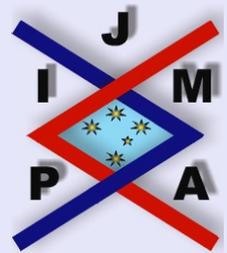


ON ERROR BOUNDS FOR GAUSS–LEGENDRE AND LOBATTO QUADRATURE RULES

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[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

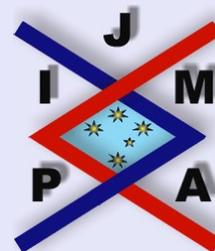
The error bounds for Gauss–Legendre and Lobatto quadratures are proved for four times differentiable functions (instead of six times differentiable functions as in the classical results). Auxiliarily we establish some inequalities for 3–convex functions.

2000 Mathematics Subject Classification: Primary: 41A55, 41A80 Secondary: 26A51, 26D15.

Key words: Convex functions of higher orders, Approximate integration, Quadrature rules.

Contents

1	Introduction	3
	1.1 Convex functions of higher orders	3
	1.2 Quadrature Rules	5
2	Inequalities for 3–Convex Functions	7
3	Error Bounds for Quadrature Rules	9
4	Error Bounds for Quadrature Rules on $[a, b]$	13
	References	



On Error Bounds for Gauss–Legendre and Lobatto Quadrature Rules

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

Page 2 of 17

1. Introduction

The classical error bounds for the Gauss–Legendre quadrature rule (with three knots) and for the Lobatto quadrature rule (with four knots) hold for six times differentiable functions. In this paper we obtain error bounds for these rules for four times differentiable functions. To prove our main results we establish some inequalities for so-called 3-convex functions. In [7] using the same technique the error bounds for Midpoint, Trapezoidal, Simpson and Radau quadrature rules were reproved. We prove our results for functions defined on $[-1, 1]$ and next we translate them to the interval $[a, b]$.

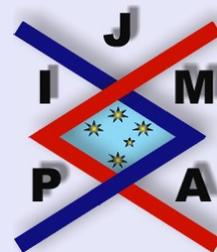
Now we would like to recall the notions and results needed in this paper (cf. also the Introduction to [7]).

1.1. Convex functions of higher orders

Hopf's thesis [2] is probably the first work devoted to higher-order convexity. This concept was also studied among others by Popoviciu [4]. Let $I \subset \mathbb{R}$ be an interval and let $n \in \mathbb{N}$. Recall that the function $f : I \rightarrow \mathbb{R}$ is called n -convex if

$$(1.1) \quad D(x_0, x_1, \dots, x_{n+1}; f) := \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & \dots & x_{n+1}^n \\ f(x_0) & f(x_1) & \dots & f(x_{n+1}) \end{vmatrix} \geq 0$$

for any $x_0, x_1, \dots, x_{n+1} \in I$ such that $x_0 < x_1 < \dots < x_{n+1}$. Obviously 1-convex functions are convex in the classical sense. More information on the



On Error Bounds for
Gauss–Legendre and Lobatto
Quadrature Rules

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

Page 3 of 17

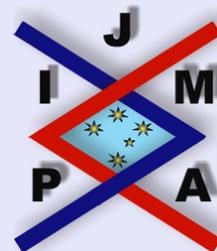
definition and properties of convex functions of higher orders can be found in [2, 3, 4, 6].

The following theorem (cf. [2, 3, 4]) characterizes n -convexity of $(n + 1)$ -times differentiable functions.

Theorem A. *Assume that $f : (a, b) \rightarrow \mathbb{R}$ is an $(n + 1)$ -times differentiable function. Then f is n -convex if and only if $f^{(n+1)}(x) \geq 0, x \in (a, b)$.*

The next result holds for the interval $[a, b]$.

