{aligned}
$$

where $M_4(p,r) = M_1(p)M_3(r)$ *and* $M_1(p)$, $M_3(r)$ *are positive constants given in (1.4) and (2.14).*

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3. Theorems

First we shall prove two theorems on the order of approximation of $f \in C_p$ by $S_{n;r}$, $r \geq 2$.

Theorem 3.1. *Let $p \in \mathbb{N}_0$ and $2 \leq r \in \mathbb{N}$ be fixed numbers. Then there exists $M_5(p,r) = const. > 0$ (depending only on p and r) such that for every $f \in C_p^1$ there holds the inequality*

$$w_p(x)\,|S_{n;r}(f;x) - f(x)| \leq M_5(p,r)\|f'\|_p\sqrt{\frac{x}{n}}\,, \tag{3.1}$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

Proof. Let $f \in C_p^1$. Then by (1.6), (1.7) and (2.14) it follows that

$$\begin{aligned}|S_{n;r}(f;x) - f(x)| &\leq S_{n;r}\left(|f(t) - f(x)|\,;x\right)\\ &\leq M_3(r)S_n\left(|f(t) - f(x)|;x\right) \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N},\end{aligned}$$

and for $t, x \in \mathbb{R}_0$

$$|f(t) - f(x)| = \left|\int_x^t f'(u)\,du\right| \leq \|f'\|_p\left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)}\right)|t-x|.$$

Using now the operator S_n, (1.1) – (1.4) and (2.1), we get

$$\begin{aligned}w_p(x)S_n\left(|f(t) - f(x)|;x\right) &\leq \|f'\|_p\left\{w_p(x)S_n\left(\frac{|t-x|}{w_p(t)};x\right) + S_n(|t-x|;x)\right\}\\ &\leq \|f'\|_p\left(S_n\left((t-x)^2;x\right)\right)^{1/2}\left\{2\|S_n(1/w_{2p})\|_{2p}^{1/2} + 1\right\}\\ &\leq \left(2\sqrt{M_1(2p)} + 1\right)\|f'\|_p\sqrt{\frac{x}{n}} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}.\end{aligned}$$

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Combining the above, we obtain the estimation (3.1). □

Theorem 3.2. *Let $p \in \mathbb{N}_0$ and $2 \le r \in \mathbb{N}$ be fixed. Then there exists $M_6(p,r) = const. > 0$ (depending only on p and r) such that for every $f \in C_p$, $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ there holds*

$$w_p(x)\left|S_{n;r}(f;x) - f(x)\right| \le M_6(p,r)\omega_1\left(f; C_p; \sqrt{\frac{x}{n}}\right), \tag{3.2}$$

where $\omega_1(f; C_p; \cdot)$ is the modulus of continuity of $f \in C_p$, i.e.

$$\omega_1(f; C_p; t) := \sup_{0 \le u \le t} \|\Delta_u f(\cdot)\|_p \quad \text{for} \quad t \ge 0, \tag{3.3}$$

and $\Delta_u f(x) = f(x+u) - f(x)$.

Proof. The inequality (3.2) for $x = 0$ follows by (1.2), (2.6) and (3.3).

Let $f \in C_p$ and $x > 0$. We use the Steklov function f_h,

$$f_h(x) := \frac{1}{h}\int_0^h f(x+t)\,dt \quad \text{for} \quad x \in \mathbb{R}_0, h > 0.$$

This f_h belongs to the space C_p^1 and by (3.3) it follows that

$$\|f - f_h\|_p \le \omega_1(f; C_p; h) \tag{3.4}$$

and

$$\|f_h'\|_p \le h^{-1}\omega_1(f; C_p; h), \quad \text{for} \quad h > 0. \tag{3.5}$$

By the above properties of f_h and (2.3) we can write

$$\begin{aligned} &\left|S_{n;r}\big(f(t);x\big) - f(x)\right| \\ &\qquad \le \left|S_{n;r}\left(f(t) - f_h(t);x\right)\right| + \left|S_{n;r}\big(f_h(t);x\big) - f_h(x)\right| + \left|f_h(x) - f(x)\right|, \end{aligned}$$

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for $n \in \mathbb{N}$ and $h > 0$. Next, by Lemma 2.4 and (3.4) we get

$$w_p(x) \left| S_{n;r}\left(f(t) - f_h(t); x\right) \right| \leq M_4(p,r) \| f - f_h \|_p \leq M_4(p,r) \omega_1(f; C_p; h).$$

In view of Theorem 3.1 and (3.5) we have

$$w_p(x) \left| S_{n;r}\left(f_h; x\right) - f_h(x) \right| \leq M_5(p,r) \| f_h' \|_p \sqrt{\frac{x}{n}} \leq M_5(p,r) h^{-1} \sqrt{\frac{x}{n}} \omega_1(f; C_p; h).$$

Consequently,

$$w_p(x) \left| S_{n;r}(f; x) - f(x) \right| \leq \omega_1(f; C_p; h) \left(M_4(p,r) + M_5(p,r) h^{-1} \sqrt{\frac{x}{n}} + 1 \right), \tag{3.6}$$

for $x > 0$, $n \in \mathbb{N}$ and $h > 0$. Putting $h = \sqrt{x/n}$ in (3.6) for given x and n, we obtain the desired estimation (3.2). □

Theorem 3.2 implies the following:

Corollary 3.3. *If $f \in C_p$, $p \in \mathbb{N}_0$, and $2 \leq r \in \mathbb{N}$, then*

$$\lim_{n \to \infty} S_{n;r}(f; x) = f(x) \quad \text{at every} \quad x \in \mathbb{R}_0.$$

This convergence is uniform on every interval $[x_1, x_2]$, $x_1 \geq 0$.