Theorem B. [7, Theorem 1.3] *Assume that $f : [a, b] \rightarrow \mathbb{R}$ is $(n + 1)$ -times differentiable on (a, b) and continuous on $[a, b]$. If $f^{(n+1)}(x) \geq 0, x \in (a, b)$, then f is n -convex.*



On Error Bounds for
Gauss–Legendre and Lobatto
Quadrature Rules

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

Page 4 of 17

1.2. Quadrature Rules

For a function $f : [-1, 1] \rightarrow \mathbb{R}$ we define some operators connected with the quadrature rules:

$$\mathcal{G}_2(f) := \frac{1}{2} \left(f \left(-\frac{\sqrt{3}}{3} \right) + f \left(\frac{\sqrt{3}}{3} \right) \right),$$

$$\mathcal{G}_3(f) := \frac{5}{18} f \left(-\frac{\sqrt{15}}{5} \right) + \frac{4}{9} f(0) + \frac{5}{18} f \left(\frac{\sqrt{15}}{5} \right),$$

$$\mathcal{L}(f) := \frac{1}{12} f(-1) + \frac{5}{12} f \left(-\frac{\sqrt{5}}{5} \right) + \frac{5}{12} f \left(\frac{\sqrt{5}}{5} \right) + \frac{1}{12} f(1),$$

$$\mathcal{S}(f) := \frac{1}{6} (f(-1) + 4f(0) + f(1)),$$

$$\mathcal{I}(f) := \frac{1}{2} \int_{-1}^1 f(x) dx.$$

The operators \mathcal{G}_2 and \mathcal{G}_3 are connected with Gauss–Legendre quadrature rules. The operators \mathcal{L} and \mathcal{S} concern Lobatto and Simpson’s quadrature rules, respectively. The operator \mathcal{I} stands for the integral mean value. Obviously all these operators are linear.

Next we recall the well known quadrature rules (cf. e.g. [5], [8], [9], [10]).

Gauss–Legendre quadratures. *If $f \in C^4([-1, 1])$ then*

$$(1.2) \quad \mathcal{I}(f) = \mathcal{G}_2(f) + \frac{f^{(4)}(\xi)}{270} \quad \text{for some } \xi \in (-1, 1).$$



On Error Bounds for
Gauss–Legendre and Lobatto
Quadrature Rules

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

Page 5 of 17

If $f \in \mathcal{C}^6([-1, 1])$ then

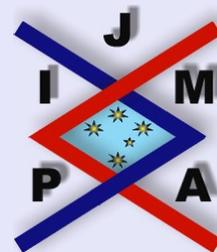
$$(1.3) \quad \mathcal{I}(f) = \mathcal{G}_3(f) + \frac{f^{(6)}(\xi)}{31500} \quad \text{for some } \xi \in (-1, 1).$$

Lobatto quadrature. If $f \in \mathcal{C}^6([-1, 1])$ then

$$(1.4) \quad \mathcal{I}(f) = \mathcal{L}(f) - \frac{f^{(6)}(\xi)}{23625} \quad \text{for some } \xi \in (-1, 1).$$

Simpson's Rule. If $f \in \mathcal{C}^4([-1, 1])$ then

$$(1.5) \quad \mathcal{I}(f) = \mathcal{S}(f) - \frac{f^{(4)}(\xi)}{180} \quad \text{for some } \xi \in (-1, 1).$$



On Error Bounds for
Gauss–Legendre and Lobatto
Quadrature Rules

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

Page 6 of 17

2. Inequalities for 3-Convex Functions

Let $V(x_1, \dots, x_n)$ be the Vandermonde determinant of the terms involved.

Lemma 2.1. *If $f : [-1, 1] \rightarrow \mathbb{R}$ is 3-convex, then the inequality*

$$v^2(f(-u) + f(u)) \leq u^2(f(-v) + f(v)) + 2(v^2 - u^2)f(0)$$

holds for any $0 < u < v \leq 1$.

Proof. Let $0 < u < v \leq 1$. Since f is 3-convex and $-1 \leq -v < -u < 0 < u < v \leq 1$, then by (1.1)

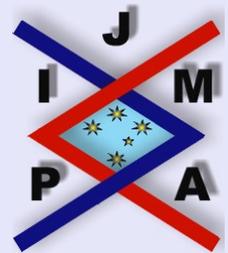
$$0 \leq D(-v, -u, 0, u, v; f) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -v & -u & 0 & u & v \\ v^2 & u^2 & 0 & u^2 & v^2 \\ -v^3 & -u^3 & 0 & u^3 & v^3 \\ f(-v) & f(-u) & f(0) & f(u) & f(v) \end{vmatrix}.$$

Expanding this determinant by the last row we obtain

$$\begin{aligned} &V(-u, 0, u, v)f(-v) - V(-v, 0, u, v)f(-u) + V(-v, -u, u, v)f(0) \\ &\quad - V(-v, -u, 0, v)f(u) + V(-v, -u, 0, u)f(v) \geq 0. \end{aligned}$$

Computing the Vandermonde determinants

$$\begin{aligned} V(-u, 0, u, v) &= V(-v, -u, 0, u) = 2u^3v(v^2 - u^2), \\ V(-v, 0, u, v) &= V(-v, -u, 0, v) = 2uv^3(v^2 - u^2), \\ V(-v, -u, u, v) &= 4uv(v^2 - u^2)^2 \end{aligned}$$



On Error Bounds for
Gauss-Legendre and Lobatto
Quadrature Rules

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

Page 7 of 17

and rearranging the above inequality we obtain

$$2uw^3(v^2 - u^2)(f(-u) + f(u)) \leq 2u^3v(v^2 - u^2)(f(-v) + f(v)) + 4uw(v^2 - u^2)^2 f(0),$$

from which, by $2uw(v^2 - u^2) > 0$, the lemma follows. \square

Proposition 2.2. *If $f : [-1, 1] \rightarrow \mathbb{R}$ is 3-convex, then $\mathcal{G}_2(f) \leq \mathcal{G}_3(f) \leq \mathcal{S}(f)$ and $\mathcal{L}(f) \leq \mathcal{S}(f)$.*

Proof. Setting in Lemma 2.1 $u = \frac{\sqrt{5}}{5}$, $v = 1$ we obtain

$$f\left(-\frac{\sqrt{5}}{5}\right) + f\left(\frac{\sqrt{5}}{5}\right) \leq \frac{1}{5}(f(-1) + f(1)) + \frac{8}{5}f(0).$$

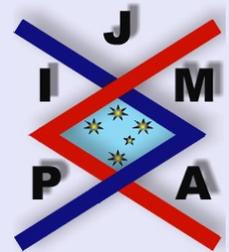
Then

$$5\left(f\left(-\frac{\sqrt{5}}{5}\right) + f\left(\frac{\sqrt{5}}{5}\right)\right) \leq f(-1) + f(1) + 8f(0),$$

whence

$$f(-1) + f(1) + 5\left(f\left(-\frac{\sqrt{5}}{5}\right) + f\left(\frac{\sqrt{5}}{5}\right)\right) \leq 2(f(-1) + f(1)) + 8f(0).$$

Dividing both sides of this inequality by 12 we get $\mathcal{L}(f) \leq \mathcal{S}(f)$. The proofs of the inequalities $\mathcal{G}_2(f) \leq \mathcal{G}_3(f)$ and $\mathcal{G}_3(f) \leq \mathcal{S}(f)$ are similar. \square



On Error Bounds for
Gauss-Legendre and Lobatto
Quadrature Rules

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

Page 8 of 17

3. Error Bounds for Quadrature Rules

In this section we assume that $f \in \mathcal{C}^4([-1, 1])$. Then

$$M_4(f) := \sup_{-1 \leq x \leq 1} |f^{(4)}(x)| < \infty.$$

The classical error bound for the Gauss–Legendre quadrature rule $\mathcal{G}_3(f)$ holds for the six times differentiable function f . This is also the case for the Lobatto quadrature formula $\mathcal{L}(f)$. We prove the error bounds for these quadratures for less regular functions, i.e. for four times differentiable functions. We start with the result for 3–convex functions.

Theorem 3.1. *If $f \in \mathcal{C}^4([-1, 1])$ is 3–convex then $|\mathcal{G}_3(f) - \mathcal{I}(f)| \leq \frac{M_4(f)}{180}$.*

Proof. On account of Theorem A, $f^{(4)} \geq 0$ on $(-1, 1)$. Therefore we conclude from (1.2) that (for some $\xi \in (-1, 1)$)

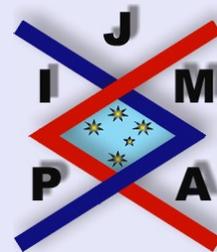
$$(3.1) \quad \mathcal{G}_2(f) - \mathcal{I}(f) = -\frac{f^{(4)}(\xi)}{270} \geq -\frac{f^{(4)}(\xi)}{180} \geq -\frac{M_4(f)}{180}.$$

By Proposition 2.2

$$(3.2) \quad \mathcal{G}_2(f) \leq \mathcal{G}_3(f) \leq \mathcal{S}(f).$$

Next, by (1.5) there exists an $\eta \in (-1, 1)$ such that

$$(3.3) \quad \mathcal{S}(f) - \mathcal{I}(f) = \frac{f^{(4)}(\eta)}{180} \leq \frac{M_4(f)}{180}.$$