The Voronovskaya type theorem given in [1] for the operators S_n can be extended to $S_{n;r}$ with $r \geq 2$.

Theorem 3.4. *Suppose that $f \in C_p^2$, $p \in \mathbb{N}_0$, and $2 \leq r \in \mathbb{N}$. Then*

$$\lim_{n \to \infty} n \left(S_{n;r}(f; x) - f(x) \right) = \frac{x}{2} f''(x) \tag{3.7}$$

at every $x \in \mathbb{R}_0$.

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Proof. The statement (3.7) for $x = 0$ is obvious by (2.6). Choosing $x > 0$, we can write the Taylor formula for $f \in C_p^2$:

$$f(t) = f(x) + f'(x) + \frac{1}{2} f''(x)(t-x)^2 + \varphi(t,x)(t-x)^2 \quad \text{for} \quad t \in \mathbb{R}_0,$$

where $\varphi(t) \equiv \varphi(t,x)$ is a function belonging to C_p and $\lim_{t \to x} \varphi(t) = \varphi(x) = 0$.

Using now the operator $S_{n;r}$ and (2.3), we get

$$\begin{aligned} S_{n;r}(f(t);x) &= f(x) + f'(x) S_{n;r}(t-x;x) \\ &\quad + \frac{1}{2} f''(x) S_{n;r}((t-x)^2;x) + S_{n;r}\left(\varphi(t)(t-x)^2;x\right), \end{aligned}$$

for $n \in \mathbb{N}$, which by Lemma 2.3 implies that

$$\lim_{n\to\infty} n\left(S_{n;r}(f(t);x) - f(x)\right) = \frac{x}{2} f''(x) + \lim_{n\to\infty} n S_{n;r}\left(\varphi(t)(t-x)^2;x\right). \tag{3.8}$$

It is clear that

$$\left| S_{n;r}\left(\varphi(t)(t-x)^2;x\right)\right| \le \left(S_{n;r}(\varphi^2(t);x) S_{n;r}((t-x)^4;x)\right)^{1/2}, \tag{3.9}$$

and by Corollary 3.3

$$\lim_{n\to\infty} S_{n;r}(\varphi^2(t);x) = \varphi^2(x) = 0. \tag{3.10}$$

Moreover, by (2.16) and (2.2) we deduce that the sequence $\left(n^2 S_{n;r}((t-x)^4;x)\right)_1^\infty$ is bounded at every fixed $x \in \mathbb{R}_0$. From this and (3.9) and (3.10) we get

$$\lim_{n\to\infty} n S_{n;r}\left(\varphi(t)(t-x)^2;x\right) = 0$$

which with (3.8) yields the statement (3.7). □

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4. Remarks

Remark 1. We observe that the estimation (1.5) for the operators S_n is better than (3.2) obtained for $S_{n;r}$ with $r \geq 2$. It is generated by formulas (2.3) – (2.5) and Lemma 2.1 which show that the operators $S_{n;r}$, $r \geq 2$, preserve only the function $e_0(x) = 1$. The operators S_n preserve the function $e_k(x) = x^k$, $k = 0, 1$.

Remark 2. In the paper [2], the approximation properties of the Szász-Mirakyan operators S_n in the exponential weight spaces C_q^*, $q > 0$, with the weight function $v_q(x) = e^{-qx}$, $x \in \mathbb{R}_0$ were examined. Obviously the operators $S_{n;r}$, $r \geq 2$, can be investigated also in these spaces.

Remark 3. G. Kirov in [8] defined the new Bernstein polynomials for m-times differentiable functions and showed that these operators have better approximation properties than classical Bernstein polynomials.

The Kirov idea was applied to the operators S_n in [10].

We mention that the Kirov method can be extended to the operators $S_{n;r}$ with $r \geq 2$, i.e. for functions $f \in C_p^m$, $m \in \mathbb{N}$, $p \in \mathbb{N}_0$, and a fixed $2 \leq r \in \mathbb{N}$ we can consider the operators

$$S_{n;r}^*(f;x) := \frac{1}{A_r(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} \sum_{j=0}^{m} \frac{f^{(j)}\left(\frac{rk}{n}\right)}{j!} \left(\frac{rk}{n} - x\right)^j,$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

In [10] it was proved that the $S_{n;1}^*$ have better approximation properties for $f \in C_p^m$, $m \geq 2$, than $S_{n;1}$.

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