On Error Bounds for
Gauss–Legendre and Lobatto
Quadrature Rules

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

Page 9 of 17

By (3.1), (3.2) and (3.3) we obtain

$$-\frac{M_4(f)}{180} \leq \mathcal{G}_2(f) - \mathcal{I}(f) \leq \mathcal{G}_3(f) - \mathcal{I}(f) \leq \mathcal{S}(f) - \mathcal{I}(f) \leq \frac{M_4(f)}{180},$$

from which the result follows. \square

To prove the next two results we need to make some observations.

Remark 1. For $f \in \mathcal{C}^4([-1, 1])$ we consider the function $g(x) = \frac{M_4(f)}{24}x^4$. Then

$$(3.4) \quad |f^{(4)}(x)| \leq M_4(f) = g^{(4)}(x), \quad -1 \leq x \leq 1.$$

Hence $(g - f)^{(4)} \geq 0$ and $(g + f)^{(4)} \geq 0$. Thus Theorem B implies that the functions $g - f$ and $g + f$ are 3-convex. Moreover, using (3.4) we obtain

$$(g - f)^{(4)}(x) = g^{(4)}(x) - f^{(4)}(x) = M_4(f) - f^{(4)}(x) \leq 2M_4(f)$$

and

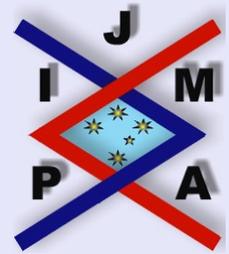
$$(g + f)^{(4)}(x) = M_4(f) + f^{(4)}(x) \leq 2M_4(f).$$

Then

$$(3.5) \quad M_4(g - f) \leq 2M_4(f) \quad \text{and} \quad M_4(g + f) \leq 2M_4(f).$$

By (1.3) and (1.4) we have also $\mathcal{G}_3(g) = \mathcal{L}(g) = \mathcal{I}(g)$.

Corollary 3.2. If $f \in \mathcal{C}^4([-1, 1])$ then $|\mathcal{G}_3(f) - \mathcal{I}(f)| \leq \frac{M_4(f)}{90}$.



On Error Bounds for
Gauss–Legendre and Lobatto
Quadrature Rules

Szymon Wąsowicz

Title Page

Contents

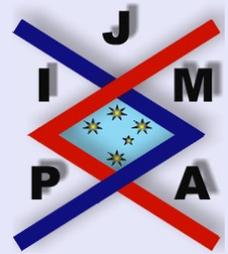


Go Back

Close

Quit

Page 10 of 17



Title Page

Contents



Go Back

Close

Quit

Page 11 of 17

Proof. By Remark 1 the function $g + f$ is 3-convex and $\mathcal{G}_3(g) = \mathcal{I}(g)$, where $g(x) = \frac{M_4(f)}{24}x^4$. Theorem 3.1 and the linearity of the operators \mathcal{G}_3 and \mathcal{I} now imply

$$\begin{aligned} |\mathcal{G}_3(f) - \mathcal{I}(f)| &= |\mathcal{G}_3(g) + \mathcal{G}_3(f) - \mathcal{I}(g) - \mathcal{I}(f)| \\ &= |\mathcal{G}_3(g + f) - \mathcal{I}(g + f)| \leq \frac{M_4(g + f)}{180}. \end{aligned}$$

This inequality together with (3.5) concludes the proof. □

Before we prove the error bound for the Lobatto quadrature rule we make the following simple observation.

Remark 2. By Proposition 2.2 and (1.5) we obtain that for a 3-convex function $f \in \mathcal{C}^4([-1, 1])$ there exists a $\xi \in (-1, 1)$ such that $\mathcal{L}(f) \leq \mathcal{S}(f) = \mathcal{I}(f) + \frac{f^{(4)}(\xi)}{180}$. This gives

$$(3.6) \quad \mathcal{L}(f) - \mathcal{I}(f) \leq \frac{M_4(f)}{180}.$$

Theorem 3.3. *If $f \in \mathcal{C}^4([-1, 1])$ then $|\mathcal{L}(f) - \mathcal{I}(f)| \leq \frac{M_4(f)}{90}$.*

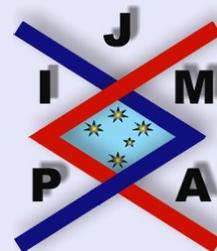
Proof. By Remark 1 the functions $g - f$ and $g + f$ are 3-convex, where $g(x) = \frac{M_4(f)}{24}x^4$. Then by (3.6)

$$\mathcal{L}(g - f) - \mathcal{I}(g - f) \leq \frac{M_4(g - f)}{180} \quad \text{and} \quad \mathcal{L}(g + f) - \mathcal{I}(g + f) \leq \frac{M_4(g + f)}{180}.$$

Because of $\mathcal{L}(g) = \mathcal{I}(g)$ and by linearity of the operators \mathcal{L} and \mathcal{I} we have

$$-(\mathcal{L}(f) - \mathcal{I}(f)) \leq \frac{M_4(g - f)}{180} \quad \text{and} \quad \mathcal{L}(f) - \mathcal{I}(f) \leq \frac{M_4(g + f)}{180}.$$

These inequalities together with (3.5) conclude the proof. \square



**On Error Bounds for
Gauss–Legendre and Lobatto
Quadrature Rules**

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

Page 12 of 17

4. Error Bounds for Quadrature Rules on $[a, b]$

In the next section we translate the quadrature rules and error bounds obtained in Theorem 3.1, Corollary 3.2 and Theorem 3.3 to the interval $[a, b]$. To do this task we use the following change of variables: for $t \in [-1, 1]$ let

$$(4.1) \quad x = \frac{1-t}{2}a + \frac{1+t}{2}b.$$

Then $x \in [a, b]$. For a function $f : [a, b] \rightarrow \mathbb{R}$ we define $F : [-1, 1] \rightarrow \mathbb{R}$ by

$$(4.2) \quad F(t) := f(x).$$

Remark 3. If f is n -convex on $[a, b]$ then the function F given by (4.2) is n -convex on $[-1, 1]$ (cf. Popoviciu [4], Chapter II, §1, point 12).

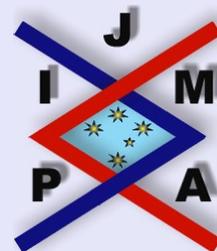
By the substitution (4.1) we obtain

$$(4.3) \quad \mathcal{I}(F) = \frac{1}{b-a} \int_a^b f(x) dx.$$

Let $f \in \mathcal{C}^4([a, b])$. Using (4.1) and considering the function F defined by (4.2) we have $F^{(4)}(t) = \left(\frac{b-a}{2}\right)^4 f^{(4)}(x)$ for $x \in [a, b]$, $t \in [-1, 1]$. It is easy to see that $F \in \mathcal{C}^4([-1, 1])$. Let

$$M_4(F) := \sup_{-1 \leq t \leq 1} |F^{(4)}(t)| \quad \text{and} \quad M_4(f) := \sup_{a \leq x \leq b} |f^{(4)}(x)|.$$

Then $M_4(F) = \left(\frac{b-a}{2}\right)^4 M_4(f)$. If moreover f is 3-convex then by Remark 3 F is also 3-convex.



On Error Bounds for
Gauss–Legendre and Lobatto
Quadrature Rules

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

Page 13 of 17

Corollary 4.1. *If $f \in C^4([a, b])$ then*

$$(4.4) \quad \left| \frac{5}{18}f\left(\frac{5+\sqrt{15}}{10}a + \frac{5-\sqrt{15}}{10}b\right) + \frac{4}{9}f\left(\frac{a+b}{2}\right) + \frac{5}{18}f\left(\frac{5-\sqrt{15}}{10}a + \frac{5+\sqrt{15}}{10}b\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^4 M_4(f)}{1440}.$$

If moreover f is 3-convex then the right hand side of (4.4) can be replaced by $\frac{(b-a)^4 M_4(f)}{2880}$.

Proof. By (4.1) and (4.2) we get

$$f\left(\frac{5+\sqrt{15}}{10}a + \frac{5-\sqrt{15}}{10}b\right) = F\left(-\frac{\sqrt{15}}{5}\right), \quad f\left(\frac{a+b}{2}\right) = F(0)$$

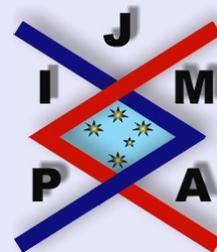
and

$$f\left(\frac{5-\sqrt{15}}{10}a + \frac{5+\sqrt{15}}{10}b\right) = F\left(\frac{\sqrt{15}}{5}\right).$$

Since $F \in C^4([-1, 1])$ then using Corollary 3.2 and (4.3) we obtain

$$|\mathcal{G}_3(F) - \mathcal{I}(F)| \leq \frac{M_4(F)}{90} = \left(\frac{b-a}{2}\right)^4 \cdot \frac{M_4(f)}{90},$$

which proves the desired inequality (4.4). For a 3-convex function f we argue similarly using Theorem 3.1. \square



On Error Bounds for
Gauss–Legendre and Lobatto
Quadrature Rules

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

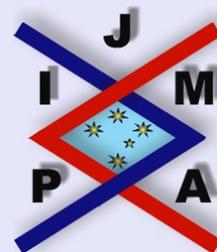
Page 14 of 17

Using Theorem 3.3 we obtain by the same reasoning

Corollary 4.2. *If $f \in C^4([a, b])$ then*

$$(4.5) \quad \left| \frac{1}{12}f(a) + \frac{5}{12}f\left(\frac{5+\sqrt{5}}{10}a + \frac{5-\sqrt{5}}{10}b\right) + \frac{5}{12}f\left(\frac{5-\sqrt{5}}{10}a + \frac{5+\sqrt{5}}{10}b\right) + \frac{1}{12}f(b) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^4 M_4(f)}{1440}.$$

Remark 4. For six times differentiable functions inequalities similar to (4.4) and (4.5) can be obtained using Bessenyei and Pales' results [1, Corollary 5] and the method of convex functions of higher orders presented in this paper (cf. also [7]). However, our results are obtained for less regular functions.



On Error Bounds for
Gauss–Legendre and Lobatto
Quadrature Rules

Szymon Wąsowicz

Title Page

Contents



Go Back

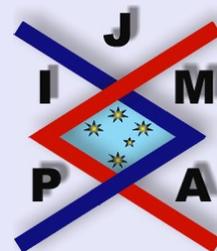
Close

Quit

Page 15 of 17

References

- [1] M. BESSENYEI AND Zs. PÁLES, Higher-order generalizations of Hadamard's inequality, *Publ. Math. Debrecen*, **61** (2002), 623–643.
- [2] E. HOPF, Über die Zusammenhänge zwischen gewissen höheren Differenzenquotienten reeller Funktionen einer reellen Variablen und deren Differenzierbarkeitseigenschaften, Dissertation, Friedrich–Wilhelms–Universität Berlin, 1926.
- [3] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Państwowe Wydawnictwo Naukowe (Polish Scientific Publishers) and Uniwersytet Śląski, Warszawa–Kraków–Katowice 1985.
- [4] T. POPOVICIU, Sur quelques propriétés des fonctions d'une ou de deux variables réelles, *Mathematica (Cluj)*, **8** (1934), 1–85.
- [5] A. RALSTON, *A First Course in Numerical Analysis*, McGraw–Hill Book Company, New York, St. Louis, San Francisco, Toronto, London, Sydney, 1965.
- [6] A.W. ROBERTS AND D.E. VARBERG, *Convex Functions*, Academic Press, New York 1973.
- [7] S. WĄSOWICZ, Some inequalities connected with an approximate integration, *J. Ineq. Pure & Appl. Math.*, **6**(2) (2005), Art. 47. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=516>].



On Error Bounds for
Gauss–Legendre and Lobatto
Quadrature Rules

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

Page 16 of 17

- [8] E.W. WEISSTEIN, Legendre–Gauss quadrature, From MathWorld–A Wolfram Web Resource. <http://mathworld.wolfram.com/Legendre-GaussQuadrature.html>
- [9] E.W. WEISSTEIN, Lobatto quadrature, From MathWorld–A Wolfram Web Resource. <http://mathworld.wolfram.com/LobattoQuadrature.html>
- [10] E.W. WEISSTEIN, Simpson’s Rule, From MathWorld–A Wolfram Web Resource. <http://mathworld.wolfram.com/SimpsonsRule.html>



**On Error Bounds for
Gauss–Legendre and Lobatto
Quadrature Rules**

Szymon Wąsowicz

Title Page

Contents



Go Back

Close

Quit

Page 17 of 17